NCERT Text Books: 11th Class Physics

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CHAPTER ONE

PHYSICAL WORLD

1.1 WHAT IS PHYSICS ?

Humans have always been curious about the world around them. The night sky with its bright celestial objects has fascinated humans since time immemorial. The regular repetitions of the day and night, the annual cycle of seasons, the eclipses, the tides, the volcanoes, the rainbow have always been a source of wonder. The world has an astonishing variety of materials and a bewildering diversity of life and behaviour. The inquiring and imaginative human mind has responded to the wonder and awe of nature in different ways. One kind of response from the earliest times has been to observe the physical environment carefully, look for any meaningful patterns and relations in natural phenomena, and build and use new tools to interact with nature. This human endeavour led, in course of time, to modern science and technology.

The word **Science** originates from the Latin verb *Scientia* meaning 'to know'. The Sanskrit word *Vijnan* and the Arabic word *Ilm* convey similar meaning, namely 'knowledge'. Science, in a broad sense, is as old as human species. The early civilisations of Egypt, India, China, Greece, Mesopotamia and many others made vital contributions to its progress. From the sixteenth century onwards, great strides were made in science in Europe. By the middle of the twentieth century, science had become a truly international enterprise, with many cultures and countries contributing to its rapid growth.

What is Science and what is the so-called **Scientific Method**? Science is a systematic attempt to understand natural phenomena in as much detail and depth as possible, and use the knowledge so gained to predict, modify and control phenomena. Science is exploring, experimenting and predicting from what we see around us. The curiosity to learn about the world, unravelling the secrets of nature is the first step towards the discovery of science. The scientific method involves several interconnected steps : Systematic observations, controlled experiments, qualitative and

quantitative reasoning, mathematical modelling, prediction and verification or falsification of theories. Speculation and conjecture also have a place in science; but ultimately, a scientific theory, to be acceptable, must be verified by relevant observations or experiments. There is much philosophical debate about the nature and method of science that we need not discuss here.

The interplay of theory and observation (or experiment) is basic to the progress of science. Science is ever dynamic. There is no 'final' theory in science and no unquestioned authority among scientists. As observations improve in detail and precision or experiments yield new results, theories must account for them, if necessary, by introducing modifications. Sometimes the modifications may not be drastic and may lie within the framework of existing theory. For example, when Johannes Kepler (1571-1630) examined the extensive data on planetary motion collected by Tycho Brahe (1546-1601), the planetary circular orbits in heliocentric theory (sun at the centre of the solar system) imagined by Nicolas Copernicus (1473–1543) had to be replaced by elliptical orbits to fit the data better. Occasionally, however, the existing theory is simply unable to explain new observations. This causes a major upheaval in science. In the beginning of the twentieth century, it was realised that Newtonian mechanics, till then a very successful theory, could not explain some of the most basic features of atomic phenomena. Similarly, the then accepted wave picture of light failed to explain the photoelectric effect properly. This led to the development of a radically new theory (*Quantum Mechanics*) to deal with atomic and molecular phenomena.

Just as a new experiment may suggest an alternative theoretical model, a theoretical advance may suggest what to look for in some experiments. The result of experiment of scattering of alpha particles by gold foil, in 1911 by Ernest Rutherford (1871–1937) established the nuclear model of the atom, which then became the basis of the quantum theory of hydrogen atom given in 1913 by Niels Bohr (1885–1962). On the other hand, the concept of antiparticle was first introduced theoretically by Paul Dirac (1902–1984) in 1930 and confirmed two years later by the experimental discovery of positron (antielectron) by Carl Anderson.

Physics is a basic discipline in the category of **Natural Sciences**, which also includes other disciplines like Chemistry and Biology. The word **Physics** comes from a Greek word meaning nature. Its Sanskrit equivalent is *Bhautiki* that is used to refer to the study of the physical world. A precise definition of this discipline is neither possible nor necessary. We can broadly describe physics as a study of the basic laws of nature and their manifestation in different natural phenomena. The scope of physics is described briefly in the next section. Here we remark on two principal thrusts in physics : **unification** and **reduction**.

In Physics, we attempt to explain diverse physical phenomena ***in terms of a few concepts and laws***. The effort is to see the physical world as manifestation of some universal laws in different domains and conditions. For example, the same law of gravitation (given by Newton) describes the fall of an apple to the ground, the motion of the moon around the earth and the motion of planets around the sun. Similarly, the basic laws of electromagnetism (Maxwell's equations) govern all electric and magnetic phenomena. The attempts to unify fundamental forces of nature (section 1.4) reflect this same quest for unification.

A related effort is to derive the properties of a bigger, more complex, system from the properties and interactions of its constituent simpler parts. This approach is called **reductionism** and is at the heart of physics. For example, the subject of thermodynamics, developed in the nineteenth century, deals with bulk systems in terms of macroscopic quantities such as temperature, internal energy, entropy, etc. Subsequently, the subjects of kinetic theory and statistical mechanics interpreted these quantities in terms of the properties of the molecular constituents of the bulk system. In particular, the temperature was seen to be related to the average kinetic energy of molecules of the system.

1.2 SCOPE AND EXCITEMENT OF PHYSICS

We can get some idea of the scope of physics by looking at its various sub-disciplines. Basically, there are two domains of interest : macroscopic and microscopic. The macroscopic domain includes phenomena at the laboratory, terrestrial and astronomical scales. The microscopic domain includes atomic, molecular and nuclear

phenomena*. **Classical Physics** deals mainly with macroscopic phenomena and includes subjects like **Mechanics**, **Electrodynamics, Optics** and **Thermodynamics**. Mechanics founded on Newton's laws of motion and the law of gravitation is concerned with the motion (or equilibrium) of particles, rigid and deformable bodies, and general systems of particles. The propulsion of a rocket by a jet of ejecting gases, propagation of water waves or sound waves in air, the equilibrium of a bent rod under a load, etc., are problems of mechanics. Electrodynamics deals with electric and magnetic phenomena associated with charged and magnetic bodies. Its basic laws were given by Coulomb, Oersted, Ampere and Faraday, and encapsulated by Maxwell in his famous set of equations. The motion of a current-carrying conductor in a magnetic field, the response of a circuit to an ac voltage (signal), the working of an antenna, the propagation of radio waves in the ionosphere, etc., are problems of electrodynamics. Optics deals with the phenomena involving light. The working of telescopes and microscopes, colours exhibited by thin films, etc., are topics in optics. Thermodynamics, in contrast to mechanics, does not deal with the motion of bodies as a whole. Rather, it deals with systems in macroscopic equilibrium and is concerned with changes in internal energy, temperature, entropy, etc., of the system through external work and transfer of heat. The efficiency of heat engines and refrigerators, the direction of a physical or chemical process, etc., are problems of interest in thermodynamics.

The microscopic domain of physics deals with the constitution and structure of matter at the minute scales of atoms and nuclei (and even lower scales of length) and their interaction with different probes such as electrons, photons and other elementary particles. Classical physics is inadequate to handle this domain and Quantum Theory is currently accepted as the proper framework for explaining microscopic phenomena. Overall, the edifice of physics is beautiful and imposing and you will appreciate it more as you pursue the subject.

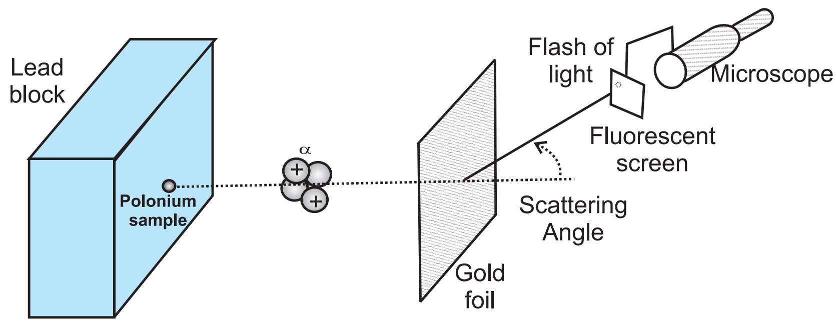


Fig. 1.1 *Theory and experiment go hand in hand in physics and help each other's progress. The alpha scattering experiments of Rutherford gave the nuclear model of the atom.*

You can now see that the scope of physics is truly vast. It covers a tremendous range of magnitude of physical quantities like length, mass, time, energy, etc. At one end, it studies phenomena at the very small scale of length (10^{-14} m or even less) involving electrons, protons, etc.; at the other end, it deals with astronomical phenomena at the scale of galaxies or even the entire universe whose extent is of the order of 10^{26} m. The two length scales differ by a factor of 10^{40} or even more. The range of time scales can be obtained by dividing the length scales by the speed of light : 10^{-22} s to 10^{18} s. The range of masses goes from, say, 10^{-30} kg (mass of an electron) to 10^{55} kg (mass of known observable universe). Terrestrial phenomena lie somewhere in the middle of this range.

* *Recently, the domain intermediate between the macroscopic and the microscopic (the so-called mesoscopic physics), dealing with a few tens or hundreds of atoms, has emerged as an exciting field of research.*

Physics is exciting in many ways. To some people the excitement comes from the elegance and universality of its basic theories, from the fact that a few basic concepts and laws can explain phenomena covering a large range of magnitude of physical quantities. To some others, the challenge in carrying out imaginative new experiments to unlock the secrets of nature, to verify or refute theories, is thrilling. Applied physics is equally demanding. Application and exploitation of physical laws to make useful devices is the most interesting and exciting part and requires great ingenuity and persistence of effort.

What lies behind the phenomenal progress of physics in the last few centuries? Great progress usually accompanies changes in our basic perceptions. First, it was realised that for scientific progress, only qualitative thinking, though no doubt important, is not enough. Quantitative measurement is central to the growth of science, especially physics, because the laws of nature happen to be expressible in precise mathematical equations. The second most important insight was that the basic laws of physics are universal — the same laws apply in widely different contexts. Lastly, the strategy of approximation turned out to be very successful. Most observed phenomena in daily life are rather complicated manifestations of the basic laws. Scientists recognised the importance of extracting the essential features of a phenomenon from its less significant aspects. It is not practical to take into account all the complexities of a phenomenon in one go. A good strategy is to focus first on the essential features, discover the basic principles and then introduce corrections to build a more refined theory of the phenomenon. For example, a stone and a feather dropped from the same height do not reach the ground at the same time. The reason is that the essential aspect of the phenomenon, namely free fall under gravity, is complicated by the presence of air resistance. To get the law of free fall under gravity, it is better to create a situation wherein the air resistance is negligible. We can, for example, let the stone and the feather fall through a long evacuated tube. In that case, the two objects will fall almost at the same rate, giving the basic law that acceleration due to gravity is independent of the mass of the object. With the basic law thus found, we can go back to the feather, introduce corrections due to air resistance, modify the existing theory and try to build a more realistic

Hypothesis, axioms and models

One should not think that everything can be proved with physics and mathematics. All physics, and also mathematics, is based on assumptions, each of which is variously called a hypothesis or axiom or postulate, etc.

For example, the universal law of gravitation proposed by Newton is an assumption or hypothesis, which he proposed out of his ingenuity. Before him, there were several observations, experiments and data, on the motion of planets around the sun, motion of the moon around the earth, pendulums, bodies falling towards the earth etc. Each of these required a separate explanation, which was more or less qualitative. What the universal law of gravitation says is that, *if we assume* that any two bodies in the universe attract each other with a force proportional to the product of their masses and inversely proportional to the square of the distance between them, then we can explain all these observations in one stroke. It not only explains these phenomena, it also allows us to predict the results of future experiments.

A hypothesis is a supposition without assuming that it is true. It would not be fair to ask anybody to prove the universal law of gravitation, because it cannot be proved. It can be verified and substantiated by experiments and observations.

An axiom is a self-evident truth while a model is a theory proposed to explain observed phenomena. But you need not worry at this stage about the nuances in using these words. For example, next year you will learn about Bohr's model of hydrogen atom, in which Bohr assumed that an electron in the hydrogen atom follows certain rules (postutates). Why did he do that? There was a large amount of spectroscopic data before him which no other theory could explain. So Bohr said that if we assume that an atom behaves in such a manner, we can explain all these things at once.

Einstein's special theory of relativity is also based on two postulates, the constancy of the speed of electromagnetic radiation and the validity of physical laws in all inertial frame of reference. It would not be wise to ask somebody to prove that the speed of light in vacuum is constant, independent of the source or observer.

In mathematics too, we need axioms and hypotheses at every stage. Euclid's statement that parallel lines never meet, is a hypothesis. This means that if we assume this statement, we can explain several properties of straight lines and two or three dimensional figures made out of them. But if you don't assume it, you are free to use a different axiom and get a new geometry, as has indeed happened in the past few centuries and decades.

theory of objects falling to the earth under gravity.

1.3 PHYSICS, TECHNOLOGY AND SOCIETY

The connection between physics, technology and society can be seen in many examples. The discipline of thermodynamics arose from the need to understand and improve the working of heat engines. The steam engine, as we know, is inseparable from the Industrial Revolution in England in the eighteenth century, which had great impact on the course of human civilisation. Sometimes technology gives rise to new physics; at other times physics generates new technology. An example of the latter is the wireless communication technology that followed the discovery of the basic laws of electricity and magnetism in the nineteenth century. The applications of physics are not always easy to foresee. As late as 1933, the great physicist Ernest Rutherford had dismissed the possibility of tapping energy from atoms. But only a few years later, in 1938, Hahn and Meitner discovered the phenomenon of neutron-induced fission of uranium, which would serve as the basis of nuclear power reactors and nuclear weapons. Yet another important example of physics giving rise to technology is the silicon 'chip' that triggered the computer revolution in the last three decades of the twentieth century.

A most significant area to which physics has and will contribute is the development of alternative energy resources. The fossil fuels of the planet are dwindling fast and there is an urgent need to discover new and affordable sources of energy. Considerable progress has already been made in this direction (for example, in conversion of solar energy, geothermal energy, etc., into electricity), but much more is still to be accomplished.

Table1.1 lists some of the great physicists, their major contribution and the country of origin. You will appreciate from this table the multi-cultural, international character of the scientific endeavour. Table 1.2 lists some important technologies and the principles of physics they are based on. Obviously, these tables are not exhaustive. We urge you to try to add many names and items to these tables with the help of your teachers, good books and websites on science. You will find that this exercise is very educative and also great fun. And, assuredly, it will never end. The progress of science is unstoppable!

Physics is the study of nature and natural phenomena. Physicists try to discover the rules that are operating in nature, on the basis of observations, experimentation and analysis. Physics deals with certain basic rules/laws governing the natural world. What is the nature

Table 1.1 **Some physicists from different countries of the world and their major contributions**

Name	Major contribution/discovery	Country of Origin
Archimedes	Principle of buoyancy; Principle of the lever	Greece
Galileo Galilei	Law of inertia	Italy
Christiaan Huygens	Wave theory of light	Holland
Isaac Newton	Universal law of gravitation; Laws of motion; Reflecting telescope	U.K.
Michael Faraday	Laws of electromagnetic induction	U.K.
James Clerk Maxwell	Electromagnetic theory; Light-an electromagnetic wave	U.K.
Heinrich Rudolf Hertz	Generation of electromagnetic waves	Germany
J.C. Bose	Ultra short radio waves	India
W.K. Roentgen	X-rays	Germany
J.J. Thomson	Electron	U.K.
Marie Sklodowska Curie	Discovery of radium and polonium; Studies on natural radioactivity	Poland
Albert Einstein	Explanation of photoelectric effect; Theory of relativity	Germany

Name	Major contribution/discovery	Country of Origin
Victor Francis Hess	Cosmic radiation	Austria
R.A. Millikan	Measurement of electronic charge	U.S.A.
Ernest Rutherford	Nuclear model of atom	New Zealand
Niels Bohr	Quantum model of hydrogen atom	Denmark
C.V. Raman	Inelastic scattering of light by molecules	India
Louis Victor de Borglie	Wave nature of matter	France
M.N. Saha	Thermal ionisation	India
S.N. Bose	Quantum statistics	India
Wolfgang Pauli	Exclusion principle	Austria
Enrico Fermi	Controlled nuclear fission	Italy
Werner Heisenberg	Quantum mechanics; Uncertainty principle	Germany
Paul Dirac	Relativistic theory of electron; Quantum statistics	U.K.
Edwin Hubble	Expanding universe	U.S.A.
Ernest Orlando Lawrence	Cyclotron	U.S.A.
James Chadwick	Neutron	U.K.
Hideki Yukawa	Theory of nuclear forces	Japan
Homi Jehangir Bhabha	Cascade process of cosmic radiation	India
Lev Davidovich Landau	Theory of condensed matter; Liquid helium	Russia
S. Chandrasekhar	Chandrasekhar limit, structure and evolution of stars	India
John Bardeen	Transistors; Theory of super conductivity	U.S.A.
C.H. Townes	Maser; Laser	U.S.A.
Abdus Salam	Unification of weak and electromagnetic interactions	Pakistan

of physical laws? We shall now discuss the nature of fundamental forces and the laws that govern the diverse phenomena of the physical world.

1.4 FUNDAMENTAL FORCES IN NATURE*

We all have an intuitive notion of force. In our experience, force is needed to push, carry or throw objects, deform or break them. We also experience the impact of forces on us, like when a moving object hits us or we are in a merry-go-round. Going from this intuitive notion to the proper scientific concept of force is not a trivial matter. Early thinkers like Aristotle had wrong ideas about it. The correct notion of force was arrived at by Isaac Newton in his famous laws of motion. He also gave an explicit form for the force for gravitational attraction between two bodies. We shall learn these matters in subsequent chapters.

In the macroscopic world, besides the gravitational force, we encounter several kinds of forces: muscular force, contact forces between bodies, friction (which is also a contact force parallel to the surfaces in contact), the forces exerted by compressed or elongated springs and taut strings and ropes (tension), the force of buoyancy and viscous force when solids are in

* *Sections 1.4 and 1.5 contain several ideas that you may not grasp fully in your first reading. However, we advise you to read them carefully to develop a feel for some basic aspects of physics. These are some of the areas which continue to occupy the physicists today.*

Table 1.2 Link between technology and physics

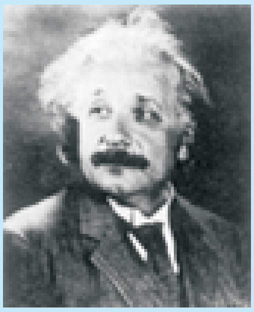
Technology	Scientific principle(s)
Steam engine	Laws of thermodynamics
Nuclear reactor	Controlled nuclear fission
Radio and Television	Generation, propagation and detection of electromagnetic waves
Computers	Digital logic
Lasers	Light amplification by stimulated emission of radiation
Production of ultra high magnetic fields	Superconductivity
Rocket propulsion	Newton's laws of motion
Electric generator	Faraday's laws of electromagnetic induction
Hydroelectric power	Conversion of gravitational potential energy into electrical energy
Aeroplane	Bernoulli's principle in fluid dynamics
Particle accelerators	Motion of charged particles in electromagnetic fields
Sonar	Reflection of ultrasonic waves
Optical fibres	Total internal reflection of light
Non-reflecting coatings	Thin film optical interference
Electron microscope	Wave nature of electrons
Photocell	Photoelectric effect
Fusion test reactor (Tokamak)	Magnetic confinement of plasma
Giant Metrewave Radio Telescope (GMRT)	Detection of cosmic radio waves
Bose-Einstein condensate	Trapping and cooling of atoms by laser beams and magnetic fields.

contact with fluids, the force due to pressure of a fluid, the force due to surface tension of a liquid, and so on. There are also forces involving charged and magnetic bodies. In the microscopic domain again, we have electric and magnetic forces, nuclear forces involving protons and neutrons, interatomic and intermolecular forces, etc. We shall get familiar with some of these forces in later parts of this course.

A great insight of the twentieth century physics is that these different forces occurring in different contexts actually arise from only a small number of fundamental forces in nature. For example, the elastic spring force arises due to the net attraction/repulsion between the neighbouring atoms of the spring when the spring is elongated/compressed. This net attraction/repulsion can be traced to the (unbalanced) sum of electric forces between the charged constituents of the atoms.

In principle, this means that the laws for 'derived' forces (such as spring force, friction) are not independent of the laws of fundamental forces in nature. The origin of these derived forces is, however, very complex.

At the present stage of our understanding, we know of four fundamental forces in nature, which are described in brief here :



Albert Einstein (1879-1955)

Albert Einstein, born in Ulm, Germany in 1879, is universally regarded as one of the greatest physicists of all time. His astonishing scientific career began with the publication of three path-breaking papers in 1905. In the first paper, he introduced the notion of light quanta (now called photons) and used it to explain the features of photoelectric effect that the classical wave theory of radiation could not account for. In the second paper, he developed a theory of Brownian motion that was confirmed experimentally a few years later and provided a convincing evidence of the atomic picture of matter. The third paper gave birth to the special theory of relativity that made Einstein a legend in his own life time. In the next decade, he explored the consequences of his new theory which included, among other things, the mass-energy equivalence enshrined in his famous equation $E = mc^2$. He also created the general version of relativity (The General Theory of Relativity), which is the modern theory of gravitation. Some of Einstein's most significant later contributions are: the notion of stimulated emission introduced in an alternative derivation of Planck's blackbody radiation law, static model of the universe which started modern cosmology, quantum statistics of a gas of massive bosons, and a critical analysis of the foundations of quantum mechanics. The year 2005 was declared as International Year of Physics, in recognition of Einstein's monumental contribution to physics, in year 1905, describing revolutionary scientific ideas that have since influenced all of modern physics.

1.4.1 Gravitational Force

The gravitational force is the force of mutual attraction between any two objects by virtue of their masses. It is a universal force. Every object experiences this force due to every other object in the universe. All objects on the earth, for example, experience the force of gravity due to the earth. In particular, gravity governs the motion of the moon and artificial satellites around the earth, motion of the earth and planets around the sun, and, of course, the motion of bodies falling to the earth. It plays a key role in the large-scale phenomena of the universe, such as formation and evolution of stars, galaxies and galactic clusters.

1.4.2 Electromagnetic Force

Electromagnetic force is the force between charged particles. In the simpler case when charges are at rest, the force is given by Coulomb's law : attractive for unlike charges and repulsive for like charges. Charges in motion produce magnetic effects and a magnetic field gives rise to a force on a moving charge. Electric and magnetic effects are, in general, inseparable – hence the name electromagnetic force. Like the gravitational force, electromagnetic force acts over large distances and does not need any intervening medium. It is enormously strong compared to gravity. The electric force between two protons, for example, is 10^{36} times the gravitational force between them, for any fixed distance.

Matter, as we know, consists of elementary charged constituents like electrons and protons. Since the electromagnetic force is so much stronger than the gravitational force, it dominates all phenomena at atomic and molecular scales. (The other two forces, as we shall see, operate only at nuclear scales.) Thus it is mainly the electromagnetic force that governs the structure of atoms and molecules, the dynamics of chemical reactions and the mechanical, thermal and other properties of materials. It underlies the macroscopic forces like 'tension', 'friction', 'normal force', 'spring force', etc.

Gravity is always attractive, while electromagnetic force can be attractive or repulsive. Another way of putting it is that mass comes only in one variety (there is no negative mass), but charge comes in two varieties : positive and negative charge. This is what makes all the difference. Matter is mostly electrically neutral (net charge is zero). Thus, electric force is largely zero and gravitational force dominates terrestrial phenomena. Electric force manifests itself in atmosphere where the atoms are ionised and that leads to lightning.

Satyendranath Bose (1894-1974)

Satyendranath Bose, born in Calcutta in 1894, is among the great Indian physicists who made a fundamental contribution to the advance of science in the twentieth century. An outstanding student throughout, Bose started his career in 1916 as a lecturer in physics in Calcutta University; five years later he joined Dacca University. Here in 1924, in a brilliant flash of insight, Bose gave a new derivation of Planck's law, treating radiation as a gas of photons and employing new statistical methods of counting of photon states. He wrote a short paper on the subject and sent it to Einstein who immediately recognised its great significance, translated it in German and forwarded it for publication. Einstein then applied the same method to a gas of molecules.

The key new conceptual ingredient in Bose's work was that the particles were regarded as indistinguishable, a radical departure from the assumption that underlies the classical Maxwell-Boltzmann statistics. It was soon realised that the new Bose-Einstein statistics was applicable to particles with integers spins, and a new quantum statistics (Fermi-Dirac statistics) was needed for particles with half integers spins satisfying Pauli's exclusion principle. Particles with integers spins are now known as bosons in honour of Bose.

An important consequence of Bose-Einstein statistics is that a gas of molecules below a certain temperature will undergo a phase transition to a state where a large fraction of atoms populate the same lowest energy state. Some seventy years were to pass before the pioneering ideas of Bose, developed further by Einstein, were dramatically confirmed in the observation of a new state of matter in a dilute gas of ultra cold alkali atoms - the Bose-Eintein condensate.

If we reflect a little, the enormous strength of the electromagnetic force compared to gravity is evident in our daily life. When we hold a book in our hand, we are balancing the gravitational force on the book due to the huge mass of the earth by the 'normal force' provided by our hand. The latter is nothing but the net electromagnetic force between the charged constituents of our hand and the book, at the surface in contact. If electromagnetic force were not intrinsically so much stronger than gravity, the hand of the strongest man would crumble under the weight of a feather ! Indeed, to be consistent, in that circumstance, we ourselves would crumble under our own weight !

1.4.3 Strong Nuclear Force

The strong nuclear force binds protons and neutrons in a nucleus. It is evident that without some attractive force, a nucleus will be unstable due to the electric repulsion between its protons. This attractive force cannot be gravitational since force of gravity is negligible compared to the electric force. A new basic force must, therefore, be invoked. The strong nuclear force is the strongest of all fundamental forces, about 100 times the electromagnetic force in strength. It is charge-independent and acts equally between a proton and a proton, a neutron and a neutron, and a proton and a neutron. Its range is, however, extremely small, of about nuclear dimensions (10^{-15}m). It is responsible for the stability of nuclei. The electron, it must be noted, does not experience this force.

Recent developments have, however, indicated that protons and neutrons are built out of still more elementary constituents called quarks.

1.4.4 Weak Nuclear Force

The weak nuclear force appears only in certain nuclear processes such as the β-decay of a nucleus. In β-decay, the nucleus emits an electron and an uncharged particle called neutrino. The weak nuclear force is not as weak as the gravitational force, but much weaker than the strong nuclear and electromagnetic forces. The range of weak nuclear force is exceedingly small, of the order of 10^{-16} m.

1.4.5 Towards Unification of Forces

We remarked in section 1.1 that unification is a basic quest in physics. Great advances in physics often amount to unification of different

Table 1.3 Fundamental forces of nature

Name	Relative strength	Range	Operates among
Gravitational force	10^{-39}	Infinite	All objects in the universe
Weak nuclear force	10^{-13}	Very short, Sub-nuclear size ($\sim 10^{-16}$m)	Some elementary particles, particularly electron and neutrino
Electromagnetic force	10^{-2}	Infinite	Charged particles
Strong nuclear force	1	Short, nuclear size ($\sim 10^{-15}$m)	Nucleons, heavier elementary particles

theories and domains. Newton unified terrestrial and celestial domains under a common law of gravitation. The experimental discoveries of Oersted and Faraday showed that electric and magnetic phenomena are in general inseparable. Maxwell unified electromagnetism and optics with the discovery that light is an electromagnetic wave. Einstein attempted to unify gravity and electromagnetism but could not succeed in this venture. But this did not deter physicists from zealously pursuing the goal of unification of forces.

Recent decades have seen much progress on this front. The electromagnetic and the weak nuclear force have now been unified and are seen as aspects of a single 'electro-weak' force. What this unification actually means cannot be explained here. Attempts have been (and are being) made to unify the electro-weak and the strong force and even to unify the gravitational force with the rest of the fundamental forces. Many of these ideas are still speculative and inconclusive. Table 1.4 summarises some of the milestones in the progress towards unification of forces in nature.

1.5 NATURE OF PHYSICAL LAWS

Physicists explore the universe. Their investigations, based on scientific processes, range from particles that are smaller than atoms in size to stars that are very far away. In addition to finding the facts by observation and experimentation, physicists attempt to discover the laws that summarise (often as mathematical equations) these facts.

In any physical phenomenon governed by different forces, several quantities may change with time. A remarkable fact is that some special physical quantities, however, remain constant in time. They are the conserved quantities of nature. Understanding these conservation principles is very important to describe the observed phenomena quantitatively.

For motion under an external conservative force, the total mechanical energy i.e. the sum of kinetic and potential energy of a body is a constant. The familiar example is the free fall of an object under gravity. Both the kinetic energy of the object and its potential energy change continuously with time, but the sum remains fixed. If the object is released from rest, the initial

Table 1.4 Progress in unification of different forces/domains in nature

Name of the physicist	Year	Achievement in unification
Isaac Newton	1687	Unified celestial and terrestrial mechanics; showed that the same laws of motion and the law of gravitation apply to both the domains.
Hans Christian Oersted Michael Faraday	1820 1830	Showed that electric and magnetic phenomena are inseparable aspects of a unified domain: electromagnetism.
James Clerk Maxwell	1873	Unified electricity, magnetism and optics; showed that light is an electromagnetic wave.
Sheldon Glashow, Abdus Salam, Steven Weinberg	1979	Showed that the 'weak' nuclear force and the electromagnetic force could be viewed as different aspects of a single electro-weak force.
Carlo Rubia, Simon Vander Meer	1984	Verified experimentally the predictions of the theory of electro-weak force.

potential energy is completely converted into the kinetic energy of the object just before it hits the ground. This law restricted for a conservative force should not be confused with the general law of conservation of energy of an isolated system (which is the basis of the First Law of Thermodynamics).

The concept of energy is central to physics and the expressions for energy can be written for every physical system. When all forms of energy e.g., heat, mechanical energy, electrical energy etc., are counted, it turns out that energy is conserved. The general law of conservation of energy is true for all forces and for any kind of transformation between different forms of energy. In the falling object example, if you include the effect of air resistance during the fall and see the situation after the object hits the ground and stays there, the total mechanical energy is obviously not conserved. The general law of energy conservation, however, is still applicable. The initial potential energy of the stone gets transformed into other forms of energy : heat and sound. (Ultimately, sound after it is absorbed becomes heat.) The total energy of the system (stone plus the surroundings) remains unchanged.

The law of conservation of energy is thought to be valid across all domains of nature, from the microscopic to the macroscopic. It is routinely applied in the analysis of atomic, nuclear and elementary particle processes. At the other end, all kinds of violent phenomena occur in the universe all the time. Yet the total energy of the universe (the most ideal isolated system possible!) is believed to remain unchanged.

Until the advent of Einstein's theory of relativity, the law of conservation of mass was regarded as another basic conservation law of nature, since matter was thought to be indestructible. It was (and still is) an important principle used, for example, in the analysis of chemical reactions. A chemical reaction is basically a rearrangement of atoms among different molecules. If the total binding energy of the reacting molecules is less than the total binding energy of the product molecules, the difference appears as heat and the reaction is exothermic. The opposite is true for energy absorbing (endothermic) reactions. However, since the atoms are merely rearranged but not destroyed, the total mass of the reactants is the same as the total mass of the products in a chemical reaction. The changes in the binding energy are too small to be measured as changes in mass.

According to Einstein's theory, mass m is equivalent to energy E given by the relation $E= mc^2$, where c is speed of light in vacuum.

In a nuclear process mass gets converted to energy (or vice-versa). This is the energy which is released in a nuclear power generation and nuclear explosions.

Sir C.V. Raman (1888-1970)

Chandrashekhara Venkata Raman was born on 07 Nov 1888 in Thiruvanaikkaval. He finished his schooling by the age of eleven. He graduated from Presidency College, Madras. After finishing his education he joined financial services of the Indian Government.

While in Kolkata, he started working on his area of interest at Indian Association for Cultivation of Science founded by Dr. Mahendra Lal Sirkar, during his evening hours. His area of interest included vibrations, variety of musical instruments, ultrasonics, diffraction and so on.

In 1917 he was offered Professorship at Calcutta University. In 1924 he was elected 'Fellow' of the Royal Society of London and received Nobel prize in Physics in 1930 for his discovery, now known as Raman Effect.

The **Raman Effect** deals with scattering of light by molecules of a medium when they are excited to vibrational energy levels. This work opened totally new avenues for research for years to come.

He spent his later years at Bangalore, first at Indian Institute of Science and then at Raman Research Institute. His work has inspired generation of young students.

Energy is a scalar quantity. But all conserved quantities are not necessarily scalars. The total linear momentum and the total angular momentum (both vectors) of an isolated system are also conserved quantities. These laws can be derived from Newton's laws of motion in mechanics. But their validity goes beyond mechanics. They are the basic conservation laws of nature in all domains, even in those where Newton's laws may not be valid.

Besides their great simplicity and generality, the conservation laws of nature are very useful in practice too. It often happens that we cannot solve the full dynamics of a complex problem involving different particles and forces. The conservation laws can still provide useful results. For example, we may not know the complicated forces that act during a collision of two automobiles; yet momentum conservation law enables us to bypass the complications and predict or rule out possible outcomes of the collision. In nuclear and elementary particle phenomena also, the conservation laws are important tools of analysis. Indeed, using the conservation laws of energy and momentum for β-decay, Wolfgang Pauli (1900-1958) correctly predicted in 1931 the existence of a new particle (now called neutrino) emitted in β-decay along with the electron.

Conservation laws have a deep connection with symmetries of nature that you will explore in more advanced courses in physics. For example, an important observation is that the laws of nature do not change with time! If you perform an experiment in your laboratory today and repeat the same experiment (on the same objects under identical conditions) after a year, the results are bound to be the same. It turns out that this symmetry of nature with respect to translation (i.e. displacement) in time is equivalent to the law of conservation of energy. Likewise, space is homogeneous and there is no (intrinsically) preferred location in the universe. To put it more clearly, the laws of nature are the same everywhere in the universe. (Caution : the phenomena may differ from place to place because of differing conditions at different locations. For example, the acceleration due to gravity at the moon is one-sixth that at the earth, but the **law of gravitation** is the same both on the moon and the earth.) This symmetry of the laws of nature with respect to translation in space gives rise to conservation of linear momentum. In the same way isotropy of space (no intrinsically preferred direction in space) underlies the law of conservation of angular momentum*. The conservation laws of charge and other attributes of elementary particles can also be related to certain abstract symmetries. Symmetries of space and time and other abstract symmetries play a central role in modern theories of fundamental forces in nature.

Conservation laws in physics

Conservation of energy, momentum, angular momentum, charge, etc are considered to be fundamental laws in physics. At this moment, there are many such conservation laws. Apart from the above four, there are others which mostly deal with quantities which have been introduced in nuclear and particle physics. Some of the conserved quantities are called spin, baryon number, strangeness, hypercharge, etc, but you need not worry about them.

A conservation law is a hypothesis, based on observations and experiments. It is important to remember that a conservation law cannot be proved. It can be verified, or disproved, by experiments. An experiment whose result is in conformity with the law verifies or substantiates the law; it does not prove the law. On the other hand, a single experiment whose result goes against the law is enough to disprove it.

It would be wrong to ask somebody to prove the law of conservation of energy. This law is an outcome of our experience over several centuries, and it has been found to be valid in all experiments, in mechanics, thermodynamics, electromagnetism, optics, atomic and nuclear physics, or any other area.

Some students feel that they can prove the conservation of mechanical energy from a body falling under gravity, by adding the kinetic and potential energies at a point and showing that it turns out to be constant. As pointed out above, this is only a verification of the law, not its proof.

* *See Chapter 7*

SUMMARY

1. Physics deals with the study of the basic laws of nature and their manifestation in different phenomena. The basic laws of physics are universal and apply in widely different contexts and conditions.
2. The scope of physics is wide, covering a tremendous range of magnitude of physical quantities.
3. Physics and technology are related to each other. Sometimes technology gives rise to new physics; at other times physics generates new technology. Both have direct impact on society.
4. There are four fundamental forces in nature that govern the diverse phenomena of the macroscopic and the microscopic world. These are the 'gravitational force', the 'electromagnetic force', the 'strong nuclear force', and the 'weak nuclear force'. Unification of different forces/domains in nature is a basic quest in physics.
5. The physical quantities that remain unchanged in a process are called conserved quantities. Some of the general conservation laws in nature include the laws of conservation of mass, energy, linear momentum, angular momentum, charge, parity, etc. Some conservation laws are true for one fundamental force but not for the other.
6. Conservation laws have a deep connection with symmetries of nature. Symmetries of space and time, and other types of symmetries play a central role in modern theories of fundamental forces in nature.

EXERCISES

Note for the student

The exercises given here are meant to enhance your awareness about the issues surrounding science, technology and society and to encourage you to think and formulate your views about them. The questions may not have clear-cut 'objective' answers.

Note for the teacher

The exercises given here are not for the purpose of a formal examination.

1.1 Some of the most profound statements on the nature of science have come from Albert Einstein, one of the greatest scientists of all time. What do you think did Einstein mean when he said : "The most incomprehensible thing about the world is that it is comprehensible"?

1.2 "Every great physical theory starts as a heresy and ends as a dogma". Give some examples from the history of science of the validity of this incisive remark.

1.3 "Politics is the art of the possible". Similarly, "Science is the art of the soluble". Explain this beautiful aphorism on the nature and practice of science.

1.4 Though India now has a large base in science and technology, which is fast expanding, it is still a long way from realising its potential of becoming a world leader in science. Name some important factors, which in your view have hindered the advancement of science in India.

1.5 No physicist has ever "seen" an electron. Yet, all physicists believe in the existence of electrons. An intelligent but superstitious man advances this analogy to argue that 'ghosts' exist even though no one has 'seen' one. How will you refute his argument?

1.6 The shells of crabs found around a particular coastal location in Japan seem mostly to resemble the legendary face of a Samurai. Given below are two explanations of this observed fact. Which of these strikes you as a scientific explanation ?

(a) A tragic sea accident several centuries ago drowned a young Samurai. As a tribute to his bravery, nature through its inscrutable ways immortalised his face by imprinting it on the crab shells in that area.

(b) After the sea tragedy, fishermen in that area, in a gesture of honour to their dead hero, let free any crab shell caught by them which accidentally had a shape resembling the face of a Samurai. Consequently, the particular shape of the crab shell survived longer and therefore in course of time the shape was genetically propagated. This is an example of evolution by artificial selection.

[Note : This interesting illustration taken from Carl Sagan's 'The Cosmos' highlights the fact that often strange and inexplicable facts which on the first sight appear 'supernatural' actually turn out to have simple scientific explanations. Try to think out other examples of this kind].

1.7 The industrial revolution in England and Western Europe more than two centuries ago was triggered by some key scientific and technological advances. What were these advances ?

1.8 It is often said that the world is witnessing now a second industrial revolution, which will transform the society as radically as did the first. List some key contemporary areas of science and technology, which are responsible for this revolution.

1.9 Write in about 1000 words a fiction piece based on your speculation on the science and technology of the twenty-second century.

1.10 Attempt to formulate your 'moral' views on the practice of science. Imagine yourself stumbling upon a discovery, which has great academic interest but is certain to have nothing but dangerous consequences for the human society. How, if at all, will you resolve your dilemma ?

1.11 Science, like any knowledge, can be put to good or bad use, depending on the user. Given below are some of the applications of science. Formulate your views on whether the particular application is good, bad or something that cannot be so clearly categorised :

(a) Mass vaccination against small pox to curb and finally eradicate this disease from the population. (This has already been successfully done in India).
(b) Television for eradication of illiteracy and for mass communication of news and ideas.
(c) Prenatal sex determination
(d) Computers for increase in work efficiency
(e) Putting artificial satellites into orbits around the Earth
(f) Development of nuclear weapons
(g) Development of new and powerful techniques of chemical and biological warfare).
(h) Purification of water for drinking
(i) Plastic surgery
(j) Cloning

1.12 India has had a long and unbroken tradition of great scholarship — in mathematics, astronomy, linguistics, logic and ethics. Yet, in parallel with this, several superstitious and obscurantistic attitudes and practices flourished in our society and unfortunately continue even today — among many educated people too. How will you use your knowledge of science to develop strategies to counter these attitudes ?

1.13 Though the law gives women equal status in India, many people hold unscientific views on a woman's innate nature, capacity and intelligence, and in practice give them a secondary status and role. Demolish this view using scientific arguments, and by quoting examples of great women in science and other spheres; and persuade yourself and others that, given equal opportunity, women are on par with men.

1.14 "It is more important to have beauty in the equations of physics than to have them agree with experiments". The great British physicist P. A. M. Dirac held this view. Criticize this statement. Look out for some equations and results in this book which strike you as beautiful.

1.15 Though the statement quoted above may be disputed, most physicists do have a feeling that the great laws of physics are at once simple and beautiful. Some of the notable physicists, besides Dirac, who have articulated this feeling, are : Einstein, Bohr, Heisenberg, Chandrasekhar and Feynman. You are urged to make special efforts to get

access to the general books and writings by these and other great masters of physics. (See the Bibliography at the end of this book.) Their writings are truly inspiring !

1.16 Textbooks on science may give you a wrong impression that studying science is dry and all too serious and that scientists are absent-minded introverts who never laugh or grin. This image of science and scientists is patently false. Scientists, like any other group of humans, have their share of humorists, and many have led their lives with a great sense of fun and adventure, even as they seriously pursued their scientific work. Two great physicists of this genre are Gamow and Feynman. You will enjoy reading their books listed in the Bibliography.

CHAPTER TWO

UNITS AND MEASUREMENT

2.1 INTRODUCTION

Measurement of any physical quantity involves comparison with a certain basic, arbitrarily chosen, internationally accepted reference standard called **unit**. The result of a measurement of a physical quantity is expressed by a number (or numerical measure) accompanied by a unit. Although the number of physical quantities appears to be very large, we need only a limited number of units for expressing all the physical quantities, since they are interrelated with one another. The units for the fundamental or base quantities are called **fundamental** or **base units**. The units of all other physical quantities can be expressed as combinations of the base units. Such units obtained for the derived quantities are called **derived units**. A complete set of these units, both the base units and derived units, is known as the **system of units**.

2.2 THE INTERNATIONAL SYSTEM OF UNITS

In earlier time scientists of different countries were using different systems of units for measurement. Three such systems, the CGS, the FPS (or British) system and the MKS system were in use extensively till recently.

The base units for length, mass and time in these systems were as follows :

- In CGS system they were centimetre, gram and second respectively.
- In FPS system they were foot, pound and second respectively.
- In MKS system they were metre, kilogram and second respectively.

The system of units which is at present internationally accepted for measurement is the *Système Internationale d' Unites* (French for International System of Units), abbreviated as SI. The SI, with standard scheme of symbols, units and abbreviations, was developed and recommended by General Conference on Weights and Measures in 1971 for

international usage in scientific, technical, industrial and commercial work. Because SI units used decimal system, conversions within the system are quite simple and convenient. We shall follow the SI units in this book.

In SI, there are seven base units as given in Table 2.1. Besides the seven base units, there are two more units that are defined for (a) plane angle dθ as the ratio of length of arc ds to the radius r and (b) solid angle dΩ as the ratio of the intercepted area dA of the spherical surface, described about the apex O as the centre, to the square of its radius r, as shown in Fig. 2.1(a) and (b) respectively. The unit for plane angle is radian with the symbol rad and the unit for the solid angle is steradian with the symbol sr. Both these are dimensionless quantities.

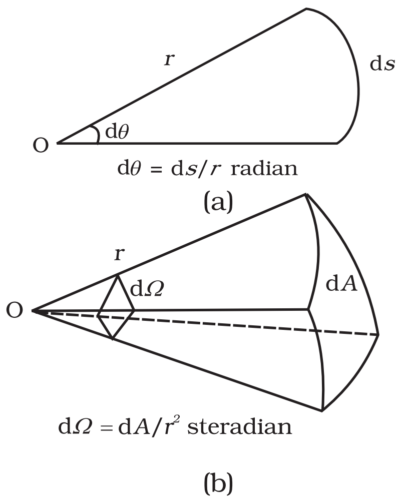


Fig. 2.1 *Description of (a) plane angle dθ and (b) solid angle* dΩ.

Table 2.1 SI Base Quantities and Units*

Base quantity	SI Units		
	Name	**Symbol**	**Definition**
Length	metre	m	The metre is the length of the path travelled by light in vacuum during a time interval of 1/299,792,458 of a second. (1983)
Mass	kilogram	kg	The kilogram is equal to the mass of the international prototype of the kilogram (a platinum-iridium alloy cylinder) kept at international Bureau of Weights and Measures, at Sevres, near Paris, France. (1889)
Time	second	s	The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom. (1967)
Electric current	ampere	A	The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per metre of length. (1948)
Thermo dynamic Temperature	kelvin	K	The kelvin, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water. (1967)
Amount of substance	mole	mol	The mole is the amount of substance of a system, which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon - 12. (1971)
Luminous intensity	candela	cd	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian. (1979)

* *The values mentioned here need not be remembered or asked in a test. They are given here only to indicate the extent of accuracy to which they are measured. With progress in technology, the measuring techniques get improved leading to measurements with greater precision. The definitions of base units are revised to keep up with this progress.*

Table 2.2 **Some units retained for general use (Though outside SI)**

Name	Symbol	Value in SI Unit
minute	min	60 s
hour	h	60 min = 3600 s
day	d	24 h = 86400 s
year	y	365.25 d = 3.156×10^7 s
degree	°	1° = (π/ 180) rad
litre	L	I $dm^3 = 10^{-3}\ m^3$
tonne	t	10^3 kg
carat	c	200 mg
bar	bar	0.1 MPa = 10^5 Pa
curie	Ci	$3.7 \times 10^{10}\ s^{-1}$
roentgen	R	2.58×10^{-4} C/kg
quintal	q	100 kg
barn	b	100 $fm^2 = 10^{-28}\ m^2$
are	a	1 $dam^2 = 10^2\ m^2$
hectare	ha	1 $hm^2 = 10^4\ m^2$
standard atmospheric pressure	atm	101325 Pa = 1.013×10^5 Pa

Note that when mole is used, the elementary entities must be specified. These entities may be atoms, molecules, ions, electrons, other particles or specified groups of such particles.

We employ units for some physical quantities that can be derived from the seven base units (Appendix A 6). Some derived units in terms of the SI base units are given in (Appendix A 6.1). Some SI derived units are given special names (Appendix A 6.2) and some derived SI units make use of these units with special names and the seven base units (Appendix A 6.3). These are given in Appendix A 6.2 and A 6.3 for your ready reference. Other units retained for general use are given in Table 2.2.

Common SI prefixes and symbols for multiples and sub-multiples are given in Appendix A2. General guidelines for using symbols for physical quantities, chemical elements and nuclides are given in Appendix A7 and those for SI units and some other units are given in Appendix A8 for your guidance and ready reference.

2.3 MEASUREMENT OF LENGTH

You are already familiar with some direct methods for the measurement of length. For example, a metre scale is used for lengths from 10^{-3} m to 10^2 m. A vernier callipers is used for lengths to an accuracy of 10^{-4} m. A screw gauge and a spherometer can be used to measure lengths as less as to 10^{-5} m. To measure lengths beyond these ranges, we make use of some special indirect methods.

2.3.1 Measurement of Large Distances

Large distances such as the distance of a planet or a star from the earth cannot be measured directly with a metre scale. An important method in such cases is the **parallax method**.

When you hold a pencil in front of you against some specific point on the background (a wall) and look at the pencil first through your left eye A (closing the right eye) and then look at the pencil through your right eye B (closing the left eye), you would notice that the position of the pencil seems to change with respect to the point on the wall. This is called **parallax**. The distance between the two points of observation is called the **basis**. In this example, the basis is the distance between the eyes.

To measure the distance D of a far away planet S by the parallax method, we observe it from two different positions (observatories) A and B on the Earth, separated by distance AB = b at the same time as shown in Fig. 2.2. We measure the angle between the two directions along which the planet is viewed at these two points. The ∠ASB in Fig. 2.2 represented by symbol θ is called the **parallax angle** or **parallactic angle**.

As the planet is very far away, $\frac{b}{D} \ll 1$, and therefore, θ is very small. Then we approximately take AB as an arc of length b of a circle with centre at S and the distance D as

the radius AS = BS so that AB = $b = D\,\theta$ where θ is in radians.

$$D = \frac{b}{\theta} \qquad (2.1)$$

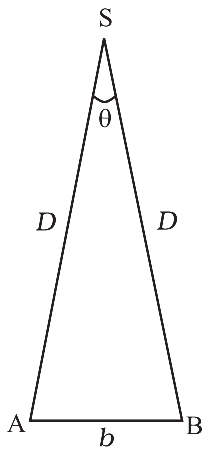


Fig. 2.2 *Parallax method.*

Having determined *D*, we can employ a similar method to determine the size or angular diameter of the planet. If *d* is the diameter of the planet and α the angular size of the planet (the angle subtended by *d* at the earth), we have

$$\alpha = d/D \qquad (2.2)$$

The angle α can be measured from the same location on the earth. It is the angle between the two directions when two diametrically opposite points of the planet are viewed through the telescope. Since *D* is known, the diameter *d* of the planet can be determined using Eq. (2.2).

Example 2.1 Calculate the angle of (a) 1^0 (degree) (b) $1'$ (minute of arc or arcmin) and (c) $1''$ (second of arc or arc second) in radians. Use $360^0 = 2\pi$ rad, $1^0 = 60'$ and $1' = 60''$

Answer (a) We have $360^0 = 2\pi$ rad

$1^0 = (\pi/180)$ rad $= 1.745 \times 10^{-2}$ rad

(b) $1^0 = 60' = 1.745 \times 10^{-2}$ rad

$1' = 2.908 \times 10^{-4}$ rad ; 2.91×10^{-4} rad

(c) $1' = 60'' = 2.908 \times 10^{-4}$ rad

$1'' = 4.847 \times 10^{-4}$ rad ; 4.85×10^{-6} rad

Example 2.2 A man wishes to estimate the distance of a nearby tower from him. He stands at a point A in front of the tower C and spots a very distant object O in line with AC. He then walks perpendicular to AC up to B, a distance of 100 m, and looks at O and C again. Since O is very distant, the direction BO is practically the same as AO; but he finds the line of sight of C shifted from the original line of sight by an angle $\theta = 40^0$ (θ is known as 'parallax') estimate the distance of the tower C from his original position A.

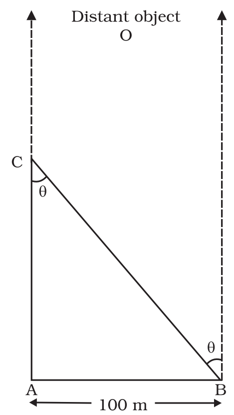


Fig. 2.3

Answer We have, parallax angle $\theta = 40^0$
From Fig. 2.3, AB = AC tan θ
AC = AB/tanθ = 100 m/tan 40^0
= 100 m/0.8391 = 119 m

Example 2.3 The moon is observed from two diametrically opposite points A and B on Earth. The angle θ subtended at the moon by the two directions of observation is $1^\circ 54'$. Given the diameter of the Earth to be about 1.276×10^7 m, compute the distance of the moon from the Earth.

Answer We have $\theta = 1^\circ 54' = 114'$

$$= (114 \times 60)'' \times (4.85 \times 10^{-6}) \text{ rad}$$

$$= 3.32 \times 10^{-2} \text{ rad},$$

since $1'' = 4.85 \times 10^{-6}$ *rad.*

Also $b = AB = 1.276 \times 10^7$ m

Hence from Eq. (2.1), we have the earth-moon distance,

$$D = b/\theta$$

$$= \frac{1.276 \times 10^7}{3.32 \times 10^{-2}}$$

$$= 3.84 \times 10^8 \text{ m}$$

Example 2.4 The Sun's angular diameter is measured to be 1920′′. The distance *D* of the Sun from the Earth is 1.496×10^{11} m. What is the diameter of the Sun ?

Answer Sun's angular diameter α

$$= 1920''$$

$$= 1920 \times 4.85 \times 10^{-6} \text{ rad}$$

$$= 9.31 \times 10^{-3} \text{ rad}$$

Sun's diameter

$$d = \alpha\, D$$

$$= \left(9.31 \times 10^{-3}\right) \times \left(1.496 \times 10^{11}\right) \text{m}$$

$$= 1.39 \times 10^{9} \text{ m}$$

2.3.2 Estimation of Very Small Distances: Size of a Molecule

To measure a very small size like that of a molecule (10^{-8} m to 10^{-10} m), we have to adopt special methods. We cannot use a screw gauge or similar instruments. Even a microscope has certain limitations. An optical microscope uses visible light to 'look' at the system under investigation. As light has wave like features, the resolution to which an optical microscope can be used is the wavelength of light (A detailed explanation can be found in the Class XII Physics textbook). For visible light the range of wavelengths is from about 4000 Å to 7000 Å (1 angstrom = 1 Å = 10^{-10} m). Hence an optical microscope cannot resolve particles with sizes smaller than this. Instead of visible light, we can use an electron beam. Electron beams can be focussed by properly designed electric and magnetic fields. The resolution of such an electron microscope is limited finally by the fact that electrons can also behave as waves ! (You will learn more about this in class XII). The wavelength of an electron can be as small as a fraction of an angstrom. Such electron microscopes with a resolution of 0.6 Å have been built. They can almost resolve atoms and molecules in a material. In recent times, tunnelling microscopy has been developed in which again the limit of resolution is better than an angstrom. It is possible to estimate the sizes of molecules.

A simple method for estimating the molecular size of oleic acid is given below. Oleic acid is a soapy liquid with large molecular size of the order of 10^{-9} m.

The idea is to first form mono-molecular layer of oleic acid on water surface.

We dissolve 1 cm^3 of oleic acid in alcohol to make a solution of 20 cm^3. Then we take 1 cm^3 of this solution and dilute it to 20 cm^3, using alcohol. So, the concentration of the solution is equal to $\left(\dfrac{1}{20 \times 20}\right) cm^3$ of oleic acid/cm^3 of solution. Next we lightly sprinkle some lycopodium powder on the surface of water in a large trough and we put one drop of this solution in the water. The oleic acid drop spreads into a thin, large and roughly circular film of molecular thickness on water surface. Then, we quickly measure the diameter of the thin film to get its area A. Suppose we have dropped n drops in the water. Initially, we determine the approximate volume of each drop (V cm^3).

Volume of n drops of solution

$$= nV \text{ cm}^3$$

Amount of oleic acid in this solution

$$= nV\left(\frac{1}{20 \times 20}\right) \text{cm}^3$$

This solution of oleic acid spreads very fast on the surface of water and forms a very thin layer of thickness t. If this spreads to form a film of area A cm^2, then the thickness of the film

$$t = \frac{\text{Volume of the film}}{\text{Area of the film}}$$

or, $$t = \frac{nV}{20 \times 20\, A} \text{ cm} \qquad (2.3)$$

If we assume that the film has mono-molecular thickness, then this becomes the size or diameter of a molecule of oleic acid. The value of this thickness comes out to be of the order of 10^{-9} m.

Example 2.5 If the size of a nucleus (in the range of 10^{-15} to 10^{-14} m) is scaled up to the tip of a sharp pin, what roughly is the size of an atom ? Assume tip of the pin to be in the range 10^{-5}m to 10^{-4}m.

Answer The size of a nucleus is in the range of 10^{-15} m and 10^{-14} m. The tip of a sharp pin is taken to be in the range of 10^{-5} m and 10^{-4} m. Thus we are scaling up by a factor of 10^{10}. An atom roughly of size 10^{-10} m will be scaled up to a size of 1 m. Thus a nucleus in an atom is as small in size as the tip of a sharp pin placed at the centre of a sphere of radius about a metre long.

2.3.3 Range of Lengths

The sizes of the objects we come across in the universe vary over a very wide range. These may vary from the size of the order of 10^{-14} m of the tiny nucleus of an atom to the size of the order of 10^{26} m of the extent of the observable universe. Table 2.6 gives the range and order of lengths and sizes of some of these objects.

We also use certain special length units for short and large lengths. These are

1 fermi = 1 f = 10^{-15} m

1 angstrom = 1 Å = 10^{-10} m

1 astronomical unit = 1 AU (average distance of the Sun from the Earth) = 1.496×10^{11} m

1 light year = 1 ly= 9.46×10^{15} m (distance that light travels with velocity of 3×10^{8} m s^{-1} in 1 year)

1 parsec = 3.08×10^{16} m (Parsec is the distance at which average radius of earth's orbit subtends an angle of 1 arc second)

2.4 MEASUREMENT OF MASS

Mass is a basic property of matter. It does not depend on the temperature, pressure or location of the object in space. The SI unit of mass is kilogram (kg). The prototypes of the International standard kilogram supplied by the International Bureau of Weights and Measures (BIPM) are available in many other laboratories of different countries. In India, this is available at the National Physical Laboratory (NPL), New Delhi.

While dealing with atoms and molecules, the kilogram is an inconvenient unit. In this case, there is an important standard unit of mass, called the **unified atomic mass unit** (u), which has been established for expressing the mass of atoms as

1 unified atomic mass unit = 1u

= (1/12) of the mass of an atom of carbon-12 isotope $\left({}^{12}_{6}C\right)$ including the mass of electrons

= 1.66×10^{-27} kg

Mass of commonly available objects can be determined by a common balance like the one used in a grocery shop. Large masses in the universe like planets, stars, etc., based on Newton's law of gravitation can be measured by using gravitational method (See Chapter 8). For measurement of small masses of atomic/sub-atomic particles etc., we make use of mass spectrograph in which radius of the trajectory is proportional to the mass of a charged particle moving in uniform electric and magnetic field.

2.4.1 Range of Masses

The masses of the objects, we come across in the universe, vary over a very wide range. These may vary from tiny mass of the order of 10^{-30} kg of an electron to the huge mass of about 10^{55} kg of the known universe. Table 2.4 gives the range and order of the typical masses of various objects.

Table 2.3 Range and order of lengths

Size of object or distance	Length (m)
Size of a proton	10^{-15}
Size of atomic nucleus	10^{-14}
Size of hydrogen atom	10^{-10}
Length of typical virus	10^{-8}
Wavelength of light	10^{-7}
Size of red blood corpuscle	10^{-5}
Thickness of a paper	10^{-4}
Height of the Mount Everest above sea level	10^{4}
Radius of the Earth	10^{7}
Distance of moon from the Earth	10^{8}
Distance of the Sun from the Earth	10^{11}
Distance of Pluto from the Sun	10^{13}
Size of our galaxy	10^{21}
Distance to Andromeda galaxy	10^{22}
Distance to the boundary of observable universe	10^{26}

Table 2.4 **Range and order of masses**

Object	Mass (kg)
Electron	10^{-30}
Proton	10^{-27}
Uranium atom	10^{-25}
Red blood cell	10^{-13}
Dust particle	10^{-9}
Rain drop	10^{-6}
Mosquito	10^{-5}
Grape	10^{-3}
Human	10^{2}
Automobile	10^{3}
Boeing 747 aircraft	10^{8}
Moon	10^{23}
Earth	10^{25}
Sun	10^{30}
Milky way galaxy	10^{41}
Observable Universe	10^{55}

2.5 MEASUREMENT OF TIME

To measure any time interval we need a clock. We now use an **atomic standard of time**, which is based on the periodic vibrations produced in a cesium atom. This is the basis of the **cesium clock**, sometimes called **atomic clock**, used in the national standards. Such standards are available in many laboratories. In the cesium atomic clock, the second is taken as the time needed for 9,192,631,770 vibrations of the radiation corresponding to the transition between the two hyperfine levels of the ground state of cesium-133 atom. The vibrations of the cesium atom regulate the rate of this cesium atomic clock just as the vibrations of a balance wheel regulate an ordinary wristwatch or the vibrations of a small quartz crystal regulate a quartz wristwatch.

The cesium atomic clocks are very accurate. In principle they provide portable standard. The national standard of time interval 'second' as well as the frequency is maintained through four cesium atomic clocks. A cesium atomic clock is used at the National Physical Laboratory (NPL), New Delhi to maintain the Indian standard of time.

In our country, the NPL has the responsibility of maintenance and improvement of physical standards, including that of time, frequency, etc. Note that the Indian Standard Time (IST) is linked to this set of atomic clocks. The efficient cesium atomic clocks are so accurate that they impart the uncertainty in time realisation as $\pm 1 \times 10^{-13}$, i.e. 1 part in 10^{13}. This implies that the uncertainty gained over time by such a device is less than 1 part in 10^{13}; they lose or gain no more than 3 μs in one year. In view of the tremendous accuracy in time measurement, the SI unit of length has been expressed in terms the path length light travels in certain interval of time (1/299, 792, 458 of a second) (Table 2.1).

The time interval of events that we come across in the universe vary over a very wide range. Table 2.5 gives the range and order of some typical time intervals.

You may notice that there is an interesting coincidence between the numbers appearing in Tables 2.3 and 2.5. Note that the ratio of the longest and shortest lengths of objects in our universe is about 10^{41}. Interestingly enough, the ratio of the longest and shortest time intervals associated with the events and objects in our universe is also about 10^{41}. This number, 10^{41} comes up again in Table 2.4, which lists typical masses of objects. The ratio of the largest and smallest masses of the objects in our universe is about $(10^{41})^2$. Is this a curious coincidence between these large numbers purely accidental ?

2.6 ACCURACY, PRECISION OF INSTRUMENTS AND ERRORS IN MEASUREMENT

Measurement is the foundation of all experimental science and technology. The result of every measurement by any measuring instrument contains some uncertainty. This uncertainty is called **error**. Every calculated quantity which is based on measured values, also has an error. We shall distinguish between two terms: **accuracy** and **precision**. The accuracy of a measurement is a measure of how close the measured value is to the true value of the quantity. Precision tells us to what resolution or limit the quantity is measured.

The accuracy in measurement may depend on several factors, including the limit or the resolution of the measuring instrument. For example, suppose the true value of a certain length is near 3.678 cm. In one experiment, using a measuring instrument of resolution 0.1 cm, the measured value is found to be 3.5 cm, while in another experiment using a measuring device of greater resolution, say 0.01 cm, the length is determined to be 3.38 cm. The first measurement has more accuracy (because it is

Table 2.5 Range and order of time intervals

Event	Time interval(s)
Life-span of most unstable particle	10^{-24}
Time required for light to cross a nuclear distance	10^{-22}
Period of x-rays	10^{-19}
Period of atomic vibrations	10^{-15}
Period of light wave	10^{-15}
Life time of an excited state of an atom	10^{-8}
Period of radio wave	10^{-6}
Period of a sound wave	10^{-3}
Wink of eye	10^{-1}
Time between successive human heart beats	10°
Travel time for light from moon to the Earth	10°
Travel time for light from the Sun to the Earth	10^{2}
Time period of a satellite	10^{4}
Rotation period of the Earth	10^{5}
Rotation and revolution periods of the moon	10^{6}
Revolution period of the Earth	10^{7}
Travel time for light from nearest star	10^{8}
Average human life-span	10^{9}
Age of Egyptian pyramids	10^{11}
Time since dinosaurs became extinct	10^{15}
Age of the universe	10^{17}

closer to the true value) but less precision (its resolution is only 0.1 cm), while the second measurement is less accurate but more precise. Thus every measurement is approximate due to errors in measurement. In general, the errors in measurement can be broadly classified as (a) systematic errors and (b) random errors.

Systematic errors

The **systematic errors** are those errors that tend to be in one direction, either positive or negative. Some of the sources of systematic errors are :

(a) **Instrumental errors** that arise from the errors due to imperfect design or calibration of the measuring instrument, zero error in the instrument, etc. For example, the temperature graduations of a thermometer may be inadequately calibrated (it may read 104 °C at the boiling point of water at STP whereas it should read 100 °C); in a vernier callipers the zero mark of vernier scale may not coincide with the zero mark of the main scale, or simply an ordinary metre scale may be worn off at one end.

(b) **Imperfection in experimental technique or procedure** To determine the temperature of a human body, a thermometer placed under the armpit will always give a temperature lower than the actual value of the body temperature. Other external conditions (such as changes in temperature, humidity, wind velocity, etc.) during the experiment may systematically affect the measurement.

(c) **Personal errors** that arise due to an individual's bias, lack of proper setting of the apparatus or individual's carelessness in taking observations without observing proper precautions, etc. For example, if you, by habit, always hold your head a bit too far to the right while reading the position of a needle on the scale, you will introduce an error due to **parallax**.

Systematic errors can be minimised by improving experimental techniques, selecting better instruments and removing personal bias as far as possible. For a given set-up, these errors may be estimated to a certain extent and the necessary corrections may be applied to the readings.

Random errors

The **random errors** are those errors, which occur irregularly and hence are random with respect

to sign and size. These can arise due to random and unpredictable fluctuations in experimental conditions (e.g. unpredictable fluctuations in temperature, voltage supply, mechanical vibrations of experimental set-ups, etc), personal (unbiased) errors by the observer taking readings, etc. For example, when the same person repeats the same observation, it is very likely that he may get different readings everytime.

Least count error

The smallest value that can be measured by the measuring instrument is called its **least count**. All the readings or measured values are good only up to this value.

The **least count error** is the error associated with the resolution of the instrument. For example, a vernier callipers has the least count as 0.01 cm; a spherometer may have a least count of 0.001 cm. Least count error belongs to the category of random errors but within a limited size; it occurs with both systematic and random errors. If we use a metre scale for measurement of length, it may have graduations at 1 mm division scale spacing or interval.

Using instruments of higher precision, improving experimental techniques, etc., we can reduce the least count error. Repeating the observations several times and taking the arithmetic mean of all the observations, the mean value would be very close to the true value of the measured quantity.

2.6.1 Absolute Error, Relative Error and Percentage Error

(a) Suppose the values obtained in several measurements are a_1, a_2, a_3...., a_n. The arithmetic mean of these values is taken as the best possible value of the quantity under the given conditions of measurement as :

$$a_{mean} = (a_1+a_2+a_3+...+a_n) / n \qquad (2.4)$$

or,

$$a_{mean} = \sum_{i=1}^{n} a_i / n \qquad (2.5)$$

This is because, as explained earlier, it is reasonable to suppose that individual measurements are as likely to overestimate as to underestimate the true value of the quantity.

The magnitude of the difference between the true value of the quantity and the individual measurement value is called the absolute error of the measurement. This is denoted by $|\Delta a|$. In absence of any other method of knowing true value, we considered arithmatic mean as the true value. Then the errors in the individual measurement values are

$$\Delta a_1 = a_{mean} - a_1,$$
$$\Delta a_2 = a_{mean} - a_2,$$
$$.... \quad .... \quad ....$$
$$.... \quad .... \quad ....$$
$$\Delta a_n = a_{mean} - a_n$$

The Δa calculated above may be positive in certain cases and negative in some other cases. But absolute error $|\Delta a|$ will always be positive.

(b) The arithmetic mean of all the *absolute errors* is taken as the *final* or *mean absolute error* of the value of the physical quantity a. It is represented by Δa_{mean}.

Thus,

$$\Delta a_{mean} = (|\Delta a_1|+|\Delta a_2|+|\Delta a_3|+...+|\Delta a_n|)/n \qquad (2.6)$$

$$= \sum_{i=1}^{n} |\Delta a_i|/n \qquad (2.7)$$

If we do a single measurement, the value we get may be in the range $a_{mean} \pm \Delta a_{mean}$

i.e. $a = a_{mean} \pm \Delta a_{mean}$

or,

$$a_{mean} - \Delta a_{mean} \le a \le a_{mean} + \Delta a_{mean} \qquad (2.8)$$

This implies that any measurement of the physical quantity a is likely to lie between $(a_{mean} + \Delta a_{mean})$ and $(a_{mean} - \Delta a_{mean})$.

(c) Instead of the absolute error, we often use the **relative error** or the **percentage error** (δa). **The relative error is the ratio of the mean absolute error Δa_{mean} to the mean value a_{mean} of the quantity measured.**

Relative error = $\Delta a_{mean} / a_{mean}$ (2.9)

When the relative error is expressed in per cent, it is called the **percentage error** (δa).

Thus, Percentage error

$$\delta a = (\Delta a_{mean} / a_{mean}) \times 100\% \quad (2.10)$$

Let us now consider an example.

Example 2.6 Two clocks are being tested against a standard clock located in a national laboratory. At 12:00:00 noon by the standard clock, the readings of the two clocks are :

	Clock 1	Clock 2
Monday	12:00:05	10:15:06
Tuesday	12:01:15	10:14:59
Wednesday	11:59:08	10:15:18
Thursday	12:01:50	10:15:07
Friday	11:59:15	10:14:53
Saturday	12:01:30	10:15:24
Sunday	12:01:19	10:15:11

If you are doing an experiment that requires precision time interval measurements, which of the two clocks will you prefer ?

Answer The range of variation over the seven days of observations is 162 s for clock 1, and 31 s for clock 2. The average reading of clock 1 is much closer to the standard time than the average reading of clock 2. The important point is that a clock's zero error is not as significant for precision work as its variation, because a 'zero-error' can always be easily corrected. Hence clock 2 is to be preferred to clock 1. ◀

Example 2.7 We measure the period of oscillation of a simple pendulum. In successive measurements, the readings turn out to be 2.63 s, 2.56 s, 2.42 s, 2.71s and 2.80 s. Calculate the absolute errors, relative error or percentage error.

Answer The mean period of oscillation of the pendulum

$$T = \frac{(2.63 + 2.56 + 2.42 + 2.71 + 2.80)\text{s}}{5}$$

$$= \frac{13.12}{5}\ \text{s}$$

$$= 2.624\ \text{s}$$

$$= 2.62\ \text{s}$$

As the periods are measured to a resolution of 0.01 s, all times are to the second decimal; it is proper to put this mean period also to the second decimal.

The errors in the measurements are

2.63 s – 2.62 s = 0.01 s
2.56 s – 2.62 s = – 0.06 s
2.42 s – 2.62 s = – 0.20 s
2.71 s – 2.62 s = 0.09 s
2.80 s – 2.62 s = 0.18 s

Note that the errors have the same units as the quantity to be measured.

The arithmetic mean of all the absolute errors (for arithmetic mean, we take only the magnitudes) is

$$\Delta T_{mean} = [(0.01 + 0.06 + 0.20 + 0.09 + 0.18)\text{s}]/5$$
$$= 0.54\ \text{s}/5$$
$$= 0.11\ \text{s}$$

That means, the period of oscillation of the simple pendulum is (2.62 ± 0.11) s i.e. it lies between (2.62 + 0.11) s and (2.62 – 0.11) s or between 2.73 s and 2.51 s. As the arithmetic mean of all the absolute errors is 0.11 s, there is already an error in the tenth of a second. Hence there is no point in giving the period to a hundredth. A more correct way will be to write

$$T = 2.6 \pm 0.1\ \text{s}$$

Note that the last numeral 6 is unreliable, since it may be anything between 5 and 7. We indicate this by saying that the measurement has two significant figures. In this case, the two significant figures are 2, which is reliable and 6, which has an error associated with it. You will learn more about the significant figures in section 2.7.

For this example, the relative error or the percentage error is

$$\delta a = \frac{0.1}{2.6} \times 100 = 4\%$$ ◀

2.6.2 Combination of Errors

If we do an experiment involving several measurements, we must know how the errors in all the measurements combine. For example,

How will you measure the length of a line?

What a naïve question, at this stage, you might say! But what if it is not a straight line? Draw a zigzag line in your copy, or on the blackboard. Well, not too difficult again. You might take a thread, place it along the line, open up the thread, and measure its length.

Now imagine that you want to measure the length of a national highway, a river, the railway track between two stations, or the boundary between two states or two nations. If you take a string of length 1 metre or 100 metre, keep it along the line, shift its position every time, the arithmetic of man-hours of labour and expenses on the project is not commensurate with the outcome. Moreover, errors are bound to occur in this enormous task. There is an interesting fact about this. France and Belgium share a common international boundary, whose length mentioned in the official documents of the two countries differs substantially!

Go one step beyond and imagine the coastline where land meets sea. Roads and rivers have fairly mild bends as compared to a coastline. Even so, all documents, including our school books, contain information on the length of the coastline of Gujarat or Andhra Pradesh, or the common boundary between two states, etc. Railway tickets come with the distance between stations printed on them. We have 'milestones' all along the roads indicating the distances to various towns. So, how is it done?

One has to decide how much error one can tolerate and optimise cost-effectiveness. If you want smaller errors, it will involve high technology and high costs. Suffice it to say that it requires fairly advanced level of physics, mathematics, engineering and technology. It belongs to the areas of fractals, which has lately become popular in theoretical physics. Even then one doesn't know how much to rely on the figure that props up, as is clear from the story of France and Belgium. Incidentally, this story of the France-Belgium discrepancy appears on the first page of an advanced Physics book on the subject of fractals and chaos!

density is the ratio of the mass to the volume of the substance. If we have errors in the measurement of mass and of the sizes or dimensions, we must know what the error will be in the density of the substance. To make such estimates, we should learn how errors combine in various mathematical operations. For this, we use the following procedure.

(a) *Error of a sum or a difference*

Suppose two physical quantities A and B have measured values $A \pm \Delta A$, $B \pm \Delta B$ respectively where ΔA and ΔB are their absolute errors. We wish to find the error ΔZ in the sum

$$Z = A + B.$$

We have by addition, $Z \pm \Delta Z$

$$= (A \pm \Delta A) + (B \pm \Delta B).$$

The maximum possible error in Z

$$\Delta Z = \Delta A + \Delta B$$

For the difference $Z = A - B$, we have

$$Z \pm \Delta Z = (A \pm \Delta A) - (B \pm \Delta B)$$

$$= (A - B) \pm \Delta A \pm \Delta B$$

or, $\pm \Delta Z = \pm \Delta A \pm \Delta B$

The maximum value of the error ΔZ is again $\Delta A + \Delta B$.

Hence the rule : When two quantities are added or subtracted, the absolute error in the final result is the sum of the absolute errors in the individual quantities.

▶ ***Example 2.8*** The temperatures of two bodies measured by a thermometer are $t_1 = 20\ ^0\text{C} \pm 0.5\ ^0\text{C}$ and $t_2 = 50\ ^0\text{C} \pm 0.5\ ^0\text{C}$. Calculate the temperature difference and the error theirin.

Answer $t' = t_2 - t_1 = (50\ ^0\text{C} \pm 0.5\ ^0\text{C}) - (20^0\text{C} \pm 0.5\ ^0\text{C})$

$t' = 30\ ^0\text{C} \pm 1\ ^0\text{C}$ ◀

(b) Error of a product or a quotient

Suppose $Z = AB$ and the measured values of A and B are $A \pm \Delta A$ and $B \pm \Delta B$. Then

$$Z \pm \Delta Z = (A \pm \Delta A)\ (B \pm \Delta B)$$

$$= AB \pm B\,\Delta A \pm A\,\Delta B \pm \Delta A\,\Delta B.$$

Dividing LHS by Z and RHS by AB we have,

$1 \pm (\Delta Z/Z) = 1 \pm (\Delta A/A) \pm (\Delta B/B) \pm (\Delta A/A)(\Delta B/B)$.

Since ΔA and ΔB are small, we shall ignore their product.

Hence the maximum relative error

$\Delta Z/\ Z = (\Delta A/A) + (\Delta B/B)$.

You can easily verify that this is true for division also.

Hence the rule : When two quantities are multiplied or divided, the relative error in the result is the sum of the relative errors in the multipliers.

Example 2.9 The resistance $R = V/I$ where $V = (100 \pm 5)$V and $I = (10 \pm 0.2)$A. Find the percentage error in R.

Answer The percentage error in V is 5% and in I it is 2%. The total error in R would therefore be 5% + 2% = 7%.

Example 2.10 Two resistors of resistances R_1 = 100 ± 3 ohm and R_2 = 200 ± 4 ohm are connected (a) in series, (b) in parallel. Find the equivalent resistance of the (a) series combination, (b) parallel combination. Use for (a) the relation $R = R_1 + R_2$, and for (b)

$$\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2} \text{ and } \frac{\Delta R'}{R'^2} = \frac{\Delta R_1}{R_1^2} + \frac{\Delta R_2}{R_2^2}.$$

Answer (a) The equivalent resistance of series combination

$R = R_1 + R_2 =$ (100 ± 3) ohm + (200 ± 4) ohm

= 300 ± 7 ohm.

(b) The equivalent resistance of parallel combination

$$R' = \frac{R_1 R_2}{R_1 + R_2} = \frac{200}{3} = 66.7 \text{ ohm}$$

Then, from $\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}$

we get,

$$\frac{\Delta R'}{R'^2} = \frac{\Delta R_1}{R_1^2} + \frac{\Delta R_2}{R_2^2}$$

$$\Delta R' = (R'^2)\frac{\Delta R_1}{R_1^2} + (R'^2)\frac{\Delta R_2}{R_2^2}$$

$$= \left(\frac{66.7}{100}\right)^2 3 + \left(\frac{66.7}{200}\right)^2 4$$

= 1.8

Then, R' = 66.7 ±1.8 ohm

(Here, ΔR is expresed as 1.8 instead of 2 to keep in confirmity with the rules of significant figures.)

(c) Error in case of a measured quantity raised to a power

Suppose $Z = A^2$,

Then,

$\Delta Z/Z = (\Delta A/A) + (\Delta A/A) = 2\,(\Delta A/A)$.

Hence, the relative error in A^2 is two times the error in A.

In general, if $Z = A^p B^q / C^r$

Then,

$\Delta Z/Z = p\,(\Delta A/A) + q\,(\Delta B/B) + r\,(\Delta C/C)$.

Hence the rule : The relative error in a physical quantity raised to the power k is the k times the relative error in the individual quantity.

Example 2.11 Find the relative error in Z, if $Z = A^4 B^{1/3} / CD^{3/2}$.

Answer The relative error in Z is $\Delta Z/Z = 4(\Delta A/A) + (1/3)\,(\Delta B/B) + (\Delta C/C) + (3/2)\,(\Delta D/D)$.

Example 2.12 The period of oscillation of a simple pendulum is $T = 2\pi\sqrt{L/g}$. Measured value of L is 20.0 cm known to 1 mm accuracy and time for 100 oscillations of the pendulum is found to be 90 s using a wrist watch of 1 s resolution. What is the accuracy in the determination of g ?

Answer $g = 4\pi^2 L/T^2$

Here, $T = \frac{t}{n}$ and $\Delta T = \frac{\Delta t}{n}$. Therefore, $\frac{\Delta T}{T} = \frac{\Delta t}{t}$.

The errors in both L and t are the least count errors. Therefore,

$(\Delta g/g) = (\Delta L/L) + 2(\Delta T/T)$

$$= \frac{0.1}{20.0} + 2\left(\frac{1}{90}\right) = 0.032$$

Thus, the percentage error in g is

$100\,(\Delta g/g) = 100(\Delta L/L) + 2 \times 100\,(\Delta T/T)$

$= 3\%$

2.7 SIGNIFICANT FIGURES

As discussed above, every measurement involves errors. Thus, the result of measurement should be reported in a way that indicates the precision of measurement. Normally, the reported result of measurement is a number that includes all digits in the number that are known reliably plus the first digit that is uncertain. The reliable digits plus

the first uncertain digit are known as **significant digits** or **significant figures**. If we say the period of oscillation of a simple pendulum is 1.62 s, the digits 1 and 6 are reliable and certain, while the digit 2 is uncertain. Thus, the measured value has three significant figures. The length of an object reported after measurement to be 287.5 cm has four significant figures, the digits 2, 8, 7 are certain while the digit 5 is uncertain. Clearly, reporting the result of measurement that includes more digits than the significant digits is superfluous and also misleading since it would give a wrong idea about the precision of measurement.

The rules for determining the number of significant figures can be understood from the following examples. Significant figures indicate, as already mentioned, the precision of measurement which depends on the least count of the measuring instrument. **A choice of change of different units does not change the number of significant digits or figures in a measurement.** This important remark makes most of the following observations clear:

(1) For example, the length 2.308 cm has four significant figures. But in different units, the same value can be written as 0.02308 m or 23.08 mm or 23080 μm.

All these numbers have the same number of significant figures (digits 2, 3, 0, 8), namely four. This shows that the location of decimal point is of no consequence in determining the number of significant figures.

The example gives the following rules :

- **All the non-zero digits are significant.**
- **All the zeros between two non-zero digits are significant, no matter where the decimal point is, if at all.**
- **If the number is less than 1, the zero(s) on the right of decimal point but to the left of the first non-zero digit are not significant.** [In <u>0</u>.<u>00</u> 2308, the underlined zeroes are not significant].
- **The terminal or trailing zero(s) in a number without a decimal point are not significant.**

  [Thus 123 m = 12300 cm = 123000 mm has *three* significant figures, the trailing zero(s) being not significant.] However, you can also see the next observation.
- **The trailing zero(s) in a number with a decimal point are significant.**

  [The numbers 3.500 or 0.06900 have four significant figures each.]

(2) There can be some confusion regarding the trailing zero(s). Suppose a length is reported to be 4.700 m. It is evident that the zeroes here are meant to convey the precision of measurement and are, therefore, significant. [If these were not, it would be superfluous to write them explicitly, the reported measurement would have been simply 4.7 m]. Now suppose we change units, then

4.700 m = 470.0 cm = 4700 mm = 0.004700 km

Since the last number has trailing zero(s) in a number with no decimal, we would conclude erroneously from observation (1) above that the number has *two* significant figures, while in fact, it has four significant figures and a mere change of units cannot change the number of significant figures.

(3) **To remove such ambiguities in determining the number of significant figures, the best way is to report every measurement in scientific notation (in the power of 10).** In this notation, every number is expressed as $a \times 10^b$, where a is a number between 1 and 10, and b is any positive or negative exponent (or power) of 10. In order to get an approximate idea of the number, we may round off the number a to 1 (for $a \le 5$) and to 10 (for $5 < a \le 10$). Then the number can be expressed approximately as 10^b in which the exponent (or power) b of 10 is called **order of magnitude** of the physical quantity. When only an estimate is required, the quantity is of the order of 10^b. For example, the diameter of the earth (1.28×10^7m) is of the order of 10^7m with the order of magnitude 7. The diameter of hydrogen atom (1.06×10^{-10}m) is of the order of 10^{-10}m, with the order of magnitude –10. Thus, the diameter of the earth is 17 orders of magnitude larger than the hydrogen atom.

It is often customary to write the decimal after the first digit. Now the confusion mentioned in (a) above disappears :

$$4.700 \text{ m} = 4.700 \times 10^2 \text{ cm}$$
$$= 4.700 \times 10^3 \text{ mm} = 4.700 \times 10^{-3} \text{ km}$$

The power of 10 is irrelevant to the determination of significant figures. However, all

zeroes appearing in the base number in the scientific notation are significant. Each number in this case has *four* significant figures.

Thus, in the scientific notation, no confusion arises about the trailing zero(s) in the base number *a*. They are always significant.

(4) The scientific notation is ideal for reporting measurement. But if this is not adopted, we use the rules adopted in the preceding example :

- **For a number greater than 1, without any decimal, the trailing zero(s) are not significant.**
- **For a number with a decimal, the trailing zero(s) are significant.**

(5) The digit 0 conventionally put on the left of a decimal for a number less than 1 (like 0.1250) is never significant. However, the zeroes at the end of such number are significant in a measurement.

(6) The multiplying or dividing factors which are neither rounded numbers nor numbers representing measured values are exact and have infinite number of significant digits. For example in $r = \frac{d}{2}$ or $s = 2\pi r$, the factor 2 is an exact number and it can be written as 2.0, 2.00 or 2.0000 as required. Similarly, in $T = \frac{t}{n}$, n is an exact number.

2.7.1 Rules for Arithmetic Operations with Significant Figures

The result of a calculation involving approximate measured values of quantities (i.e. values with limited number of significant figures) must reflect the uncertainties in the original measured values. It cannot be more accurate than the original measured values themselves on which the result is based. In general, the final result should not have more significant figures than the original data from which it was obtained. Thus, if mass of an object is measured to be, say, 4.237 g (four significant figures) and its volume is measured to be 2.51 cm^3, then its density, by mere arithmetic division, is 1.68804780876 g/cm^3 upto 11 decimal places. It would be clearly absurd and irrelevant to record the calculated value of density to such a precision when the measurements on which the value is based, have much less precision. The following rules for arithmetic operations with significant figures ensure that the final result of a calculation is shown with the precision that is consistent with the precision of the input measured values :

(1) **In multiplication or division, the final result should retain as many significant figures as are there in the original number with the least significant figures.**

Thus, in the example above, density should be reported to *three* significant figures.

$$\text{Density} = \frac{4.237\text{g}}{2.51\ \text{cm}^3} = 1.69\ \text{g cm}^{-3}$$

Similarly, if the speed of light is given as 3.00×10^8 m s^{-1} (*three* significant figures) and one year (1y = 365.25 d) has 3.1557×10^7 s (*five* significant figures), the light year is 9.47×10^{15} m (*three* significant figures).

(2) **In addition or subtraction, the final result should retain as many decimal places as are there in the number with the least decimal places.**

For example, the sum of the numbers 436.32 g, 227.2 g and 0.301 g by mere arithmetic addition, is 663.821 g. But the least precise measurement (227.2 g) is correct to only one decimal place. The final result should, therefore, be rounded off to 663.8 g.

Similarly, the difference in length can be expressed as :

$0.307\ \text{m} - 0.304\ \text{m} = 0.003\ \text{m} = 3 \times 10^{-3}\ \text{m}$.

Note that we should not use the *rule* (1) applicable for multiplication and division and write 664 g as the result in the example of **addition** and 3.00×10^{-3} m in the example of **subtraction**. They do not convey the precision of measurement properly. For addition and subtraction, the rule is in terms of decimal places.

2.7.2 Rounding off the Uncertain Digits

The result of computation with approximate numbers, which contain more than one uncertain digit, should be rounded off. The rules for rounding off numbers to the appropriate significant figures are obvious in most cases. A number 2.74<u>6</u> rounded off to three significant figures is 2.75, while the number 2.743 would be 2.74. The *rule* by convention is that the **preceding digit is raised by 1 if the**

insignificant digit to be dropped (the underlined digit in this case) is more than 5, and is left unchanged if the latter is less than 5. But what if the number is 2.745 in which the insignificant digit is 5. Here, the convention is that **if the preceding digit is even, the insignificant digit is simply dropped and, if it is odd, the preceding digit is raised by** 1. Then, the number 2.745 rounded off to three significant figures becomes 2.74. On the other hand, the number 2.735 rounded off to three significant figures becomes 2.74 since the preceding digit is odd.

In any involved or complex multi-step calculation, you should retain, in intermediate steps, one digit more than the significant digits and round off to proper significant figures at the end of the calculation. Similarly, a number known to be within many significant figures, such as in 2.99792458×10^8 m/s for the speed of light in vacuum, is rounded off to an approximate value 3×10^8 m/s , which is often employed in computations. Finally, remember that exact numbers that appear in formulae like $2\,\pi$ in $T = 2\pi\sqrt{\frac{L}{g}}$, have a large (infinite) number of significant figures. The value of π = 3.1415926.... is known to a large number of significant figures. You may take the value as 3.142 or 3.14 for π, with limited number of significant figures as required in specific cases.

▶ ***Example 2.13*** Each side of a cube is measured to be 7.203 m. What are the total surface area and the volume of the cube to appropriate significant figures?

Answer The number of significant figures in the measured length is 4. The calculated area and the volume should therefore be rounded off to 4 significant figures.

Surface area of the cube $= 6(7.203)^2\ \text{m}^2$
$= 311.299254\ \text{m}^2$
$= 311.3\ \text{m}^2$

Volume of the cube $= (7.203)^3\ \text{m}^3$
$= 373.714754\ \text{m}^3$
$= 373.7\ \text{m}^3$ ◀

▶ ***Example 2.14*** 5.74 g of a substance occupies 1.2 cm^3. Express its density by keeping the significant figures in view.

Answer There are 3 significant figures in the measured mass whereas there are only 2 significant figures in the measured volume. Hence the density should be expressed to only 2 significant figures.

$$\text{Density} = \frac{5.74}{1.2}\ \text{g cm}^{-3}$$
$$= 4.8\ \text{g cm}^{-3}.$$ ◀

2.7.3 Rules for Determining the Uncertainty in the Results of Arithmatic Calculations

The rules for determining the uncertainty or error in the number/measured quantity in arithmetic operations can be understood from the following examples.

(1) If the length and breadth of a thin rectangular sheet are measured, using a metre scale as 16.2 cm and, 10.1 cm respectively, there are three significant figures in each measurement. It means that the length l may be written as

$$l = 16.2 \pm 0.1\ \text{cm}$$
$$= 16.2\ \text{cm} \pm 0.6\ \%.$$

Similarly, the breadth b may be written as

$$b = 10.1 \pm 0.1\ \text{cm}$$
$$= 10.1\ \text{cm} \pm 1\ \%$$

Then, the error of the product of two (or more) experimental values, using the combination of errors rule, will be

$$l\,b = 163.62\ \text{cm}^2 \pm 1.6\%$$
$$= 163.62 \pm 2.6\ \text{cm}^2$$

This leads us to quote the final result as

$$l\,b = 164 \pm 3\ \text{cm}^2$$

Here 3 cm^2 is the uncertainty or error in the estimation of area of rectangular sheet.

(2) **If a set of experimental data is specified to n significant figures, a result obtained by combining the data will also be valid to n significant figures.**

However, if data are subtracted, the number of significant figures can be reduced.

For example, 12.9 g – 7.06 g, both specified to three significant figures, cannot properly be evaluated as 5.84 g but only as 5.8 g, as uncertainties in subtraction or addition combine in a different fashion (smallest number of decimal places rather than the number of significant figures in any of the number added or subtracted).

(3) **The relative error of a value of number specified to significant figures depends not only on n but also on the number itself.**

For example, the accuracy in measurement of mass 1.02 g is ± 0.01 g whereas another measurement 9.89 g is also accurate to ± 0.01 g. The relative error in 1.02 g is

$= (\pm 0.01/1.02) \times 100\ \%$

$= \pm 1\%$

Similarly, the relative error in 9.89 g is

$= (\pm 0.01/9.89) \times 100\ \%$

$= \pm 0.1\ \%$

Finally, remember that **intermediate results in a multi-step computation should be calculated to one more significant figure in every measurement than the number of digits in the least precise measurement.** These should be justified by the data and then the arithmetic operations may be carried out; otherwise rounding errors can build up. For example, the reciprocal of 9.58, calculated (after rounding off) to the same number of significant figures (three) is 0.104, but the reciprocal of 0.104 calculated to three significant figures is 9.62. However, if we had written 1/9.58 = 0.1044 and then taken the reciprocal to three significant figures, we would have retrieved the original value of 9.58.

This example justifies the idea to retain one more extra digit (than the number of digits in the least precise measurement) in intermediate steps of the complex multi-step calculations in order to avoid additional errors in the process of rounding off the numbers.

2.8 DIMENSIONS OF PHYSICAL QUANTITIES

The nature of a physical quantity is described by its dimensions. All the physical quantities represented by derived units can be expressed in terms of some combination of seven fundamental or base quantities. We shall call these base quantities as the seven dimensions of the physical world, which are denoted with square brackets [ ]. Thus, length has the dimension [L], mass [M], time [T], electric current [A], thermodynamic temperature [K], luminous intensity [cd], and amount of substance [mol]. **The dimensions of a physical quantity are the powers (or exponents) to which the base quantities are raised to represent that quantity**. Note that using the square brackets [ ] round a quantity means that we are dealing with '**the dimensions of**' the quantity.

In mechanics, all the physical quantities can be written in terms of the dimensions [L], [M] and [T]. For example, the volume occupied by an object is expressed as the product of length, breadth and height, or three lengths. Hence the dimensions of volume are $[L] \times [L] \times [L] = [L]^3 = [L^3]$. As the volume is independent of mass and time, it is said to possess zero dimension in mass [M°], zero dimension in time [T°] and three dimensions in length.

Similarly, force, as the product of mass and acceleration, can be expressed as

Force = mass × acceleration

= mass × (length)/(time)^2

The dimensions of force are $[M]\ [L]/[T]^2 = [M\ L\ T^{-2}]$. Thus, the force has one dimension in mass, one dimension in length, and –2 dimensions in time. The dimensions in all other base quantities are zero.

Note that in this type of representation, the magnitudes are not considered. It is the quality of the type of the physical quantity that enters. Thus, a change in velocity, initial velocity, average velocity, final velocity, and speed are all equivalent in this context. Since all these quantities can be expressed as length/time, their dimensions are $[L]/[T]$ or $[L\ T^{-1}]$.

2.9 DIMENSIONAL FORMULAE AND DIMENSIONAL EQUATIONS

The expression which shows how and which of the base quantities represent the dimensions of a physical quantity is called the *dimensional formula* of the given physical quantity. For example, the dimensional formula of the volume is $[M^{\circ}\ L^3\ T^{\circ}]$, and that of speed or velocity is $[M^{\circ}\ L\ T^{-1}]$. Similarly, $[M^{\circ}\ L\ T^{-2}]$ is the dimensional formula of acceleration and $[M\ L^{-3}\ T^{\circ}]$ that of mass density.

An equation obtained by equating a physical quantity with its dimensional formula is called the **dimensional equation** of the physical

quantity. Thus, the dimensional equations are the equations, which represent the dimensions of a physical quantity in terms of the base quantities. For example, the dimensional equations of volume [V], speed [v], force [F] and mass density [ρ] may be expressed as

$[V] = [M^0 L^3 T^0]$
$[v] = [M^0 L T^{-1}]$
$[F] = [M L T^{-2}]$
$[\rho] = [M L^{-3} T^0]$

The dimensional equation can be obtained from the equation representing the relations between the physical quantities. The dimensional formulae of a large number and wide variety of physical quantities, derived from the equations representing the relationships among other physical quantities and expressed in terms of base quantities are given in Appendix 9 for your guidance and ready reference.

2.10 DIMENSIONAL ANALYSIS AND ITS APPLICATIONS

The recognition of concepts of dimensions, which guide the description of physical behaviour is of basic importance as only those physical quantities can be added or subtracted which have the same dimensions. A thorough understanding of dimensional analysis helps us in deducing certain relations among different physical quantities and checking the derivation, accuracy and dimensional consistency or homogeneity of various mathematical expressions. When magnitudes of two or more physical quantities are multiplied, their units should be treated in the same manner as ordinary algebraic symbols. We can cancel identical units in the numerator and denominator. The same is true for dimensions of a physical quantity. Similarly, physical quantities represented by symbols on both sides of a mathematical equation must have the same dimensions.

2.10.1 Checking the Dimensional Consistency of Equations

The magnitudes of physical quantities may be added together or subtracted from one another only if they have the same dimensions. In other words, we can add or subtract similar physical quantities. Thus, velocity cannot be added to force, or an electric current cannot be subtracted from the thermodynamic temperature. This simple principle called **the principle of homogeneity of dimensions** in an equation is extremely useful in checking the correctness of an equation. If the dimensions of all the terms are not same, the equation is wrong. Hence, if we derive an expression for the length (or distance) of an object, regardless of the symbols appearing in the original mathematical relation, when all the individual dimensions are simplified, the remaining dimension must be that of length. Similarly, if we derive an equation of speed, the dimensions on both the sides of equation, when simplified, must be of length/time, or $[L T^{-1}]$.

Dimensions are customarily used as a preliminary test of the consistency of an equation, when there is some doubt about the correctness of the equation. However, the dimensional consistency does not guarantee correct equations. It is uncertain to the extent of dimensionless quantities or functions. The arguments of special functions, such as the trigonometric, logarithmic and exponential functions must be dimensionless. A pure number, ratio of similar physical quantities, such as angle as the ratio (length/length), refractive index as the ratio (speed of light in vacuum/speed of light in medium) etc., has no dimensions.

Now we can test the dimensional consistency or homogeneity of the equation

$$x = x_0 + v_0\, t + (1/2)\, a\, t^2$$

for the distance x travelled by a particle or body in time t which starts from the position x_0 with an initial velocity v_0 at time $t = 0$ and has uniform acceleration a along the direction of motion.

The dimensions of each term may be written as

$$[x] = [L]$$
$$[x_0] = [L]$$
$$[v_0\, t] = [L T^{-1}]\ [T] = [L]$$
$$[(1/2)\, a\, t^2] = [L T^{-2}]\ [T^2] = [L]$$

As each term on the right hand side of this equation has the same dimension, namely that of length, which is same as the dimension of left hand side of the equation, hence this equation is a dimensionally correct equation.

It may be noted that a test of consistency of dimensions tells us no more and no less than a

test of consistency of units, but has the advantage that we need not commit ourselves to a particular choice of units, and we need not worry about conversions among multiples and sub-multiples of the units. It may be borne in mind that **if an equation fails this consistency test, it is proved wrong, but if it passes, it is not proved right. Thus, a dimensionally correct equation need not be actually an exact (correct) equation, but a dimensionally wrong (incorrect) or inconsistent equation must be wrong**.

Example 2.15 Let us consider an equation

$$\frac{1}{2} m\, v^2 = m\, g\, h$$

where m is the mass of the body, v its velocity, g is the acceleration due to gravity and h is the height. Check whether this equation is dimensionally correct.

Answer The dimensions of LHS are

$$[M]\ [L\,T^{-1}]^2 = [M]\ [L^2 T^{-2}] = [M\,L^2\,T^{-2}]$$

The dimensions of RHS are

$$[M][L\,T^{-2}]\ [L] = [M][L^2\,T^{-2}] = [M\,L^2 T^{-2}]$$

The dimensions of LHS and RHS are the same and hence the equation is dimensionally correct.

Example 2.16 The SI unit of energy is J = kg m^2 s^{-2}; that of speed v is m s^{-1} and of acceleration a is m s^{-2}. Which of the formulae for kinetic energy (K) given below can you rule out on the basis of dimensional arguments (m stands for the mass of the body) :

(a) $K = m^2 v^3$
(b) $K = (1/2)mv^2$
(c) $K = ma$
(d) $K = (3/16)mv^2$
(e) $K = (1/2)mv^2 + ma$

Answer Every correct formula or equation must have the same dimensions on both sides of the equation. Also, only quantities with the same physical dimensions can be added or subtracted. The dimensions of the quantity on the right side are [M^2 L^3 T^{-3}] for (a); [M L^2 T^{-2}] for (b) and (d); [M L T^{-2}] for (c). The quantity on the right side of (e) has no proper dimensions since two quantities of different dimensions have been added. Since the kinetic energy K has the dimensions of [M L^2 T^{-2}], formulas (a), (c) and (e) are ruled out. Note that dimensional arguments cannot tell which of the two, (b) or (d), is the correct formula. For this, one must turn to the actual definition of kinetic energy (see Chapter 6). The correct formula for kinetic energy is given by (b).

2.10.2 Deducing Relation among the Physical Quantities

The method of dimensions can sometimes be used to deduce relation among the physical quantities. For this we should know the dependence of the physical quantity on other quantities (upto three physical quantities or linearly independent variables) and consider it as a product type of the dependence. Let us take an example.

Example 2.17 Consider a simple pendulum, having a bob attached to a string, that oscillates under the action of the force of gravity. Suppose that the period of oscillation of the simple pendulum depends on its length (l), mass of the bob (m) and acceleration due to gravity (g). Derive the expression for its time period using method of dimensions.

Answer The dependence of time period T on the quantities l, g and m as a product may be written as :

$$T = k\, l^x\, g^y\, m^z$$

where k is dimensionless constant and x, y and z are the exponents.

By considering dimensions on both sides, we have

$$[L^o M^o T^1] = [L^1]^x [L^1 T^{-2}]^y [M^1]^z$$

$$= L^{x+y}\, T^{-2y}\ M^z$$

On equating the dimensions on both sides, we have

$$x + y = 0;\ -2y = 1;\ \text{and}\ z = 0$$

So that $x = \frac{1}{2}, y = -\frac{1}{2}, z = 0$

Then, $T = k\, l^{½}\, g^{-½}$

or, $T = k\sqrt{\frac{l}{g}}$

Note that value of constant k can not be obtained by the method of dimensions. Here it does not matter if some number multiplies the right side of this formula, because that does not affect its dimensions.

Actually, $k = 2\pi$ so that $T = 2\pi\sqrt{\frac{l}{g}}$ ◀

Dimensional analysis is very useful in deducing relations among the interdependent physical quantities. However, dimensionless constants cannot be obtained by this method. The method of dimensions can only test the dimensional validity, but not the exact relationship between physical quantities in any equation. It does not distinguish between the physical quantities having same dimensions.

A number of exercises at the end of this chapter will help you develop skill in dimensional analysis.

SUMMARY

1. Physics is a quantitative science, based on measurement of physical quantities. Certain physical quantities have been chosen as fundamental or base quantities (such as length, mass, time, electric current, thermodynamic temperature, amount of substance, and luminous intensity).
2. Each base quantity is defined in terms of a certain basic, arbitrarily chosen but properly standardised reference standard called unit (such as metre, kilogram, second, ampere, kelvin, mole and candela). The units for the fundamental or base quantities are called fundamental or base units.
3. Other physical quantities, derived from the base quantities, can be expressed as a combination of the base units and are called derived units. A complete set of units, both fundamental and derived, is called a system of units.
4. The International System of Units (SI) based on seven base units is at present internationally accepted unit system and is widely used throughout the world.
5. The SI units are used in all physical measurements, for both the base quantities and the derived quantities obtained from them. Certain derived units are expressed by means of SI units with special names (such as joule, newton, watt, etc).
6. The SI units have well defined and internationally accepted unit symbols (such as m for metre, kg for kilogram, s for second, A for ampere, N for newton etc.).
7. Physical measurements are usually expressed for small and large quantities in scientific notation, with powers of 10. Scientific notation and the prefixes are used to simplify measurement notation and numerical computation, giving indication to the precision of the numbers.
8. Certain general rules and guidelines must be followed for using notations for physical quantities and standard symbols for SI units, some other units and SI prefixes for expressing properly the physical quantities and measurements.
9. In computing any physical quantity, the units for derived quantities involved in the relationship(s) are treated as though they were algebraic quantities till the desired units are obtained.
10. Direct and indirect methods can be used for the measurement of physical quantities. In measured quantities, while expressing the result, the accuracy and precision of measuring instruments along with errors in measurements should be taken into account.
11. In measured and computed quantities proper significant figures only should be retained. Rules for determining the number of significant figures, carrying out arithmetic operations with them, and 'rounding off ' the uncertain digits must be followed.
12. The dimensions of base quantities and combination of these dimensions describe the nature of physical quantities. Dimensional analysis can be used to check the dimensional consistency of equations, deducing relations among the physical quantities, etc. A dimensionally consistent equation need not be actually an exact (correct) equation, but a dimensionally wrong or inconsistent equation must be wrong.

EXERCISES

Note : In stating numerical answers, take care of significant figures.

2.1 Fill in the blanks

(a) The volume of a cube of side 1 cm is equal to .....m^3

(b) The surface area of a solid cylinder of radius 2.0 cm and height 10.0 cm is equal to ...$(mm)^2$

(c) A vehicle moving with a speed of 18 km h^{-1} covers....m in 1 s

(d) The relative density of lead is 11.3. Its density is ....g cm^{-3} or ....kg m^{-3}.

2.2 Fill in the blanks by suitable conversion of units

(a) 1 kg m^2 s^{-2} = ...g cm^2 s^{-2}

(b) 1 m = ... ly

(c) 3.0 m s^{-2} = ... km h^{-2}

(d) G = 6.67 × 10^{-11} N m^2 $(kg)^{-2}$ = ... $(cm)^3$ s^{-2} g^{-1}.

2.3 A calorie is a unit of heat or energy and it equals about 4.2 J where 1J = 1 kg m^2 s^{-2}. Suppose we employ a system of units in which the unit of mass equals α kg, the unit of length equals β m, the unit of time is γ s. Show that a calorie has a magnitude 4.2 $\alpha^{-1} \beta^{-2} \gamma^{2}$ in terms of the new units.

2.4 Explain this statement clearly :

"To call a dimensional quantity 'large' or 'small' is meaningless without specifying a standard for comparison". In view of this, reframe the following statements wherever necessary :

(a) atoms are very small objects

(b) a jet plane moves with great speed

(c) the mass of Jupiter is very large

(d) the air inside this room contains a large number of molecules

(e) a proton is much more massive than an electron

(f) the speed of sound is much smaller than the speed of light.

2.5 A new unit of length is chosen such that the speed of light in vacuum is unity. What is the distance between the Sun and the Earth in terms of the new unit if light takes 8 min and 20 s to cover this distance ?

2.6 Which of the following is the most precise device for measuring length :

(a) a vernier callipers with 20 divisions on the sliding scale

(b) a screw gauge of pitch 1 mm and 100 divisions on the circular scale

(c) an optical instrument that can measure length to within a wavelength of light ?

2.7 A student measures the thickness of a human hair by looking at it through a microscope of magnification 100. He makes 20 observations and finds that the average width of the hair in the field of view of the microscope is 3.5 mm. What is the estimate on the thickness of hair ?

2.8 Answer the following :

(a) You are given a thread and a metre scale. How will you estimate the diameter of the thread ?

(b) A screw gauge has a pitch of 1.0 mm and 200 divisions on the circular scale. Do you think it is possible to increase the accuracy of the screw gauge arbitrarily by increasing the number of divisions on the circular scale ?

(c) The mean diameter of a thin brass rod is to be measured by vernier callipers. Why is a set of 100 measurements of the diameter expected to yield a more reliable estimate than a set of 5 measurements only ?

2.9 The photograph of a house occupies an area of 1.75 cm^2 on a 35 mm slide. The slide is projected on to a screen, and the area of the house on the screen is 1.55 m^2. What is the linear magnification of the projector-screen arrangement.

2.10 State the number of significant figures in the following :

(a) 0.007 m^2

(b) 2.64 × 10^{24} kg

(c) 0.2370 g cm^{-3}

(d) 6.320 J
(e) 6.032 N m^{-2}
(f) 0.0006032 m^2

2.11 The length, breadth and thickness of a rectangular sheet of metal are 4.234 m, 1.005 m, and 2.01 cm respectively. Give the area and volume of the sheet to correct significant figures.

2.12 The mass of a box measured by a grocer's balance is 2.300 kg. Two gold pieces of masses 20.15 g and 20.17 g are added to the box. What is (a) the total mass of the box, (b) the difference in the masses of the pieces to correct significant figures ?

2.13 A physical quantity P is related to four observables a, b, c and d as follows :

$$P = a^3 b^2 / \left(\sqrt{c}\, d\right)$$

The percentage errors of measurement in a, b, c and d are 1%, 3%, 4% and 2%, respectively. What is the percentage error in the quantity P? If the value of P calculated using the above relation turns out to be 3.763, to what value should you round off the result ?

2.14 A book with many printing errors contains four different formulas for the displacement y of a particle undergoing a certain periodic motion :
(a) $y = a \sin 2\pi\, t/T$
(b) $y = a \sin vt$
(c) $y = (a/T) \sin t/a$
(d) $y = (a\sqrt{2})\ (\sin 2\pi t / T + \cos 2\pi t / T)$
(a = maximum displacement of the particle, v = speed of the particle. T = time-period of motion). Rule out the wrong formulas on dimensional grounds.

2.15 A famous relation in physics relates 'moving mass' m to the 'rest mass' m_o of a particle in terms of its speed v and the speed of light, c. (This relation first arose as a consequence of special relativity due to Albert Einstein). A boy recalls the relation almost correctly but forgets where to put the constant c. He writes :

$$m = \frac{m_0}{\left(1 - v^2\right)^{1/2}}.$$

Guess where to put the missing c.

2.16 The unit of length convenient on the atomic scale is known as an angstrom and is denoted by **Å**: 1 **Å** = 10^{-10} m. The size of a hydrogen atom is about 0.5 **Å**. What is the total atomic volume in m^3 of a mole of hydrogen atoms ?

2.17 One mole of an ideal gas at standard temperature and pressure occupies 22.4 L (molar volume). What is the ratio of molar volume to the atomic volume of a mole of hydrogen ? (Take the size of hydrogen molecule to be about 1 **Å**). Why is this ratio so large ?

2.18 Explain this common observation clearly : If you look out of the window of a fast moving train, the nearby trees, houses etc. seem to move rapidly in a direction opposite to the train's motion, but the distant objects (hill tops, the Moon, the stars etc.) seem to be stationary. (In fact, since you are aware that you are moving, these distant objects seem to move with you).

2.19 The principle of 'parallax' in section 2.3.1 is used in the determination of distances of very distant stars. The baseline *AB* is the line joining the Earth's two locations six months apart in its orbit around the Sun. That is, the baseline is about the diameter of the Earth's orbit $\approx 3 \times 10^{11}$m. However, even the nearest stars are so distant that with such a long baseline, they show parallax only of the order of 1" (second) of arc or so. A *parsec* is a convenient unit of length on the astronomical scale. It is the distance of an object that will show a parallax of 1" (second) of arc from opposite ends of a baseline equal to the distance from the Earth to the Sun. How much is a parsec in terms of metres ?

2.20 The nearest star to our solar system is 4.29 light years away. How much is this distance in terms of parsecs? How much parallax would this star (named Alpha Centauri) show when viewed from two locations of the Earth six months apart in its orbit around the Sun ?

2.21 Precise measurements of physical quantities are a *need* of science. For example, to ascertain the speed of an aircraft, one must have an accurate method to find its positions at closely separated instants of time. This was the actual motivation behind the discovery of radar in World War II. Think of different examples in modern science where precise measurements of length, time, mass etc. are needed. Also, wherever you can, give a quantitative idea of the precision needed.

2.22 Just as precise measurements are necessary in science, it is equally important to be able to make rough estimates of quantities using rudimentary ideas and common observations. Think of ways by which you can estimate the following (where an estimate is difficult to obtain, try to get an upper bound on the quantity) :

(a) the total mass of rain-bearing clouds over India during the Monsoon

(b) the mass of an elephant

(c) the wind speed during a storm

(d) the number of strands of hair on your head

(e) the number of air molecules in your classroom.

2.23 The Sun is a hot plasma (ionized matter) with its inner core at a temperature exceeding 10^7 K, and its outer surface at a temperature of about 6000 K. At these high temperatures, no substance remains in a solid or liquid phase. In what range do you expect the mass density of the Sun to be, in the range of densities of solids and liquids or gases ? Check if your guess is correct from the following data : mass of the Sun = 2.0×10^{30} kg, radius of the Sun = 7.0×10^8 m.

2.24 When the planet Jupiter is at a distance of 824.7 million kilometers from the Earth, its angular diameter is measured to be 35.72" of arc. Calculate the diameter of Jupiter.

Additional Exercises

2.25 A man walking briskly in rain with speed v must slant his umbrella forward making an angle θ with the vertical. A student derives the following relation between θ and v : $\tan\theta = v$ and checks that the relation has a correct limit: as $v \to 0$, $\theta \to 0$, as expected. (We are assuming there is no strong wind and that the rain falls vertically for a stationary man). Do you think this relation can be correct ? If not, guess the correct relation.

2.26 It is claimed that two cesium clocks, if allowed to run for 100 years, free from any disturbance, may differ by only about 0.02 s. What does this imply for the accuracy of the standard cesium clock in measuring a time-interval of 1 s ?

2.27 Estimate the average mass density of a sodium atom assuming its size to be about 2.5 Å. (Use the known values of Avogadro's number and the atomic mass of sodium). Compare it with the density of sodium in its crystalline phase : 970 kg m^{-3}. Are the two densities of the same order of magnitude ? If so, why ?

2.28 The unit of length convenient on the nuclear scale is a fermi : 1 f = 10^{-15} m. Nuclear sizes obey roughly the following empirical relation :

$$r = r_0 A^{1/3}$$

where r is the radius of the nucleus, A its mass number, and r_0 is a constant equal to about, 1.2 f. Show that the rule implies that nuclear mass density is nearly constant for different nuclei. Estimate the mass density of sodium nucleus. Compare it with the average mass density of a sodium atom obtained in Exercise. 2.27.

2.29 A LASER is a source of very intense, monochromatic, and unidirectional beam of light. These properties of a laser light can be exploited to measure long distances. The distance of the Moon from the Earth has been already determined very precisely using a laser as a source of light. A laser light beamed at the Moon takes 2.56 s to

return after reflection at the Moon's surface. How much is the radius of the lunar orbit around the Earth ?

2.30 A SONAR (sound navigation and ranging) uses ultrasonic waves to detect and locate objects under water. In a submarine equipped with a SONAR the time delay between generation of a probe wave and the reception of its echo after reflection from an enemy submarine is found to be 77.0 s. What is the distance of the enemy submarine? (Speed of sound in water = 1450 m s^{-1}).

2.31 The farthest objects in our Universe discovered by modern astronomers are so distant that light emitted by them takes billions of years to reach the Earth. These objects (known as quasars) have many puzzling features, which have not yet been satisfactorily explained. What is the distance in km of a quasar from which light takes 3.0 billion years to reach us ?

2.32 It is a well known fact that during a total solar eclipse the disk of the moon almost completely covers the disk of the Sun. From this fact and from the information you can gather from examples 2.3 and 2.4, determine the approximate diameter of the moon.

2.33 A great physicist of this century (P.A.M. Dirac) loved playing with numerical values of Fundamental constants of nature. This led him to an interesting observation. Dirac found that from the basic constants of atomic physics (c, e, mass of electron, mass of proton) and the gravitational constant G, he could arrive at a number with the dimension of time. Further, it was a very large number, its magnitude being close to the present estimate on the age of the universe (~15 billion years). From the table of fundamental constants in this book, try to see if you too can construct this number (or any other interesting number you can think of). If its coincidence with the age of the universe were significant, what would this imply for the constancy of fundamental constants ?

CHAPTER THREE

MOTION IN A STRAIGHT LINE

3.1 INTRODUCTION

Motion is common to everything in the universe. We walk, run and ride a bicycle. Even when we are sleeping, air moves into and out of our lungs and blood flows in arteries and veins. We see leaves falling from trees and water flowing down a dam. Automobiles and planes carry people from one place to the other. The earth rotates once every twenty-four hours and revolves round the sun once in a year. The sun itself is in motion in the Milky Way, which is again moving within its local group of galaxies.

Motion is change in position of an object with time. How does the position change with time ? In this chapter, we shall learn how to describe motion. For this, we develop the concepts of velocity and acceleration. We shall confine ourselves to the study of motion of objects along a straight line, also known as **rectilinear motion**. For the case of rectilinear motion with uniform acceleration, a set of simple equations can be obtained. Finally, to understand the relative nature of motion, we introduce the concept of relative velocity.

In our discussions, we shall treat the objects in motion as point objects. This approximation is valid so far as the size of the object is much smaller than the distance it moves in a reasonable duration of time. In a good number of situations in real-life, the size of objects can be neglected and they can be considered as point-like objects without much error.

In **Kinematics**, we study ways to describe motion without going into the causes of motion. What causes motion described in this chapter and the next chapter forms the subject matter of Chapter 5.

3.2 POSITION, PATH LENGTH AND DISPLACEMENT

Earlier you learnt that motion is change in position of an object with time. In order to specify position, we need to use a reference point and a set of axes. It is convenient to choose

a rectangular coordinate system consisting of three mutually perpendicular axes, labelled X-, Y-, and Z- axes. The point of intersection of these three axes is called origin (O) and serves as the **reference point**. The coordinates (x, y. z) of an object describe the position of the object with respect to this coordinate system. To measure time, we position a clock in this system. This coordinate system along with a clock constitutes a **frame of reference**.

If one or more coordinates of an object change with time, we say that the object is in motion. Otherwise, the object is said to be at rest with respect to this frame of reference.

The choice of a set of axes in a frame of reference depends upon the situation. For example, for describing motion in one dimension, we need only one axis. To describe motion in two/three dimensions, we need a set of two/three axes.

Description of an event depends on the frame of reference chosen for the description. For example, when you say that a car is moving on a road, you are describing the car with respect to a frame of reference attached to you or to the ground. But with respect to a frame of reference attached with a person sitting in the car, the car is at rest.

To describe motion along a straight line, we can choose an axis, say *X*-axis, so that it coincides with the path of the object. We then measure the position of the object with reference to a conveniently chosen origin, say O, as shown in Fig. 3.1. Positions to the right of O are taken as positive and to the left of O, as negative. Following this convention, the position coordinates of point P and Q in Fig. 3.1 are +360 m and +240 m. Similarly, the position coordinate of point R is –120 m.

Path length

Consider the motion of a car along a straight line. We choose the *x*-axis such that it coincides with the path of the car's motion and origin of the axis as the point from where the car started moving, i.e. the car was at $x = 0$ at $t = 0$ (Fig. 3.1). Let P, Q and R represent the positions of the car at different instants of time. Consider two cases of motion. In the first case, the car moves from O to P. Then the distance moved by the car is OP = +360 m. **This distance is called the path length** traversed by the car**.** In the second case, the car moves from O to P and then moves back from P to Q. During this course of motion, the path length traversed is OP + PQ = + 360 m + (+120 m) = + 480 m. Path length is a scalar quantity — a quantity that has a magnitude only and no direction (see Chapter 4).

Displacement

It is useful to define another quantity displacement as the change in position. Let x_1 and x_2 be the positions of an object at time t_1 and t_2. Then its displacement, denoted by Δx, in time $\Delta t = (t_2 - t_1)$, is given by the difference between the final and initial positions :

$$\Delta x = x_2 - x_1$$

(We use the Greek letter delta (Δ) to denote a change in a quantity.)

If $x_2 > x_1$, Δx is positive; and if $x_2 < x_1$, Δx is negative.

Displacement has both magnitude and direction. Such quantities are represented by vectors. You will read about vectors in the next chapter. Presently, we are dealing with motion along a straight line (also called **rectilinear motion**) only. In one-dimensional motion, there are *only two* directions (backward and forward, upward and downward) in which an object can move, and these two directions can easily be specified by + and – signs. For example, displacement of the car in moving from O to P is :

$$\Delta x = x_2 - x_1 = (+360 \text{ m}) - 0 \text{ m} = +360 \text{ m}$$

The displacement has a magnitude of 360 m and is directed in the positive x direction as indicated by the + sign. Similarly, the displacement of the car from P to Q is 240 m – 360 m = – 120 m. The

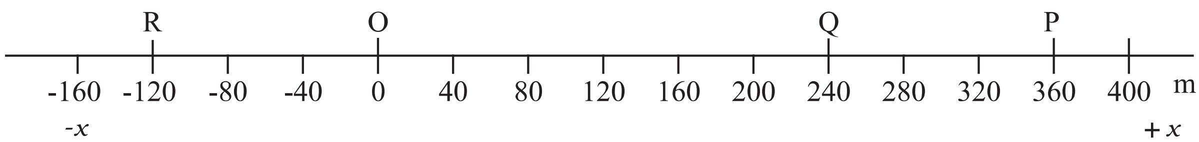


Fig. 3.1 *x-axis, origin and positions of a car at different times.*

negative sign indicates the direction of displacement. Thus, it is not necessary to use vector notation for discussing motion of objects in one-dimension.

The magnitude of displacement may or may not be equal to the path length traversed by an object. For example, for motion of the car from O to P, the path length is +360 m and the displacement is +360 m. In this case, the magnitude of displacement (360 m) is equal to the path length (360 m). But consider the motion of the car from O to P and back to Q. In this case, the path length = (+360 m) + (+120 m) = + 480 m. However, the displacement = (+240 m) − (0 m) = + 240 m. Thus, the magnitude of displacement (240 m) is not equal to the path length (480 m).

The magnitude of the displacement for a course of motion may be zero but the corresponding path length is **not zero**. For example, if the car starts from O, goes to P and then returns to O, the final position coincides with the initial position and the displacement is zero. However, the path length of this journey is OP + PO = 360 m + 360 m = 720 m.

Motion of an object can be represented by a position-time graph as you have already learnt about it. Such a graph is a powerful tool to represent and analyse different aspects of motion of an object. For motion along a straight line, say *X*-axis, only *x*-coordinate varies with time and we have an *x-t* graph. Let us first consider the simple case in which an object is stationary, e.g. a car standing still at x = 40 m. The position-time graph is a straight line parallel to the time axis, as shown in Fig. 3.2(a).

If an object moving along the straight line covers equal distances in equal intervals of time, it is said to be in **uniform motion** along a straight line. Fig. 3.2(b) shows the position-time graph of such a motion.

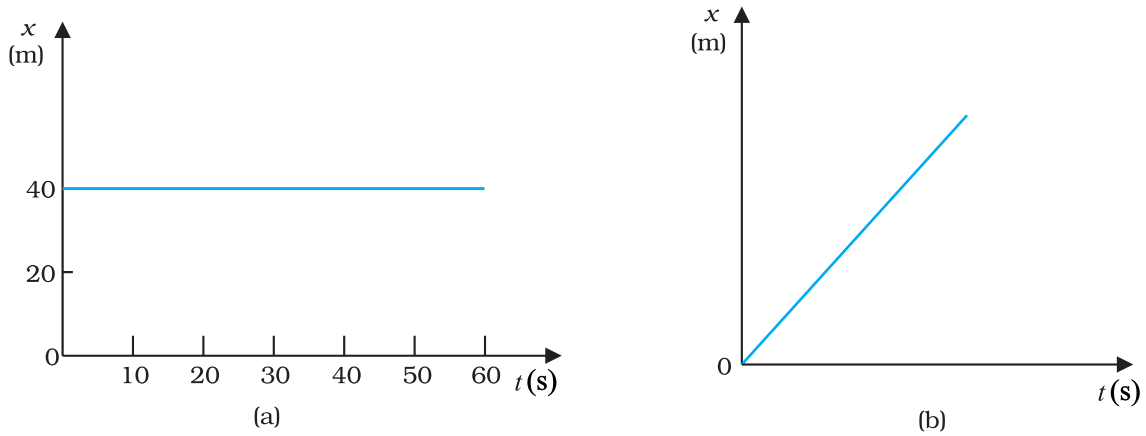


***Fig. 3.2** Position-time graph of (a) stationary object, and (b) an object in uniform motion.*

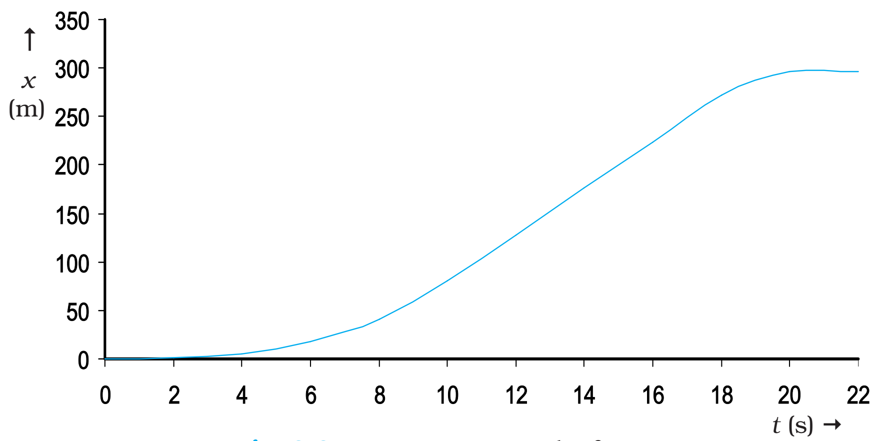


***Fig. 3.3** Position-time graph of a car.*

Now, let us consider the motion of a car that starts from rest at time $t = 0$ s from the origin O and picks up speed till $t = 10$ s and thereafter moves with uniform speed till $t = 18$ s. Then the brakes are applied and the car stops at $t = 20$ s and $x = 296$ m. The position-time graph for this case is shown in Fig. 3.3. We shall refer to this graph in our discussion in the following sections.

3.3 AVERAGE VELOCITY AND AVERAGE SPEED

When an object is in motion, its position changes with time. But how fast is the position changing with time and in what direction? To describe this, we define the quantity **average velocity**. Average velocity is defined as the change in position or displacement (Δx) divided by the time intervals (Δt), in which the displacement occurs :

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \qquad (3.1)$$

where x_2 and x_1 are the positions of the object at time t_2 and t_1, respectively. Here the bar over the symbol for velocity is a standard notation used to indicate an average quantity. The SI unit for velocity is m/s or m s^{-1}, although km h^{-1} is used in many everyday applications.

Like displacement, average velocity is also a vector quantity. But as explained earlier, for motion in a straight line, the directional aspect of the vector can be taken care of by + and – signs and we do not have to use the vector notation for velocity in this chapter.

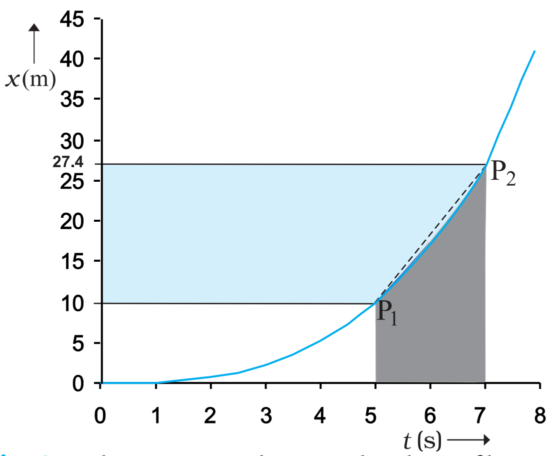


Fig. 3.4 *The average velocity is the slope of line* P_1P_2.

Consider the motion of the car in Fig. 3.3. The portion of the x-t graph between $t = 0$ s and $t = 8$ s is blown up and shown in Fig. 3.4. As seen from the plot, the average velocity of the car between time $t = 5$ s and $t = 7$ s is :

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{(27.4 - 10.0)\,\text{m}}{(7-5)\,\text{s}} = 8.7\ \text{m s}^{-1}$$

Geometrically, this is the slope of the straight line P_1P_2 connecting the initial position P_1 to the final position P_2 as shown in Fig. 3.4.

The average velocity can be positive or negative depending upon the sign of the displacement. It is zero if the displacement is zero. Fig. 3.5 shows the x-t graphs for an object, moving with positive velocity (Fig. 3.5a), moving with negative velocity (Fig. 3.5b) and at rest (Fig. 3.5c).

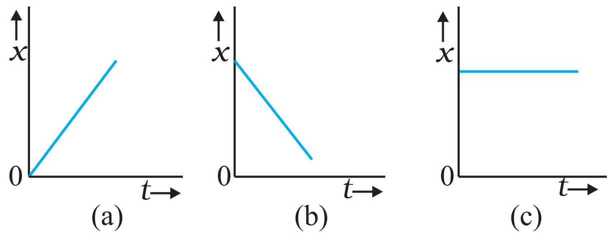


Fig. 3.5 *Position-time graph for an object (a) moving with positive velocity, (b) moving with negative velocity, and (c) at rest.*

Average velocity as defined above involves only the displacement of the object. We have seen earlier that the magnitude of displacement may be different from the actual path length. To describe the rate of motion over the actual path, we introduce another quantity called **average speed**.

Average speed is defined as the total path length travelled divided by the total time interval during which the motion has taken place :

$$\textit{Average speed} = \frac{\textit{Total path length}}{\textit{Total time interval}} \qquad (3.2)$$

Average speed has obviously the same unit (m s^{-1}) as that of velocity. But it does not tell us in what direction an object is moving. Thus, it is always positive (in contrast to the average velocity which can be positive or negative). If the motion of an object is along a straight line and in the **same direction**, the magnitude of displacement is equal to the total path length. In that case, the magnitude of average velocity

is equal to the average speed. This is not always the case, as you will see in the following example.

> ▶ ***Example 3.1*** A car is moving along a straight line, say OP in Fig. 3.1. It moves from O to P in 18 s and returns from P to Q in 6.0 s. What are the average velocity and average speed of the car in going (a) from O to P ? and (b) from O to P and back to Q ?

Answer (a)

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time interval}}$$

$$\bar{v} = \frac{+360 \text{ m}}{18 \text{ s}} = +20 \text{ m s}^{-1}$$

$$\text{Average speed} = \frac{\text{Path length}}{\text{Time interval}}$$

$$= \frac{360 \text{ m}}{18 \text{ s}} = 20 \text{ m s}^{-1}$$

Thus, in this case the average speed is equal to the magnitude of the average velocity.
(b) In this case,

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time interval}} = \frac{+240 \text{ m}}{(18+6.0) \text{ s}}$$

$$= +10 \text{ m s}^{-1}$$

$$\text{Average speed} = \frac{\text{Path length}}{\text{Time interval}} = \frac{\text{OP} + \text{PQ}}{\Delta t}$$

$$= \frac{(360+120) \text{ m}}{24 \text{ s}} = 20 \text{ m s}^{-1}$$

Thus, in this case the average speed is *not* equal to the magnitude of the average velocity. This happens because the motion here involves change in direction so that the path length is greater than the magnitude of displacement. This shows that **speed is, in general, greater than the magnitude of the velocity.** ◀

If the car in Example 3.1 moves from O to P and comes back to O in the same time interval, average speed is 20 m/s but the average velocity is zero !

3.4 INSTANTANEOUS VELOCITY AND SPEED

The average velocity tells us how fast an object has been moving over a given time interval but does not tell us how fast it moves at different instants of time during that interval. For this, we define **instantaneous velocity** or simply velocity v at an instant t.

The velocity at an instant is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally small. In other words,

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{3.3a}$$

$$= \frac{dx}{dt} \tag{3.3b}$$

where the symbol $\lim_{\Delta t \to 0}$ stands for the operation of taking limit as $\Delta t \to 0$ of the quantity on its right. In the language of calculus, the quantity on the right hand side of Eq. (3.3a) is the differential coefficient of x with respect to t and is denoted by $\frac{dx}{dt}$ (see Appendix 3.1). It is the rate of change of position with respect to time, at that instant.

We can use Eq. (3.3a) for obtaining the value of velocity at an instant either **graphically** or **numerically**. Suppose that we want to obtain graphically the value of velocity at time $t = 4$ s (point P) for the motion of the car represented in Fig. 3.3. The figure has been redrawn in Fig. 3.6 choosing different scales to facilitate the

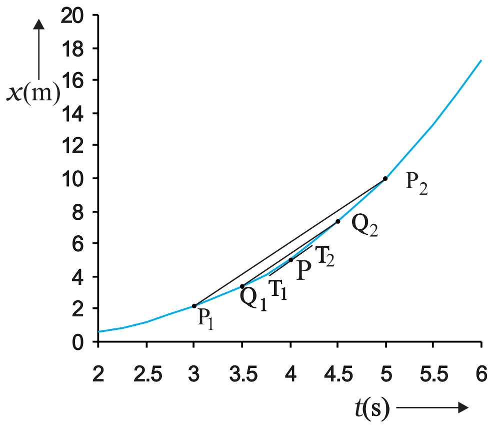


Fig. 3.6 *Determining velocity from position-time graph. Velocity at t = 4 s is the slope of the tangent to the graph at that instant.*

calculation. Let us take Δt = 2 s centred at t = 4 s. Then, by the definition of the average velocity, the slope of line P_1P_2 (Fig. 3.6) gives the value of average velocity over the interval 3 s to 5 s. Now, we decrease the value of Δt from 2 s to 1 s. Then line P_1P_2 becomes Q_1Q_2 and its slope gives the value of the average velocity over the interval 3.5 s to 4.5 s. In the limit $\Delta t \to 0$, the line P_1P_2 becomes tangent to the position-time curve at the point P and the velocity at t = 4 s is given by the slope of the tangent at that point. **It is difficult to show this process graphically. But if we use numerical method to obtain the value of the velocity, the meaning of the limiting process becomes clear.** For the graph shown in Fig. 3.6, $x = 0.08\ t^3$. Table 3.1 gives the value of $\Delta x/\Delta t$ calculated for Δt equal to 2.0 s, 1.0 s, 0.5 s, 0.1 s and 0.01 s centred at t = 4.0 s. The second and third columns give the value of $t_1 = \left(t - \frac{\Delta t}{2}\right)$ and $t_2 = \left(t + \frac{\Delta t}{2}\right)$ and the fourth and the fifth columns give the corresponding values of x, i.e. $x(t_1) = 0.08\ t_1^3$ and $x(t_2) = 0.08\ t_2^3$. The sixth column lists the difference $\Delta x = x(t_2) - x(t_1)$ and the last column gives the ratio of Δx and Δt, i.e. the average velocity corresponding to the value of Δt listed in the first column.

We see from Table 3.1 that as we decrease the value of Δt from 2.0 s to 0.010 s, the value of the average velocity approaches the limiting value 3.84 m s^{-1} which is the value of velocity at t = 4.0 s, i.e. the value of $\frac{dx}{dt}$ at t = 4.0 s. In this manner, we can calculate velocity at each instant for motion of the car shown in Fig. 3.3. For this case, the variation of velocity with time is found to be as shown in Fig. 3.7.

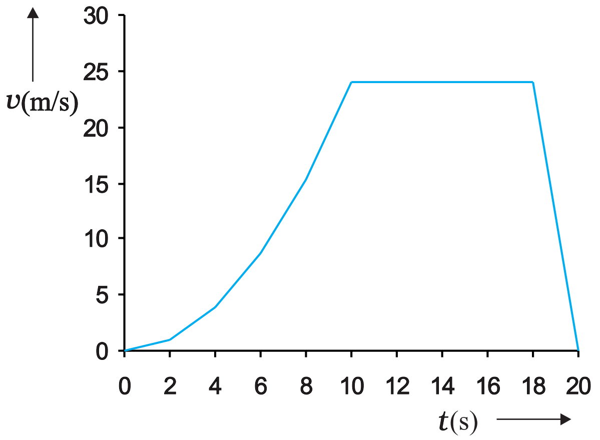


Fig. 3.7 *Velocity–time graph corresponding to motion shown in Fig. 3.3.*

The graphical method for the determination of the instantaneous velocity is always not a convenient method. For this, we must carefully plot the position–time graph and calculate the value of average velocity as Δt becomes smaller and smaller. It is easier to calculate the value of velocity at different instants if we have data of positions at different instants or exact expression for the position as a function of time. Then, we calculate $\Delta x/\Delta t$ from the data for decreasing the value of Δt and find the limiting value as we have done in Table 3.1 or use differential calculus for the given expression and calculate $\frac{dx}{dt}$ at different instants as done in the following example.

Table 3.1 Limiting value of $\frac{\Delta x}{\Delta t}$ at t = 4 s

Δt (s)	t_1 (s)	t_2 (s)	$x(t_1)$ (m)	$x(t_2)$ (m)	Δx (m)	$\Delta x / \Delta t$ (m s^{-1})
2.0	3.0	5.0	2.16	10.0	7.84	3.92
1.0	3.5	4.5	3.43	7.29	3.86	3.86
0.5	3.75	4.25	4.21875	6.14125	1.9225	3.845
0.1	3.95	4.05	4.93039	5.31441	0.38402	3.8402
0.01	3.995	4.005	5.100824	5.139224	0.0384	3.8400

> ▶ ***Example 3.2*** *The position of an object moving along x-axis is given by $x = a + bt^2$ where $a = 8.5$ m, $b = 2.5$ m s^{-2} and t is measured in seconds. What is its velocity at t = 0 s and t = 2.0 s. What is the average velocity between t = 2.0 s and t = 4.0 s ?*

Answer In notation of differential calculus, the velocity is

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(a + bt^2\right) = 2b\,t = 5.0\ t\ \text{m s}^{-1}$$

At $t = 0$ s, $v = 0$ m s^{-1} and at $t = 2.0$ s, $v = 10$ m s^{-1} .

$$\textit{Average velocity} = \frac{x(4.0) - x(2.0)}{4.0 - 2.0}$$

$$= \frac{a + 16b - a - 4b}{2.0} = 6.0 \times b$$

$$= 6.0 \times 2.5 = 15\ \text{m s}^{-1}$$ ◀

From Fig. 3.7, we note that during the period t = 10 s to 18 s the velocity is constant. Between period t = 18 s to t = 20 s, it is uniformly decreasing and during the period t = 0 s to t = 10 s, it is increasing. **Note that for uniform motion, velocity is the same as the average velocity at all instants.**

Instantaneous speed or simply speed is the magnitude of velocity. For example, a velocity of + 24.0 m s^{-1} and a velocity of – 24.0 m s^{-1} — both have an associated speed of 24.0 m s^{-1}. It should be noted that though average speed over a finite interval of time is greater or equal to the magnitude of the average velocity, instantaneous speed at an instant is equal to the magnitude of the instantaneous velocity at that instant. Why so ?

3.5 ACCELERATION

The velocity of an object, in general, changes during its course of motion. How to describe this change? Should it be described as the rate of change in velocity **with distance** or **with time** ? This was a problem even in Galileo's time. It was first thought that this change could be described by the rate of change of velocity with distance. But, through his studies of motion of freely falling objects and motion of objects on an inclined plane, Galileo concluded that the rate of change of velocity with time is a constant of motion for all objects in free fall. On the other hand, the change in velocity with distance is not constant – it decreases with the increasing distance of fall.

This led to the concept of acceleration as the rate of change of velocity with time.

The average acceleration $\bar{a}$ over a time interval is defined as the change of velocity divided by the time interval :

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t} \qquad (3.4)$$

where v_2 and v_1 are the instantaneous velocities or simply velocities at time t_2 and t_1 . It is the average change of velocity per unit time. The SI unit of acceleration is m s^{-2} .

On a plot of velocity versus time, the average acceleration is the slope of the straight line connecting the points corresponding to (v_2, t_2) and (v_1, t_1). The average acceleration for velocity-time graph shown in Fig. 3.7 for different time intervals 0 s - 10 s, 10 s – 18 s, and 18 s – 20 s are :

0 s - 10 s $\quad \bar{a} = \frac{(24 - 0)\,\text{m s}^{-1}}{(10 - 0)\,\text{s}} = 2.4\ \text{m s}^{-2}$

10 s - 18 s $\quad \bar{a} = \frac{(24 - 24)\,\text{m s}^{-1}}{(18 - 10)\,\text{s}} = 0\ \text{m s}^{-2}$

18 s - 20 s $\quad \bar{a} = \frac{(0 - 24)\,\text{m s}^{-1}}{(20 - 18)\,\text{s}} = -12\ \text{m s}^{-2}$

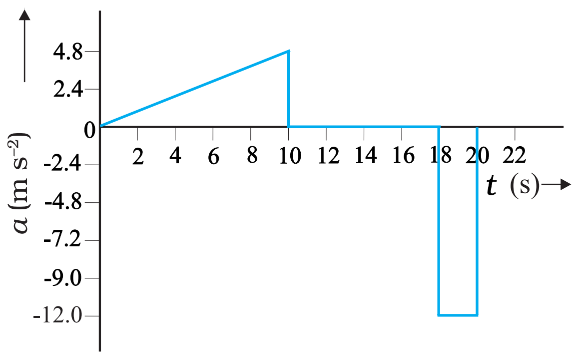


Fig. 3.8 *Acceleration as a function of time for motion represented in Fig. 3.3.*

Instantaneous acceleration is defined in the same way as the instantaneous velocity :

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \qquad (3.5)$$

The acceleration at an instant is the slope of the tangent to the v–t curve at that instant. For the v–t curve shown in Fig. 3.7, we can obtain acceleration at every instant of time. The resulting $a - t$ curve is shown in Fig. 3.8. We see

that the acceleration is nonuniform over the period 0 s to 10 s. It is zero between 10 s and 18 s and is constant with value –12 m s^{-2} between 18 s and 20 s. When the acceleration is uniform, obviously, it equals the average acceleration over that period.

Since velocity is a quantity having both magnitude and direction, a change in velocity may involve either or both of these factors. Acceleration, therefore, may result from a change in speed (magnitude), a change in direction or changes in both. Like velocity, acceleration can also be positive, negative or zero. Position-time graphs for motion with positive, negative and zero acceleration are shown in Figs. 3.9 (a), (b) and (c), respectively. Note that the graph curves upward for positive acceleration; downward for negative acceleration and it is a straight line for zero acceleration. As an exercise, identify in Fig. 3.3, the regions of the curve that correspond to these three cases.

Although acceleration can vary with time, our study in this chapter will be restricted to motion with constant acceleration. In this case, the average acceleration equals the constant value of acceleration during the interval. If the velocity of an object is v_o at $t = 0$ and v at time t, we have

$$\bar{a} = \frac{v - v_0}{t - 0} \quad \text{or,} \quad v = v_0 + a\,t \qquad (3.6)$$

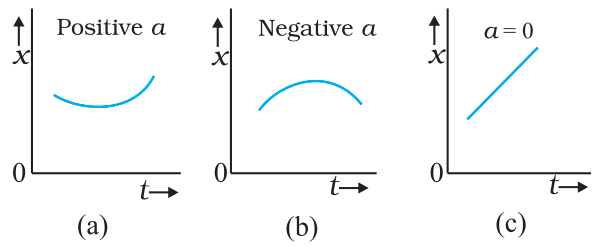


Fig. 3.9 *Position-time graph for motion with (a) positive acceleration; (b) negative acceleration, and (c) zero acceleration.*

Let us see how velocity-time graph looks like for some simple cases. Fig. 3.10 shows velocity-time graph for motion with constant acceleration for the following cases :

(a) An object is moving in a positive direction with a positive acceleration, for example the motion of the car in Fig. 3.3 between t = 0 s and t = 10 s.

(b) An object is moving in positive direction with a negative acceleration, for example, motion of the car in Fig 3.3 between t = 18 s and 20 s.

(c) An object is moving in negative direction with a negative acceleration, for example the motion of a car moving from O in Fig. 3.1 in negative x-direction with increasing speed.

(d) An object is moving in positive direction till time t_1, and then turns back with the same negative acceleration, for example the motion of a car from point O to point Q in Fig. 3.1 till time t_1 with decreasing speed and turning back and moving with the same negative acceleration.

An interesting feature of a velocity-time graph for any moving object is that **the area under the curve represents the displacement over a given time interval**. A general proof of this

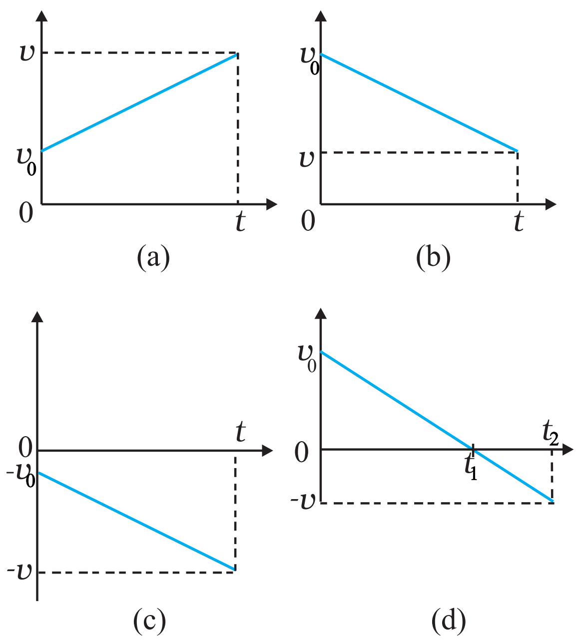


Fig. 3.10 *Velocity–time graph for motions with constant acceleration. (a) Motion in positive direction with positive acceleration, (b) Motion in positive direction with negative acceleration, (c) Motion in negative direction with negative acceleration, (d) Motion of an object with negative acceleration that changes direction at time t_1. Between times 0 to t_1, its moves in positive x - direction and between t_1 and t_2 it moves in the opposite direction.*

statement requires use of calculus. We can, however, see that it is true for the simple case of an object moving with constant velocity u. Its velocity-time graph is as shown in Fig. 3.11.

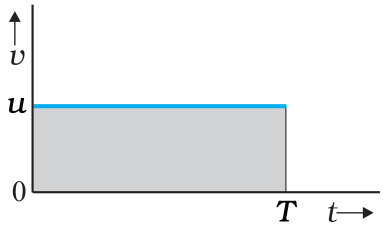


Fig. 3.11 *Area under v–t curve equals displacement of the object over a given time interval.*

The v-t curve is a straight line parallel to the time axis and the area under it between $t = 0$ and $t = T$ is the area of the rectangle of height u and base T. Therefore, area $= u \times T = uT$ which is the displacement in this time interval. How come in this case an area is equal to a distance? Think! Note the dimensions of quantities on the two coordinate axes, and you will arrive at the answer.

Note that the *x-t*, *v-t*, and *a-t* graphs shown in several figures in this chapter have sharp kinks at some points implying that the functions are not differentiable at these points. In any realistic situation, the functions will be differentiable at all points and the graphs will be smooth.

What this means physically is that acceleration and velocity cannot change values abruptly at an instant. Changes are always continuous.

3.6 KINEMATIC EQUATIONS FOR UNIFORMLY ACCELERATED MOTION

For uniformly accelerated motion, we can derive some simple equations that relate displacement (x), time taken (t), initial velocity (v_0), final velocity (v) and acceleration (a). Equation (3.6) already obtained gives a relation between final and initial velocities v and v_0 of an object moving with uniform acceleration a :

$$v = v_0 + at \tag{3.6}$$

This relation is graphically represented in Fig. 3.12. The area under this curve is :

Area between instants 0 and t = Area of triangle ABC + Area of rectangle OACD

$$= \frac{1}{2}(v-v_0)t + v_0 t$$

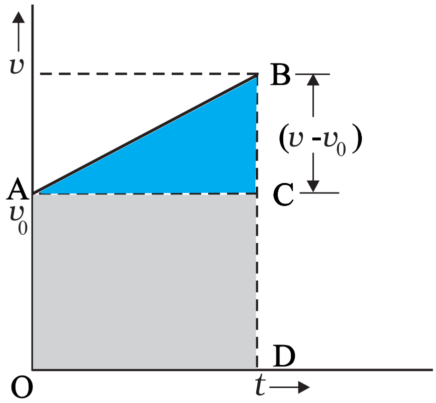


Fig. 3.12 *Area under v-t curve for an object with uniform acceleration.*

As explained in the previous section, the area under v-t curve represents the displacement. Therefore, the displacement x of the object is :

$$x = \frac{1}{2}(v-v_0)t + v_0 t \tag{3.7}$$

But $\quad v - v_0 = a\,t$

Therefore, $x = \frac{1}{2}a\,t^2 + v_0 t$

or, $$x = v_0 t + \frac{1}{2}at^2 \tag{3.8}$$

Equation (3.7) can also be written as

$$x = \frac{v+v_0}{2}t = \bar{v}\,t \tag{3.9a}$$

where,

$$\bar{v} = \frac{v+v_0}{2} \quad \textbf{(constant acceleration only)} \tag{3.9b}$$

Equations (3.9a) and (3.9b) mean that the object has undergone displacement x with an average velocity equal to the arithmetic average of the initial and final velocities.

From Eq. (3.6), $t = (v - v_0)/a$. Substituting this in Eq. (3.9a), we get

$$x = \bar{v}\,t = \left(\frac{v+v_0}{2}\right)\left(\frac{v-v_0}{a}\right) = \frac{v^2 - v_0^2}{2a}$$

$$v^2 = v_0^2 + 2ax \tag{3.10}$$

This equation can also be obtained by substituting the value of t from Eq. (3.6) into Eq. (3.8). Thus, we have obtained three important equations :

$$v = v_0 + at$$

$$x = v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2ax \qquad (3.11a)$$

connecting five quantities v_0, v, a, t and x. These are kinematic equations of rectilinear motion for constant acceleration.

The set of Eq. (3.11a) were obtained by assuming that at t = 0, the position of the particle, x is 0. We can obtain a more general equation if we take the position coordinate at t = 0 as non-zero, say x_0. Then Eqs. (3.11a) are modified (replacing x by $x - x_0$) to :

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad (3.11b)$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad (3.11c)$$

Example 3.3 Obtain equations of motion for constant acceleration using method of calculus.

Answer By definition

$$a = \frac{dv}{dt}$$

$$dv = a\,dt$$

Integrating both sides

$$\int_{v_0}^{v} dv = \int_0^t a\ dt$$

$$= a\int_0^t dt \qquad (a \text{ is constant})$$

$$v - v_0 = at$$

$$v = v_0 + at$$

Further, $$v = \frac{dx}{dt}$$

$$dx = v\,dt$$

Integrating both sides

$$\int_{x_0}^{x} dx = \int_0^t v\ dt$$

$$= \int_0^t (v_0 + at)\ dt$$

$$x - x_0 = v_0\ t + \frac{1}{2}a\ t^2$$

$$x = x_0 + v_0\ t + \frac{1}{2}a\ t^2$$

We can write

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

or, $v\,dv = a\,dx$

Integrating both sides,

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a\,dx$$

$$\frac{v^2 - v_0^2}{2} = a(x - x_0)$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

The advantage of this method is that it can be used for motion with non-uniform acceleration also.

Now, we shall use these equations to some important cases. ◀

Example 3.4 A ball is thrown vertically upwards with a velocity of 20 m s^{-1} from the top of a multistorey building. The height of the point from where the ball is thrown is 25.0 m from the ground. (a) How high will the ball rise ? and (b) how long will it be before the ball hits the ground? Take g = 10 m s^{-2}.

Answer (a) Let us take the y-axis in the vertically upward direction with zero at the ground, as shown in Fig. 3.13.

Now v_o = + 20 m s^{-1},

$a = -g = -10$ m s^{-2},

$v = 0$ m s^{-1}

If the ball rises to height y from the point of launch, then using the equation

$$v^2 = v_0^2 + 2\ a\ (y - y_0)$$

we get

$$0 = (20)^2 + 2(-10)(y - y_0)$$

Solving, we get, $(y - y_0)$ = 20 m.

(b) We can solve this part of the problem in two ways. **Note carefully the methods used**.

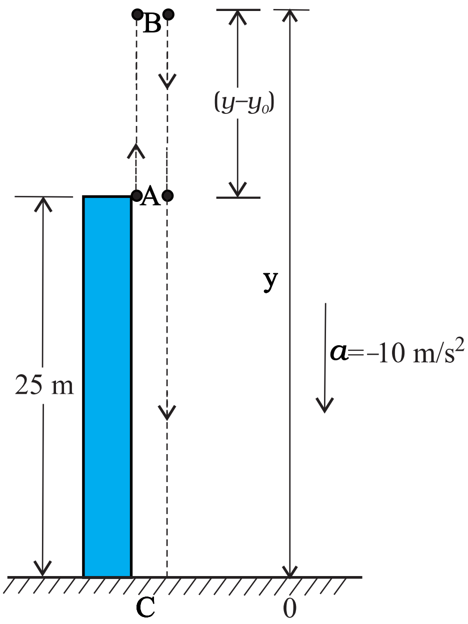


Fig. 3.13

FIRST METHOD : In the first method, we split the path in two parts : the upward motion (A to B) and the downward motion (B to C) and calculate the corresponding time taken t_1 and t_2. Since the velocity at B is zero, we have :

$$v = v_o + at$$
$$0 = 20 - 10t_1$$
Or, $$t_1 = 2 \text{ s}$$

This is the time in going from *A* to *B*. From *B*, or the point of the maximum height, the ball falls freely under the acceleration due to gravity. The ball is moving in negative *y* direction. We use equation

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

We have, $y_0 = 45$ m, $y = 0$, $v_0 = 0$, $a = -g = -10$ m s^{-2}

$$0 = 45 + (½)(-10)\, t_2^2$$

Solving, we get $t_2 = 3$ s

Therefore, the total time taken by the ball before it hits the ground $= t_1 + t_2 = 2$ s + 3 s = 5 s.

SECOND METHOD : The total time taken can also be calculated by noting the coordinates of initial and final positions of the ball with respect to the origin chosen and using equation

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

Now $y_0 = 25$ m $\quad y = 0$ m
$v_o = 20$ m s^{-1}, $\quad a = -10$m s^{-2}, $\quad t = ?$

$$0 = 25 + 20\,t + (½)(-10)\,t^2$$

Or, $$5t^2 - 20t - 25 = 0$$

Solving this quadratic equation for *t*, we get

$$t = 5\text{s}$$

Note that the second method is better since we do not have to worry about the path of the motion as the motion is under constant acceleration.

Example 3.5 Free-fall : Discuss the motion of an object under free fall. Neglect air resistance.

Answer An object released near the surface of the Earth is accelerated downward under the influence of the force of gravity. The magnitude of acceleration due to gravity is represented by ***g***. If air resistance is neglected, the object is said to be in **free fall**. If the height through which the object falls is small compared to the earth's radius, *g* can be taken to be constant, equal to 9.8 m s^{-2}. Free fall is thus a case of motion with uniform acceleration.

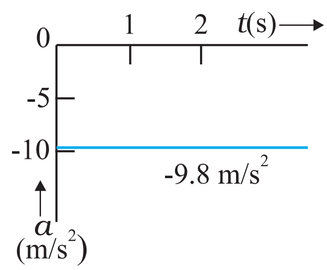
We assume that the motion is in *y*-direction, more correctly in –*y*-direction because we choose upward direction as positive. Since the acceleration due to gravity is always downward, it is in the negative direction and we have

$$a = -g = -9.8 \text{ m s}^{-2}$$

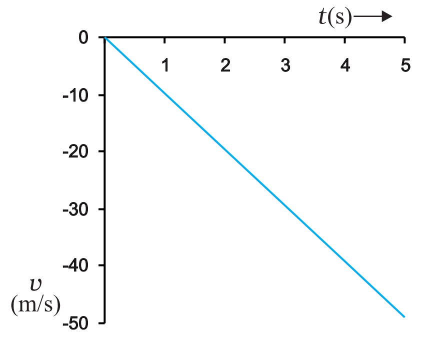
The object is released from rest at $y = 0$. Therefore, $v_0 = 0$ and the equations of motion become:

$$v = 0 - g\,t \qquad = -9.8\,t \quad \text{m s}^{-1}$$
$$y = 0 - ½\,g\,t^2 \qquad = -4.9\,t^2 \quad \text{m}$$
$$v^2 = 0 - 2\,g\,y \qquad = -19.6\,y \quad \text{m}^2\text{ s}^{-2}$$

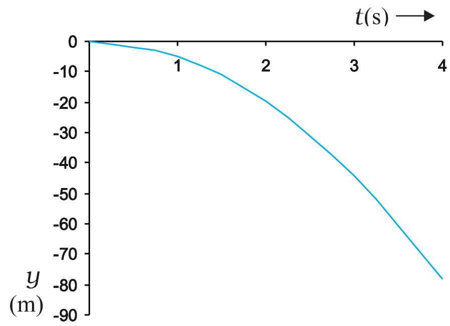
These equations give the velocity and the distance travelled as a function of time and also the variation of velocity with distance. The variation of acceleration, velocity, and distance, with time have been plotted in Fig. 3.14(a), (b) and (c).



(a)



(b)



(c)

Fig. 3.14 *Motion of an object under free fall.*
(a) Variation of acceleration with time.
(b) Variation of velocity with time.
(c) Variation of distance with time

▶ ***Example 3.6*** **Galileo's law of odd numbers** : *"The distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity [namely, 1: 3: 5: 7......]." Prove it.*

Answer Let us divide the time interval of motion of an object under free fall into many equal intervals τ and find out the distances traversed during successive intervals of time. Since initial velocity is zero, we have

$$y = -\frac{1}{2} g t^2$$

Using this equation, we can calculate the position of the object after different time intervals, 0, τ, 2τ, 3τ... which are given in second column of Table 3.2. If we take $(-1/2)\, g\tau^2$ as y_0 — the position coordinate after first time interval τ, then third column gives the positions in the unit of y_o. The fourth column gives the distances traversed in successive τs. We find that the distances are in the simple ratio 1: 3: 5: 7: 9: 11... as shown in the last column. This law was established by Galileo Galilei (1564-1642) who was the first to make quantitative studies of free fall. ◀

▶ ***Example 3.7*** **Stopping distance of vehicles :** When brakes are applied to a moving vehicle, the distance it travels before stopping is called stopping distance. It is an important factor for road safety and depends on the initial velocity (υ_0) and the braking capacity, or deceleration, $-a$ that is caused by the braking. Derive an expression for stopping distance of a vehicle in terms of υ_o and a.

Answer Let the distance travelled by the vehicle before it stops be d_s. Then, using equation of motion $\upsilon^2 = \upsilon_o^{\,2} + 2\,ax$, and noting that $\upsilon = 0$, we have the stopping distance

Thus, the stopping distance is proportional to the square of the initial velocity. Doubling the

Table 3.2

t	y	y in terms of y_o [=(- ½) g τ^2]	Distance traversed in successive intervals	Ratio of distances traversed
0	0	0		
τ	$-(1/2)\, g\, \tau^2$	y_o	y_o	1
$2\,\tau$	$-4(1/2)\, g\, \tau^2$	$4\, y_o$	$3\, y_o$	3
$3\,\tau$	$-9(1/2)\, g\, \tau^2$	$9\, y_o$	$5\, y_o$	5
$4\,\tau$	$-16(1/2)\, g\, \tau^2$	$16\, y_o$	$7\, y_o$	7
$5\,\tau$	$-25(1/2)\, g\, \tau^2$	$25\, y_o$	$9\, y_o$	9
$6\,\tau$	$-36(1/2)\, g\, \tau^2$	$36\, y_o$	$11\, y_o$	11

initial velocity increases the stopping distance by a factor of 4 (for the same deceleration).

For the car of a particular make, the braking distance was found to be 10 m, 20 m, 34 m and 50 m corresponding to velocities of 11, 15, 20 and 25 m/s which are nearly consistent with the above formula.

Stopping distance is an important factor considered in setting speed limits, for example, in school zones. ◀

▶ ***Example 3.8*** **Reaction time :** When a situation demands our immediate action, it takes some time before we really respond. Reaction time is the time a person takes to observe, think and act. For example, if a person is driving and suddenly a boy appears on the road, then the time elapsed before he slams the brakes of the car is the reaction time. Reaction time depends on complexity of the situation and on an individual.

You can measure your reaction time by a simple experiment. Take a ruler and ask your friend to drop it vertically through the gap between your thumb and forefinger (Fig. 3.15). After you catch it, find the distance d travelled by the ruler. In a particular case, d was found to be 21.0 cm. Estimate reaction time.

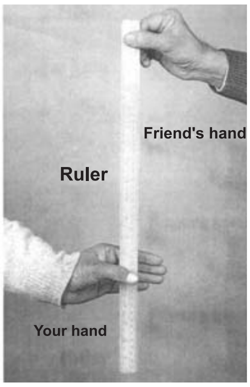


Fig. 3.15 *Measuring the reaction time.*

Answer The ruler drops under free fall. Therefore, $v_o = 0$, and $g = -9.8$ m s^{-2}. The distance travelled d and the reaction time t_r are related by

$$d = -\frac{1}{2} g t_r^2$$

Or, $$t_r = \sqrt{\frac{2d}{g}} \text{ s}$$

Given $d = 21.0$ cm and $g = 9.8$ m s^{-2} the reaction time is

$$t_r = \sqrt{\frac{2 \times 0.21}{9.8}} \text{ s} \cong 0.2 \text{ s.}$$ ◀

3.7 RELATIVE VELOCITY

You must be familiar with the experience of travelling in a train and being overtaken by another train moving in the same direction as you are. While that train must be travelling faster than you to be able to pass you, it does seem slower to you than it would be to someone standing on the ground and watching both the trains. In case both the trains have the same velocity with respect to the ground, then to you the other train would seem to be not moving at all. To understand such observations, we now introduce the concept of relative velocity.

Consider two objects A and B moving uniformly with average velocities v_A and v_B in one dimension, say along x-axis. (Unless otherwise specified, the velocities mentioned in this chapter are measured with reference to the ground). If $x_A(0)$ and $x_B(0)$ are positions of objects A and B, respectively at time $t = 0$, their positions $x_A(t)$ and $x_B(t)$ at time t are given by:

$$x_A(t) = x_A(0) + v_A t \qquad (3.12a)$$
$$x_B(t) = x_B(0) + v_B t \qquad (3.12b)$$

Then, the displacement from object A to object B is given by

$$x_{BA}(t) = x_B(t) - x_A(t)$$
$$= [\, x_B(0) - x_A(0)\,] + (v_B - v_A)\, t. \qquad (3.13)$$

Equation (3.13) is easily interpreted. It tells us that as seen from object A, object B has a velocity $v_B - v_A$ because the displacement from A to B changes steadily by the amount $v_B - v_A$ in each unit of time. We say that the velocity of object B *relative to object* A is $v_B - v_A$:

$$v_{BA} = v_B - v_A \qquad (3.14a)$$

Similarly, velocity of object A *relative to object* B is:

$$v_{AB} = v_A - v_B \qquad (3.14b)$$

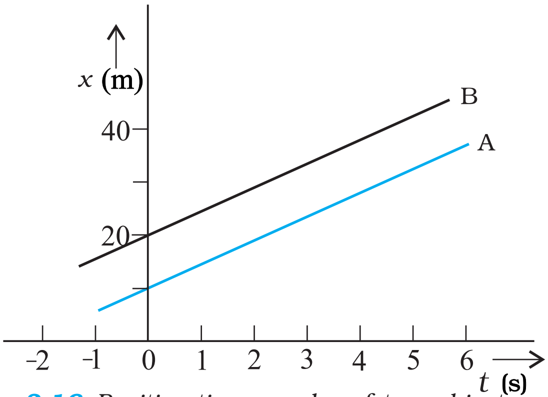


Fig. 3.16 *Position-time graphs of two objects with equal velocities.*

This shows: $\upsilon_{BA} = -\ \upsilon_{AB}$ (3.14c)

Now we consider some special cases :

(a) If $\upsilon_B = \upsilon_A$, $\upsilon_B - \upsilon_A = 0$. Then, from Eq. (3.13), $x_B(t) - x_A(t) = x_B(0) - x_A(0)$. Therefore, the two objects stay at a constant distance ($x_B(0) - x_A(0)$) apart, and their position–time graphs are straight lines parallel to each other as shown in Fig. 3.16. The relative velocity υ_{AB} or υ_{BA} is zero in this case.

(b) If $\upsilon_A > \upsilon_B$, $\upsilon_B - \upsilon_A$ is negative. One graph is steeper than the other and they meet at a common point. For example, suppose $\upsilon_A = 20$ m s^{-1} and $x_A(0) = 10$ m; and $\upsilon_B = 10$ m s^{-1}, $x_B(0) = 40$ m; then the time at which they meet is $t = 3$ s (Fig. 3.17). At this instant they are both at a position $x_A(t) = x_B(t) = 70$ m. Thus, object *A* overtakes object *B* at this time. In this case, υ_{BA} = 10 m s^{-1} – 20 m s^{-1} = – 10 m s^{-1} = $-\ \upsilon_{AB}$.

(c) Suppose υ_A and υ_B are of opposite signs. For example, if in the above example object *A* is moving with 20 m s^{-1} starting at $x_A(0) = 10$ m and object *B* is moving with – 10 m s^{-1} starting at $x_B(0) = 40$ m, the two objects meet at $t = 1$ s (Fig. 3.18). The velocity of *B* relative to *A*, υ_{BA} = [–10 – (20)] m s^{-1} = –30 m s^{-1} = $-\ \upsilon_{AB}$. In this case, the magnitude of υ_{BA} or υ_{AB} (= 30 m s^{-1}) is greater than the magnitude of velocity of *A* or that of *B*. If the objects under consideration are two trains, then for a person sitting on either of the two, the other train seems to go very fast.

Note that Eq. (3.14) are valid even if υ_A and υ_B represent instantaneous velocities.

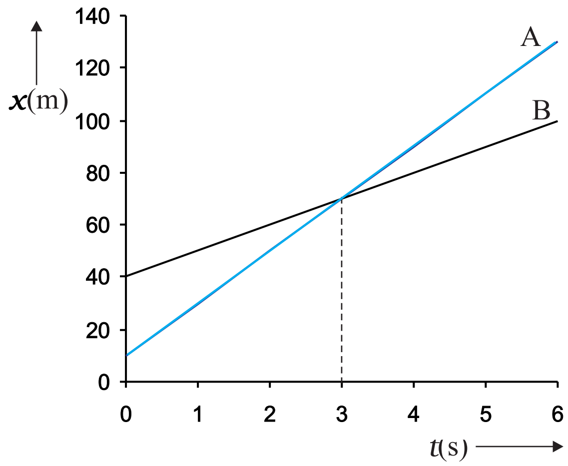


Fig. 3.17 *Position-time graphs of two objects with unequal velocities, showing the time of meeting.*

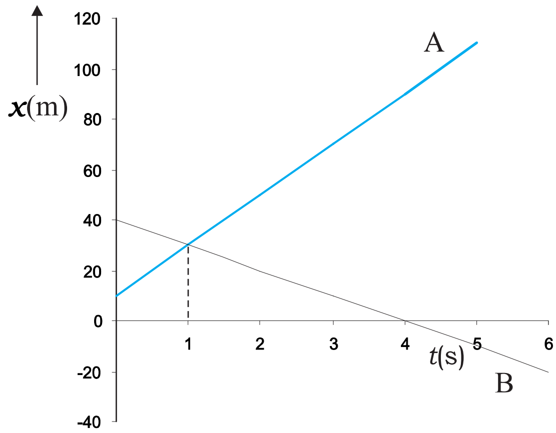


Fig. 3.18 *Position-time graphs of two objects with velocities in opposite directions, showing the time of meeting.*

▶ ***Example 3.9*** *Two parallel rail tracks run north-south. Train A moves north with a speed of 54 km h^{-1}, and train B moves south with a speed of 90 km h^{-1}. What is the*
(a) *velocity of B with respect to A ?,*
(b) *velocity of ground with respect to B ?, and*
(c) *velocity of a monkey running on the roof of the train A against its motion (with a velocity of 18 km h^{-1} with respect to the train A) as observed by a man standing on the ground ?*

Answer Choose the positive direction of *x*-axis to be from south to north. Then,

v_A = + 54 km h^{-1} = 15 m s^{-1}
v_B = – 90 km h^{-1} = – 25 m s^{-1}

Relative velocity of *B* with respect to *A* = $v_B - v_A$= – 40 m s^{-1} , i.e. the train *B* appears to *A* to move with a speed of 40 m s^{-1} from north to south.

Relative velocity of ground with respect to *B* = 0 – v_B = 25 m s^{-1}.

In (c), let the velocity of the monkey with respect to ground be v_M. Relative velocity of the monkey with respect to *A*,

$v_{MA} = v_M - v_A$ = –18 km h^{-1} =–5 m s^{-1}. Therefore, v_M = (15 – 5) m s^{-1} = 10 m s^{-1}.

◀

SUMMARY

1. An object is said to be in *motion* if its position changes with time. The position of the object can be specified with reference to a conveniently chosen origin. For motion in a straight line, position to the right of the origin is taken as positive and to the left as negative.
2. *Path length* is defined as the total length of the path traversed by an object.
3. *Displacement* is the change in position : $\Delta x = x_2 - x_1$. Path length is greater or equal to the magnitude of the displacement between the same points.
4. An object is said to be in *uniform motion* in a straight line if its displacement is equal in equal intervals of time. Otherwise, the motion is said to be *non-uniform.*
5. *Average velocity* is the displacement divided by the time interval in which the displacement occurs :

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

On an *x-t* graph, the average velocity over a time interval is the slope of the line connecting the initial and final positions corresponding to that interval.

6. *Average Speed* is the ratio of total path length traversed and the corresponding time interval.

The average speed of an object is greater or equal to the magnitude of the average velocity over a given time interval.

7. *Instantaneous velocity* or simply velocity is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally small :

$$v = \lim_{\Delta t \to 0} \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

The velocity at a particular instant is equal to the slope of the tangent drawn on position-time graph at that instant.

8. *Average acceleration* is the change in velocity divided by the time interval during which the change occurs :

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

9. *Instantaneous acceleration* is defined as the limit of the average acceleration as the time interval Δt goes to zero :

$$a = \lim_{\Delta t \to 0} \bar{a} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

The acceleration of an object at a particular time is the slope of the velocity-time graph at that instant of time. For uniform motion, acceleration is zero and the *x-t* graph is a straight line inclined to the time axis and the *v-t* graph is a straight line

parallel to the time axis. For motion with uniform acceleration, *x-t* graph is a parabola while the *v-t* graph is a straight line inclined to the time axis.

10. The area under the velocity-time curve between times t_1 and t_2 is equal to the displacement of the object during that interval of time.
11. For objects in uniformly accelerated rectilinear motion, the five quantities, displacement *x*, time taken *t*, initial velocity v_0, final velocity *v* and acceleration *a* are related by a set of simple equations called *kinematic equations of motion* :

$$v = v_0 + at$$

$$x = v_0 t + \frac{1}{2} a t^2$$

$$v^2 = v_0^2 + 2ax$$

if the position of the object at time $t = 0$ is 0. If the particle starts at $x = x_0$, *x* in above equations is replaced by $(x - x_0)$.

Physical quantity	Symbol	Dimensions	Unit	Remarks
Path length		[L]	m	
Displacement	Δx	[L]	m	$= x_2 - x_1$ In one dimension, its sign indicates the direction.
Velocity		$[LT^{-1}]$	m s^{-1}	
(a) Average	$\bar{v}$			$= \frac{\Delta x}{\Delta t}$
(b) Instantaneous	v			$= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$ In one dimension, its sign indicates the direction.
Speed		$[LT^{-1}]$	m s^{-1}	
(a) Average				$= \frac{\text{Path length}}{\text{Time interval}}$
(b) Instantaneous				$= \frac{dx}{dt}$
Acceleration		$[LT^{-2}]$	m s^{-2}	
(a) Average	$\bar{a}$			$= \frac{\Delta v}{\Delta t}$
(b) Instantaneous	a			$= \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$ In one dimension, its sign indicates the direction.

POINTS TO PONDER

1. The path length traversed by an object between two points is, in general, not the same as the magnitude of displacement. The displacement depends only on the end points; the path length (as the name implies) depends on the actual path. In one dimension, the two quantities are equal only if the object does not change its direction during the course of motion. In all other cases, the path length is greater than the magnitude of displacement.
2. In view of point 1 above, the average speed of an object is greater than or equal to the magnitude of the average velocity over a given time interval. The two are equal only if the path length is equal to the magnitude of displacement.
3. The origin and the positive direction of an axis are a matter of choice. You should first specify this choice before you assign signs to quantities like displacement, velocity and acceleration.
4. If a particle is speeding up, acceleration is in the direction of velocity; if its speed is decreasing, acceleration is in the direction opposite to that of the velocity. This statement is independent of the choice of the origin and the axis.
5. The sign of acceleration does not tell us whether the particle's speed is increasing or decreasing. The sign of acceleration (as mentioned in point 3) depends on the choice of the positive direction of the axis. For example, if the vertically upward direction is chosen to be the positive direction of the axis, the acceleration due to gravity is negative. If a particle is falling under gravity, this acceleration, though negative, results in increase in speed. For a particle thrown upward, the same negative acceleration (of gravity) results in decrease in speed.
6. The zero velocity of a particle at any instant does not necessarily imply zero acceleration at that instant. A particle may be momentarily at rest and yet have non-zero acceleration. For example, a particle thrown up has zero velocity at its uppermost point but the acceleration at that instant continues to be the acceleration due to gravity.
7. In the kinematic equations of motion [Eq. (3.11)], the various quantities are algebraic, i.e. they may be positive or negative. The equations are applicable in all situations (for one dimensional motion with constant acceleration) provided the values of different quantities are substituted in the equations with proper signs.
8. The definitions of instantaneous velocity and acceleration (Eqs. (3.3) and (3.5)) are exact and are always correct while the kinematic equations (Eq. (3.11)) are true only for motion in which the magnitude and the direction of acceleration are constant during the course of motion.

EXERCISES

3.1 In which of the following examples of motion, can the body be considered approximately a point object:
(a) a railway carriage moving without jerks between two stations.
(b) a monkey sitting on top of a man cycling smoothly on a circular track.
(c) a spinning cricket ball that turns sharply on hitting the ground.
(d) a tumbling beaker that has slipped off the edge of a table.

3.2 The position-time (x-t) graphs for two children A and B returning from their school O to their homes P and Q respectively are shown in Fig. 3.19. Choose the correct entries in the brackets below ;
(a) (A/B) lives closer to the school than (B/A)
(b) (A/B) starts from the school earlier than (B/A)
(c) (A/B) walks faster than (B/A)
(d) A and B reach home at the (same/different) time
(e) (A/B) overtakes (B/A) on the road (once/twice).

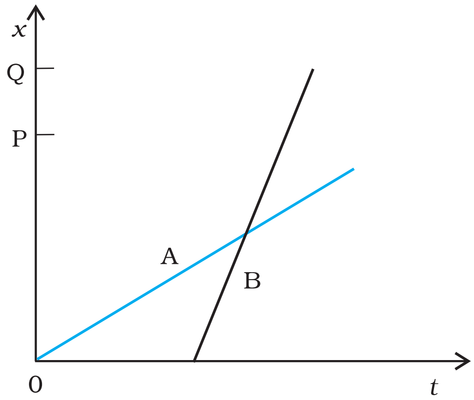


Fig. 3.19

3.3 A woman starts from her home at 9.00 am, walks with a speed of 5 km h^{-1} on a straight road up to her office 2.5 km away, stays at the office up to 5.00 pm, and returns home by an auto with a speed of 25 km h^{-1}. Choose suitable scales and plot the *x-t* graph of her motion.

3.4 A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again by 5 steps forward and 3 steps backward, and so on. Each step is 1 m long and requires 1 s. Plot the *x-t* graph of his motion. Determine graphically and otherwise how long the drunkard takes to fall in a pit 13 m away from the start.

3.5 A jet airplane travelling at the speed of 500 km h^{-1} ejects its products of combustion at the speed of 1500 km h^{-1} relative to the jet plane. What is the speed of the latter with respect to an observer on the ground ?

3.6 A car moving along a straight highway with speed of 126 km h^{-1} is brought to a stop within a distance of 200 m. What is the retardation of the car (assumed uniform), and how long does it take for the car to stop ?

3.7 Two trains A and B of length 400 m each are moving on two parallel tracks with a uniform speed of 72 km h^{-1} in the same direction, with A ahead of B. The driver of B decides to overtake A and accelerates by 1 m s^{-2}. If after 50 s, the guard of B just brushes past the driver of A, what was the original distance between them ?

3.8 On a two-lane road, car A is travelling with a speed of 36 km h^{-1}. Two cars B and C approach car A in opposite directions with a speed of 54 km h^{-1} each. At a certain instant, when the distance AB is equal to AC, both being 1 km, B decides to overtake A before C does. What minimum acceleration of car B is required to avoid an accident ?

3.9 Two towns A and B are connected by a regular bus service with a bus leaving in either direction every *T* minutes. A man cycling with a speed of 20 km h^{-1} in the direction A to B notices that a bus goes past him every 18 min in the direction of his motion, and every 6 min in the opposite direction. What is the period *T* of the bus service and with what speed (assumed constant) do the buses ply on the road?

3.10 A player throws a ball upwards with an initial speed of 29.4 m s^{-1}.

(a) What is the direction of acceleration during the upward motion of the ball ?

(b) What are the velocity and acceleration of the ball at the highest point of its motion ?

(c) Choose the x = 0 m and t = 0 s to be the location and time of the ball at its highest point, vertically downward direction to be the positive direction of *x*-axis, and give the signs of position, velocity and acceleration of the ball during its upward, and downward motion.

(d) To what height does the ball rise and after how long does the ball return to the player's hands ? (Take g = 9.8 m s^{-2} and neglect air resistance).

3.11 Read each statement below carefully and state with reasons and examples, if it is true or false ;

A particle in one-dimensional motion

(a) with zero speed at an instant may have non-zero acceleration at that instant

(b) with zero speed may have non-zero velocity,

(c) with constant speed must have zero acceleration,

(d) with positive value of acceleration *must* be speeding up.

3.12 A ball is dropped from a height of 90 m on a floor. At each collision with the floor, the ball loses one tenth of its speed. Plot the speed-time graph of its motion between $t = 0$ to 12 s.

3.13 Explain clearly, with examples, the distinction between :

(a) magnitude of displacement (sometimes called distance) over an interval of time, and the total length of path covered by a particle over the same interval;

(b) magnitude of average velocity over an interval of time, and the average speed over the same interval. [Average speed of a particle over an interval of time is defined as the total path length divided by the time interval]. Show in both (a) and (b) that the second quantity is either greater than or equal to the first. When is the equality sign true ? [For simplicity, consider one-dimensional motion only].

3.14 A man walks on a straight road from his home to a market 2.5 km away with a speed of 5 km h^{-1}. Finding the market closed, he instantly turns and walks back home with a speed of 7.5 km h^{-1}. What is the

(a) magnitude of average velocity, and

(b) average speed of the man over the interval of time (i) 0 to 30 min, (ii) 0 to 50 min, (iii) 0 to 40 min ? [Note: You will appreciate from this exercise why it is better to define average speed as total path length divided by time, and not as magnitude of average velocity. You would not like to tell the tired man on his return home that his average speed was zero !]

3.15 In Exercises 3.13 and 3.14, we have carefully distinguished between *average* speed and magnitude of *average* velocity. No such distinction is necessary when we consider instantaneous speed and magnitude of velocity. The instantaneous speed is always equal to the magnitude of instantaneous velocity. Why ?

3.16 Look at the graphs (a) to (d) (Fig. 3.20) carefully and state, with reasons, which of these *cannot* possibly represent one-dimensional motion of a particle.

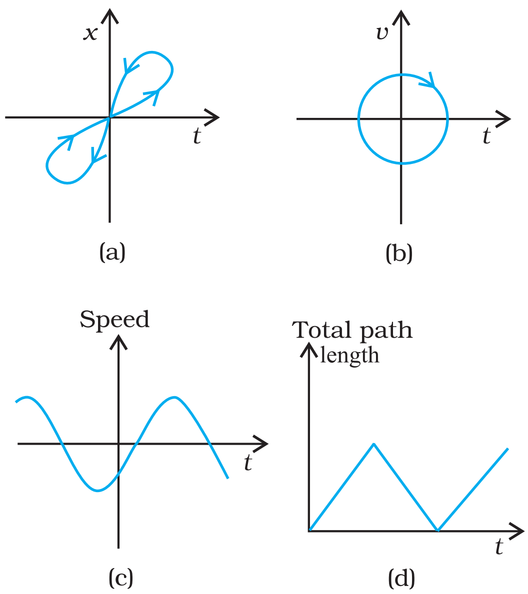


Fig. 3.20

3.17 Figure 3.21 shows the *x-t* plot of one-dimensional motion of a particle. Is it correct to say from the graph that the particle moves in a straight line for $t < 0$ and on a parabolic path for $t > 0$? If not, suggest a suitable physical context for this graph.

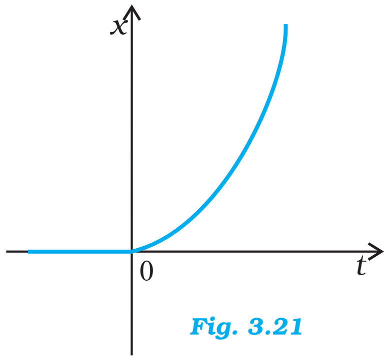


Fig. 3.21

3.18 A police van moving on a highway with a speed of 30 km h^{-1} fires a bullet at a thief's car speeding away in the same direction with a speed of 192 km h^{-1}. If the muzzle speed of the bullet is 150 m s^{-1}, with what speed does the bullet hit the thief's car ? (Note: Obtain that speed which is relevant for damaging the thief's car).

3.19 Suggest a suitable physical situation for each of the following graphs (Fig 3.22):

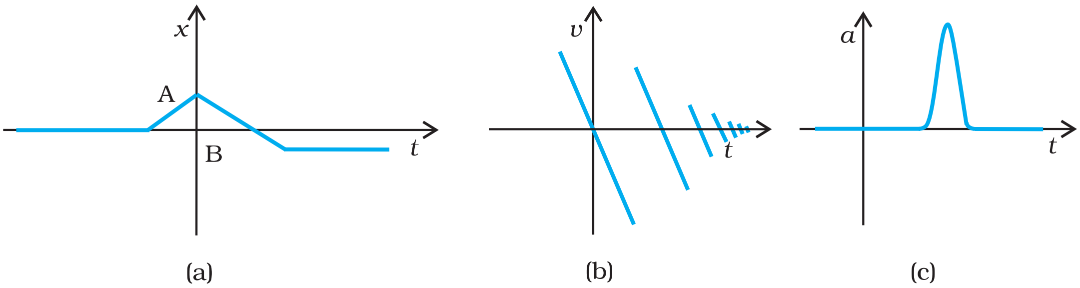


Fig. 3.22

3.20 Figure 3.23 gives the *x-t* plot of a particle executing one-dimensional simple harmonic motion. (You will learn about this motion in more detail in Chapter14). Give the signs of position, velocity and acceleration variables of the particle at $t = 0.3$ s, 1.2 s, – 1.2 s.

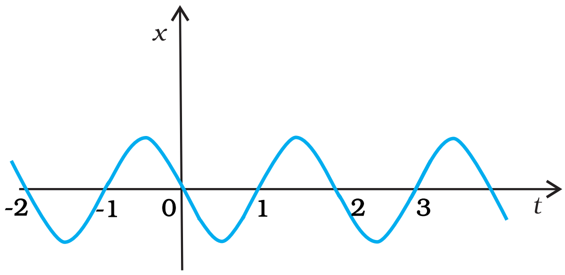


Fig. 3.23

3.21 Figure 3.24 gives the *x-t* plot of a particle in one-dimensional motion. Three different equal intervals of time are shown. In which interval is the average speed greatest, and in which is it the least ? Give the sign of average velocity for each interval.

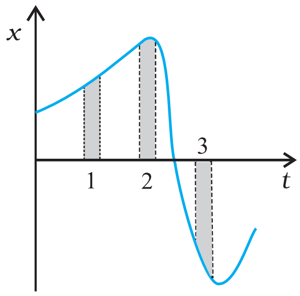


Fig. 3.24

3.22 Figure 3.25 gives a speed-time graph of a particle in motion along a constant direction. Three equal intervals of time are shown. In which interval is the average acceleration greatest in magnitude ? In which interval is the average speed greatest ? Choosing the positive direction as the constant direction of motion, give the signs of v and a in the three intervals. What are the accelerations at the points A, B, C and D ?

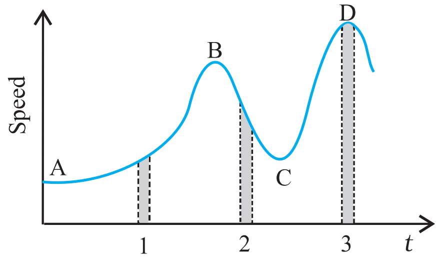


Fig. 3.25

Additional Exercises

3.23 A three-wheeler starts from rest, accelerates uniformly with 1 m s^{-2} on a straight road for 10 s, and then moves with uniform velocity. Plot the distance covered by the vehicle during the n^{th} second (n = 1,2,3….) versus n. What do you expect this plot to be during accelerated motion : a straight line or a parabola ?

3.24 A boy standing on a stationary lift (open from above) throws a ball upwards with the maximum initial speed he can, equal to 49 m s^{-1}. How much time does the ball take to return to his hands? If the lift starts moving up with a uniform speed of 5 m s^{-1} and the boy again throws the ball up with the maximum speed he can, how long does the ball take to return to his hands ?

3.25 On a long horizontally moving belt (Fig. 3.26), a child runs to and fro with a speed 9 km h^{-1} (with respect to the belt) between his father and mother located 50 m apart on the moving belt. The belt moves with a speed of 4 km h^{-1}. For an observer on a stationary platform outside, what is the
(a) speed of the child running in the direction of motion of the belt ?.
(b) speed of the child running opposite to the direction of motion of the belt ?
(c) time taken by the child in (a) and (b) ?
Which of the answers alter if motion is viewed by one of the parents ?

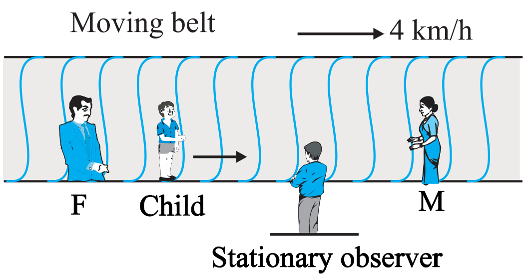


Fig. 3.26

3.26 Two stones are thrown up simultaneously from the edge of a cliff 200 m high with initial speeds of 15 m s^{-1} and 30 m s^{-1}. Verify that the graph shown in Fig. 3.27 correctly represents the time variation of the relative position of the second stone with respect to the first. Neglect air resistance and assume that the stones do not rebound after hitting the ground. Take g = 10 m s^{-2}. Give the equations for the linear and curved parts of the plot.

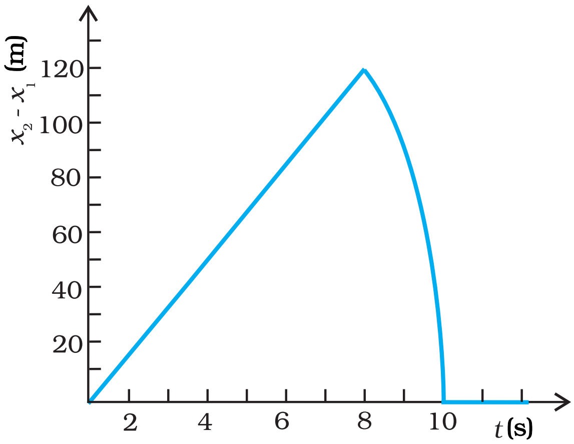


Fig. 3.27

3.27 The speed-time graph of a particle moving along a fixed direction is shown in Fig. 3.28. Obtain the distance traversed by the particle between (a) $t = 0$ s to 10 s, (b) $t = 2$ s to 6 s.

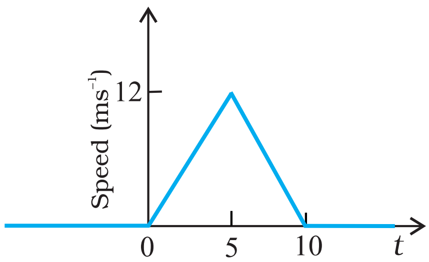


Fig. 3.28

What is the average speed of the particle over the intervals in (a) and (b) ?

3.28 The velocity-time graph of a particle in one-dimensional motion is shown in Fig. 3.29 :

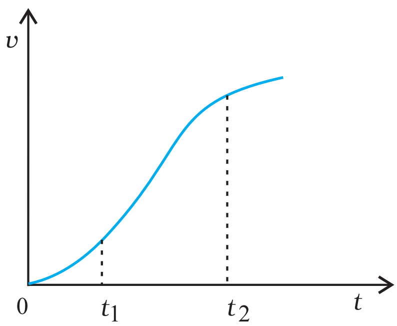


Fig. 3.29

Which of the following formulae are correct for describing the motion of the particle over the time-interval t_1 to t_2:

(a) $x(t_2) = x(t_1) + v(t_1)(t_2 - t_1) + (½)\, a\, (t_2 - t_1)^2$
(b) $v(t_2) = v(t_1) + a\,(t_2 - t_1)$
(c) $v_{average} = (x(t_2) - x(t_1))/(t_2 - t_1)$
(d) $a_{average} = (v(t_2) - v(t_1))/(t_2 - t_1)$
(e) $x(t_2) = x(t_1) + v_{average}(t_2 - t_1) + (½)\, a_{average}\, (t_2 - t_1)^2$
(f) $x(t_2) - x(t_1)$ = area under the v-t curve bounded by the t-axis and the dotted line shown.

APPENDIX 3.1 : ELEMENTS OF CALCULUS

Differential Calculus

Using the concept of 'differential coefficient' or 'derivative', we can easily define velocity and acceleration. Though you will learn in detail in mathematics about derivatives, we shall introduce this concept in brief in this Appendix so as to facilitate its use in describing physical quantities involved in motion.

Suppose we have a quantity y whose value depends upon a single variable x, and is expressed by an equation defining y as some specific function of x. This is represented as:

$$y = f(x) \tag{1}$$

This relationship can be visualised by drawing a graph of function $y = f(x)$ regarding y and x as Cartesian coordinates, as shown in Fig. 3.30 (a).

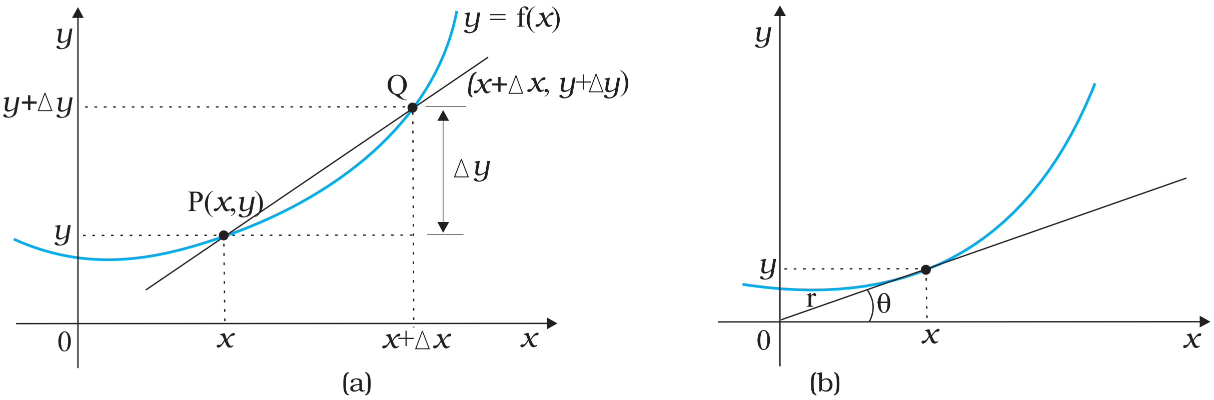


Fig. 3.30

Consider the point P on the curve $y = f(x)$ whose coordinates are (x, y) and another point Q where coordinates are $(x + \Delta x, y + \Delta y)$. The slope of the line joining P and Q is given by:

$$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{(y + \Delta y) - y}{\Delta x} \tag{2}$$

Suppose now that the point Q moves along the curve towards P. In this process, Δy *and* Δx decrease and approach zero; though their ratio $\frac{\Delta y}{\Delta x}$ will not necessarily vanish. What happens to the line PQ as $\Delta y \to 0$, $\Delta x \to 0$. You can see that this line becomes a tangent to the curve at point P as shown in Fig. 3.30(b). This means that tan θ approaches the slope of the tangent at P, denoted by m:

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(y + \Delta y) - y}{\Delta x} \tag{3}$$

The limit of the ratio $\Delta y/\Delta x$ as Δx approaches zero is called the derivative of y with respect to x and is written as dy/dx. It represents the slope of the tangent line to the curve $y = f(x)$ at the point (x, y).

Since $y = f(x)$ and $y + \Delta y = f(x + \Delta x)$, we can write the definition of the derivative as:

$$\frac{dy}{dx} = \frac{d\boldsymbol{f}(\boldsymbol{x})}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

Given below are some elementary formulae for derivatives of functions. In these $u(x)$ and $v(x)$ represent arbitrary functions of x, and a and b denote constant quantities that are independent of x. Derivatives of some common functions are also listed .

$$\frac{d(a\,u)}{dx} = a\frac{du}{dx} \quad ; \quad \frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt}$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad ; \quad \frac{d(u/v)}{dx} = \frac{1}{v^2}\frac{du}{dx} - u\frac{dv}{dx}$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$\frac{d}{dx}(\sin x) = \cos x \quad ; \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad ; \quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \tan x \ \sec x \quad ; \quad \frac{d}{dx}(\operatorname{cosec}^2 x) = -\cot x \ \operatorname{cosec} x$$

$$\frac{d}{dx}(u)^n = n\,u^{n-1}\frac{du}{dx} \quad ; \quad \frac{d}{du}(\ln u) = \frac{1}{u}$$

$$\frac{d}{du}(e^u) = e^u$$

In terms of derivatives, instantaneous velocity and acceleration are defined as

$$v = \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$a = \lim_{\Delta t \to 0}\frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Integral Calculus

You are familiar with the notion of area. The formulae for areas of simple geometrical figures are also known to you. For example, the area of a rectangle is length times breadth and that of a triangle is half of the product of base and height. But how to deal with the problem of determination of area of an irregular figure? The mathematical notion of integral is necessary in connection with such problems.

Let us take a concrete example. Suppose a variable force $f(x)$ acts on a particle in its motion along x - axis from $x = a$ *to* $x = b$. The problem is to determine the work done (W) by the force on the particle during the motion. This problem is discussed in detail in Chapter 6.

Figure 3.31 shows the variation of $F(x)$ with x. If the force were constant, work would be simply the area $F(b\text{-}a)$ as shown in Fig. 3.31(i). But in the general case, force is varying .

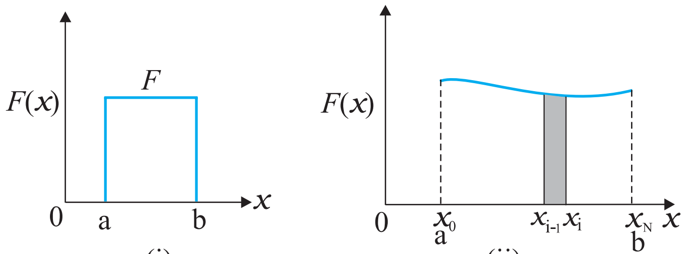


Fig. 3.31

To calculate the area under this curve [Fig. 3.31 (ii)], let us employ the following trick. Divide the interval on *x-axis* from *a to b* into a large number (*N*) of small intervals: $x_0(=a)$ *to* x_1, x_1 *to* x_2 ; x_2 *to* x_3, ... x_{N-1} *to* x_N $(=b)$. The area under the curve is thus divided into *N* strips. Each strip is approximately a rectangle, since the variation of $F(x)$ over a strip is negligible. The area of the i^{th} strip shown [Fig. 3.31(ii)] is then approximately

$$\Delta A_i = F(x_i)(x_i - x_{i-1}) = F(x_i)\Delta x$$

where Δx is the width of the strip which we have taken to be the same for all the strips. You may wonder whether we should put $F(x_{i-1})$ or the mean of $F(x_i)$ and $F(x_{i-1})$ in the above expression. If we take *N* to be very very large ($N \to \infty$), it does not really matter, since then the strip will be so thin that the difference between $F(x_i)$ and $F(x_{i-1})$ is vanishingly small. The total area under the curve then is:

$$A = \sum_{i=1}^{N} \Delta A_i = \sum_{i=1}^{N} F(x_i)\Delta x$$

The limit of this sum as $N \to \infty$ is known as the integral of $F(x)$ over x from a to b. It is given a special symbol as shown below:

$$A = \int_a^b F(x)dx$$

The integral sign $\int$ looks like an elongated S, reminding us that it basically is the limit of the sum of an infinite number of terms.

A most significant mathematical fact is that integration is, in a sense, an inverse of differentiation. Suppose we have a function $g(x)$ whose derivative is $f(x)$, i.e. $f(x) = \dfrac{dg(x)}{dx}$

The function g (x) is known as the indefinite integral of $f(x)$ and is denoted as:

$$g(x) = \int f(x)dx$$

An integral with lower and upper limits is known as a definite integral. It is a number. Indefinite integral has no limits; it is a function.

A fundamental theorem of mathematics states that

$$\int_a^b f(x)\,dx = g(x)\Big|_a^b \equiv g(b) - g(a)$$

As an example, suppose $f(x) = x^2$ and we wish to determine the value of the definite integral from $x = 1$ to $x = 2$. The function $g(x)$ whose derivative is x^2 is $x^3/3$. Therefore,

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Bigg|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Clearly, to evaluate definite integrals, we need to know the corresponding indefinite integrals. Some common indefinite integrals are

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} \qquad (n \neq -1)$$

$$\int \left(\frac{1}{x}\right)\mathrm{d}x = \ln x \qquad (x > 0)$$

$$\int \sin x \ \mathrm{d}x = -\cos x \qquad \int \cos x \ \mathrm{d}x = \sin x$$

$$\int \mathrm{e}^x \mathrm{d}x = \mathrm{e}^x$$

This introduction to differential and integral calculus is not rigorous and is intended to convey to you the basic notions of calculus.

CHAPTER FOUR

MOTION IN A PLANE

4.1 INTRODUCTION

In the last chapter we developed the concepts of position, displacement, velocity and acceleration that are needed to describe the motion of an object along a straight line. We found that the directional aspect of these quantities can be taken care of by + and – signs, as in one dimension only two directions are possible. But in order to describe motion of an object in two dimensions (a plane) or three dimensions (space), we need to use vectors to describe the above-mentioned physical quantities. Therefore, it is first necessary to learn the language of vectors. What is a vector? How to add, subtract and multiply vectors ? What is the result of multiplying a vector by a real number ? We shall learn this to enable us to use vectors for defining velocity and acceleration in a plane. We then discuss motion of an object in a plane. As a simple case of motion in a plane, we shall discuss motion with constant acceleration and treat in detail the projectile motion. Circular motion is a familiar class of motion that has a special significance in daily-life situations. We shall discuss uniform circular motion in some detail.

The equations developed in this chapter for motion in a plane can be easily extended to the case of three dimensions.

4.2 SCALARS AND VECTORS

In physics, we can classify quantities as scalars or vectors. Basically, the difference is that a **direction** is associated with a vector but not with a scalar. A scalar quantity is a quantity with magnitude only. It is specified completely by a single number, along with the proper unit. Examples are : the distance between two points, mass of an object, the temperature of a body and the time at which a certain event happened. The rules for combining scalars are the rules of ordinary algebra. Scalars can be added, subtracted, multiplied and divided

just as the ordinary numbers*. For example, if the length and breadth of a rectangle are 1.0 m and 0.5 m respectively, then its perimeter is the sum of the lengths of the four sides, 1.0 m + 0.5 m +1.0 m + 0.5 m = 3.0 m. The length of each side is a scalar and the perimeter is also a scalar. Take another example: the maximum and minimum temperatures on a particular day are 35.6 °C and 24.2 °C respectively. Then, the difference between the two temperatures is 11.4 °C. Similarly, if a uniform solid cube of aluminium of side 10 cm has a mass of 2.7 kg, then its volume is 10^{-3} m^3 (a scalar) and its density is 2.7×10^3 kg m^{-3} (a scalar).

A **vector** quantity is a quantity that has both a magnitude and a direction and obeys the **triangle law of addition** or equivalently the **parallelogram law of addition**. So, a vector is specified by giving its magnitude by a number and its direction. Some physical quantities that are represented by vectors are displacement, velocity, acceleration and force.

To represent a vector, we use a bold face type in this book. Thus, a velocity vector can be represented by a symbol **v**. Since bold face is difficult to produce, when written by hand, a vector is often represented by an arrow placed over a letter, say $\vec{v}$. Thus, both **v** and $\vec{v}$ represent the velocity vector. The magnitude of a vector is often called its absolute value, indicated by $|\mathbf{v}| = v$. Thus, a vector is represented by a bold face, e.g. by **A, a, p, q, r, ... x, y**, with respective magnitudes denoted by light face *A, a, p, q, r, ... x, y*.

4.2.1 Position and Displacement Vectors

To describe the position of an object moving in a plane, we need to choose a convenient point, say O as origin. Let P and P′ be the positions of the object at time *t* and *t*′, respectively [Fig. 4.1(a)]. We join O and P by a straight line. Then, **OP** is the position vector of the object at time *t*. An arrow is marked at the head of this line. It is represented by a symbol **r**, i.e. **OP** = **r**. Point P′ is represented by another position vector, **OP′** denoted by **r′**. The length of the vector **r** represents the magnitude of the vector and its direction is the direction in which P lies as seen from O. If the object moves from P to P′, the vector **PP′** (with tail at P and tip at P′) is called the **displacement vector** corresponding to motion from point P (at time *t*) to point P′ (at time *t*′).

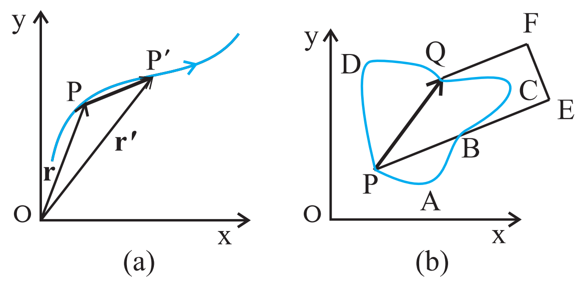


Fig. 4.1 (a) *Position and displacement vectors.* (b) *Displacement vector* **PQ** *and different courses of motion.*

It is important to note that displacement vector is the straight line joining the initial and final positions and does not depend on the actual path undertaken by the object between the two positions. For example, in Fig. 4.1b, given the initial and final positions as P and Q, the displacement vector is the same **PQ** for different paths of journey, say PABCQ, PDQ, and PBEFQ. Therefore, the **magnitude of displacement is either less or equal to the path length of an object between two points**. This fact was emphasised in the previous chapter also while discussing motion along a straight line.

4.2.2 Equality of Vectors

Two vectors **A** and **B** are said to be equal if, and only if, they have the same magnitude and the same direction.**

Figure 4.2(a) shows two equal vectors **A** and **B**. We can easily check their equality. Shift **B** parallel to itself until its tail Q coincides with that of *A*, i.e. Q coincides with O. Then, since their tips S and P also coincide, the two vectors are said to be equal. In general, equality is indicated

* *Addition and subtraction of scalars make sense only for quantities with same units. However, you can multiply and divide scalars of different units.*

** *In our study, vectors do not have fixed locations. So displacing a vector parallel to itself leaves the vector unchanged. Such vectors are called* free vectors. *However, in some physical applications, location or line of application of a vector is important. Such vectors are called* localised vectors.

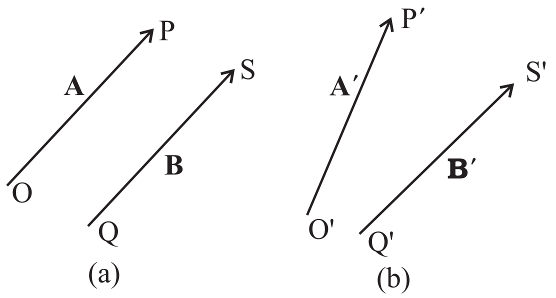


Fig. 4.2 (a) *Two equal vectors* **A** *and* **B**. (b) *Two vectors* **A′** *and* **B′** *are unequal though they are of the same length.*

as **A** = **B**. Note that in Fig. 4.2(b), vectors **A′** and **B′** have the same magnitude but they are not equal because they have different directions. Even if we shift **B′** parallel to itself so that its tail Q′ coincides with the tail O′ of **A′**, the tip S′ of **B′** does not coincide with the tip P′ of **A′**.

4.3 MULTIPLICATION OF VECTORS BY REAL NUMBERS

Multiplying a vector **A** with a positive number λ gives a vector whose magnitude is changed by the factor λ but the direction is the same as that of **A** :

$$|\lambda \mathbf{A}| = \lambda |\mathbf{A}| \quad \text{if} \ \lambda > 0.$$

For example, if **A** is multiplied by 2, the resultant vector 2**A** is in the same direction as **A** and has a magnitude twice of |**A**| as shown in Fig. 4.3(a).

Multiplying a vector **A** by a negative number λ gives a vector λ**A** whose direction is opposite to the direction of **A** and whose magnitude is –λ times |**A**|.

Multiplying a given vector **A** by negative numbers, say –1 and –1.5, gives vectors as shown in Fig 4.3(b).

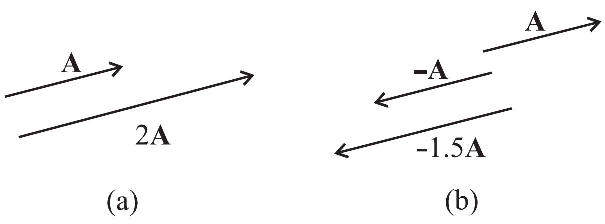


Fig. 4.3 (a) *Vector* **A** *and the resultant vector after multiplying* **A** *by a positive number 2.* (b) *Vector* **A** *and resultant vectors after multiplying it by a negative number –1 and –1.5.*

The factor λ by which a vector **A** is multiplied could be a scalar having its own physical dimension. Then, the dimension of λ **A** is the product of the dimensions of λ and **A**. For example, if we multiply a constant velocity vector by duration (of time), we get a displacement vector.

4.4 ADDITION AND SUBTRACTION OF VECTORS — GRAPHICAL METHOD

As mentioned in section 4.2, vectors, by definition, obey the triangle law or equivalently, the parallelogram law of addition. We shall now describe this law of addition using the graphical method. Let us consider two vectors **A** and **B** that lie in a plane as shown in Fig. 4.4(a). The lengths of the line segments representing these vectors are proportional to the magnitude of the vectors. To find the sum **A** + **B**, we place vector **B** so that its tail is at the head of the vector **A**, as in Fig. 4.4(b). Then, we join the tail of **A** to the head of **B**. This line OQ represents a vector **R,** that is, the sum of the vectors **A** and **B**. Since, in this procedure of vector addition, vectors are

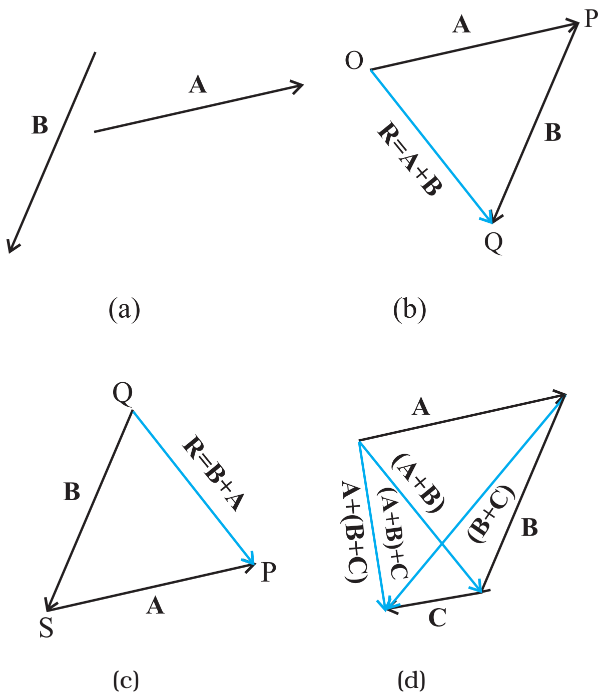


Fig. 4.4 (a) *Vectors* **A** *and* **B**. (b) *Vectors* **A** *and* **B** *added graphically.* (c) *Vectors* **B** *and* **A** *added graphically.* (d) *Illustrating the associative law of vector addition.*

arranged head to tail, this graphical method is called the **head-to-tail method**. The two vectors and their resultant form three sides of a triangle, so this method is also known as **triangle method of vector addition**. If we find the resultant of **B** + **A** as in Fig. 4.4(c), the same vector **R** is obtained. Thus, vector addition is **commutative**:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad (4.1)$$

The addition of vectors also obeys the associative law as illustrated in Fig. 4.4(d). The result of adding vectors **A** and **B** first and then adding vector **C** is the same as the result of adding **B** and **C** first and then adding vector **A** :

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \qquad (4.2)$$

What is the result of adding two equal and opposite vectors ? Consider two vectors **A** and –**A** shown in Fig. 4.3(b). Their sum is **A** + (–**A**). Since the magnitudes of the two vectors are the same, but the directions are opposite, the resultant vector has zero magnitude and is represented by **0** called a **null vector** or a **zero vector** :

$$\mathbf{A} - \mathbf{A} = \mathbf{0} \qquad |\mathbf{0}| = 0 \qquad (4.3)$$

Since the magnitude of a null vector is zero, its direction cannot be specified.

The null vector also results when we multiply a vector **A** by the number zero. The main properties of **0** are :

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$
$$\lambda\, \mathbf{0} = \mathbf{0}$$
$$0\, \mathbf{A} = \mathbf{0} \qquad (4.4)$$

What is the physical meaning of a zero vector? Consider the position and displacement vectors in a plane as shown in Fig. 4.1(a). Now suppose that an object which is at P at time *t*, moves to P′ and then comes back to P. Then, what is its displacement? Since the initial and final positions coincide, the displacement is a "null vector".

Subtraction of vectors can be defined in terms of addition of vectors. We define the difference of two vectors **A** and **B** as the sum of two vectors **A** and –**B** :

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \qquad (4.5)$$

It is shown in Fig 4.5. The vector –**B** is added to vector **A** to get $\mathbf{R}_2 = (\mathbf{A} - \mathbf{B})$. The vector $\mathbf{R}_1 = \mathbf{A} + \mathbf{B}$ is also shown in the same figure for comparison. We can also use the **parallelogram method** to find the sum of two vectors. Suppose we have two vectors **A** and **B**. To add these vectors, we bring their tails to a common origin O as shown in Fig. 4.6(a). Then we draw a line from the head of **A** parallel to **B** and another line from the head of **B** parallel to **A** to complete a parallelogram OQSP. Now we join the point of the intersection of these two lines to the origin O. The resultant vector **R** is directed from the common origin O along the diagonal (OS) of the parallelogram [Fig. 4.6(b)]. In Fig.4.6(c), the triangle law is used to obtain the resultant of **A** and **B** and we see that the two methods yield the same result. Thus, the two methods are equivalent.

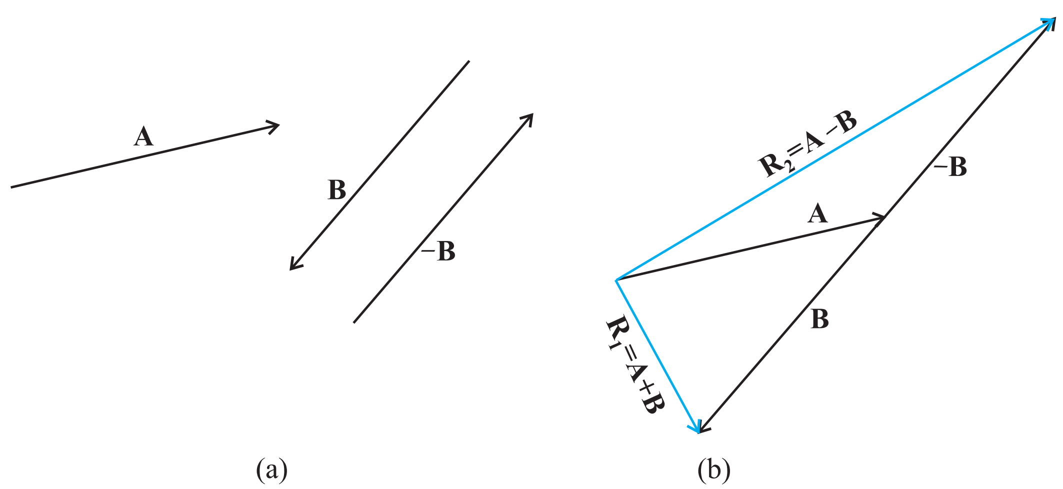


Fig. 4.5 (a) *Two vectors* **A** *and* **B**, – **B** *is also shown.* (b) *Subtracting vector* **B** *from vector* **A** *– the result is* $\mathbf{R}_2$. *For comparison, addition of vectors* **A** *and* **B**, *i.e.* $\mathbf{R}_1$ *is also shown.*

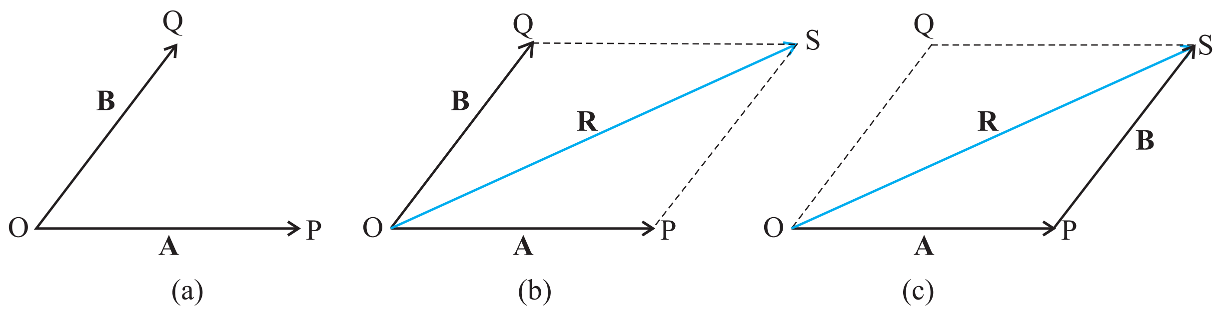


Fig. 4.6 (a) *Two vectors* **A** *and* **B** *with their tails brought to a common origin.* (b) *The sum* **A** + **B** *obtained using the parallelogram method.* (c) *The parallelogram method of vector addition is equivalent to the triangle method.*

> **Example 4.1** Rain is falling vertically with a speed of 35 m s^{-1}. Winds starts blowing after sometime with a speed of 12 m s^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella ?

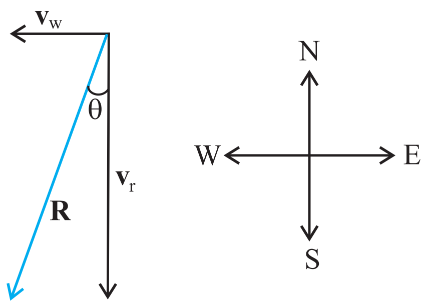


Fig. 4.7

Answer The velocity of the rain and the wind are represented by the vectors $\mathbf{v}_r$ and $\mathbf{v}_w$ in Fig. 4.7 and are in the direction specified by the problem. Using the rule of vector addition, we see that the resultant of $\mathbf{v}_r$ and $\mathbf{v}_w$ is **R** as shown in the figure. The magnitude of **R** is

$$R = \sqrt{v_r^2 + v_w^2} = \sqrt{35^2 + 12^2} \text{ m s}^{-1} = 37 \text{ m s}^{-1}$$

The direction θ that R makes with the vertical is given by

$$\tan\theta = \frac{v_w}{v_r} = \frac{12}{35} = 0.343$$

Or, $\theta = \tan^{-1}(0.343) = 19°$

Therefore, the boy should hold his umbrella in the vertical plane at an angle of about 19° with the vertical towards the east. ◀

4.5 RESOLUTION OF VECTORS

Let **a** and **b** be any two non-zero vectors in a plane with different directions and let **A** be another vector in the same plane(Fig. 4.8). **A** can be expressed as a sum of two vectors – one obtained by multiplying **a** by a real number and the other obtained by multiplying **b** by another real number. To see this, let O and P be the tail and head of the vector **A**. Then, through O, draw a straight line parallel to **a**, and through P, a straight line parallel to **b**. Let them intersect at Q. Then, we have

$$\mathbf{A} = \mathbf{OP} = \mathbf{OQ} + \mathbf{QP} \qquad (4.6)$$

But since **OQ** is parallel to **a**, and **QP** is parallel to **b**, we can write :

$$\mathbf{OQ} = \lambda\,\mathbf{a}, \text{ and } \mathbf{QP} = \mu\,\mathbf{b} \qquad (4.7)$$

where λ and μ are real numbers.

Therefore, $\mathbf{A} = \lambda\,\mathbf{a} + \mu\,\mathbf{b}$ (4.8)

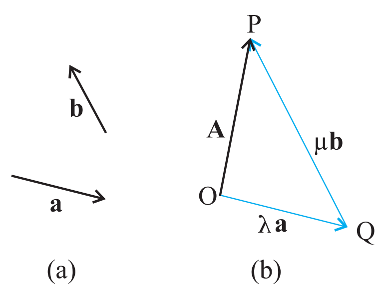


Fig. 4.8 (a) *Two non-colinear vectors* **a** *and* **b**. (b) *Resolving a vector* **A** *in terms of vectors* **a** *and* **b**.

We say that **A** has been resolved into two component vectors λ **a** and μ **b** along **a** and **b** respectively. Using this method one can resolve

a given vector into two component vectors along a set of two vectors – all the three lie in the same plane. It is convenient to resolve a general vector along the axes of a rectangular coordinate system using vectors of unit magnitude. These are called unit vectors that we discuss now.

Unit vectors: A unit vector is a vector of unit magnitude and points in a particular direction. It has no dimension and unit. It is used to specify a direction only. Unit vectors along the *x*-, *y*- and *z*-axes of a rectangular coordinate system are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively, as shown in Fig. 4.9(a).

Since these are unit vectors, we have

$$|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1 \quad (4.9)$$

These unit vectors are perpendicular to each other. In this text, they are printed in bold face with a cap (^) to distinguish them from other vectors. Since we are dealing with motion in two dimensions in this chapter, we require use of only two unit vectors. If we multiply a unit vector, say $\hat{\mathbf{n}}$ by a scalar, the result is a vector $\boldsymbol{\lambda} = \lambda\,\hat{\mathbf{n}}$. In general, a vector **A** can be written as

$$\mathbf{A} = |\mathbf{A}|\,\hat{\mathbf{n}} \quad (4.10)$$

where $\hat{\mathbf{n}}$ is a unit vector along **A**.

We can now resolve a vector **A** in terms of component vectors that lie along unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Consider a vector **A** that lies in *x-y* plane as shown in Fig. 4.9(b). We draw lines from the head of **A** perpendicular to the coordinate axes as in Fig. 4.9(b), and get vectors $\mathbf{A_1}$ and $\mathbf{A_2}$ such that $\mathbf{A_1} + \mathbf{A_2} = \mathbf{A}$. Since $\mathbf{A_1}$ is parallel to $\hat{\mathbf{i}}$ and $\mathbf{A_2}$ is parallel to $\hat{\mathbf{j}}$, we have :

$$\mathbf{A_1} = A_x\,\hat{\mathbf{i}},\quad \mathbf{A_2} = A_y\,\hat{\mathbf{j}} \quad (4.11)$$

where A_x and A_y are real numbers.

Thus, $$\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} \quad (4.12)$$

This is represented in Fig. 4.9(c). The quantities A_x and A_y are called *x*-, and *y*- components of the vector **A**. Note that A_x is itself not a vector, but $A_x\,\hat{\mathbf{i}}$ is a vector, and so is $A_y\,\hat{\mathbf{j}}$. Using simple trigonometry, we can express A_x and A_y in terms of the magnitude of **A** and the angle θ it makes with the *x*-axis :

$$A_x = A\cos\theta$$
$$A_y = A\sin\theta \quad (4.13)$$

As is clear from Eq. (4.13), a component of a vector can be positive, negative or zero depending on the value of θ.

Now, we have two ways to specify a vector **A** in a plane. It can be specified by :

(i) its magnitude *A* and the direction θ it makes with the *x*-axis; or

(ii) its components A_x and A_y

If A and θ are given, A_x and A_y can be obtained using Eq. (4.13). If A_x and A_y are given, *A* and θ can be obtained as follows :

$$A_x^2 + A_y^2 = A^2\cos^2\theta + A^2\sin^2\theta$$
$$= A^2$$

Or, $$A = \sqrt{A_x^2 + A_y^2} \quad (4.14)$$

And $$\tan\theta = \frac{A_y}{A_x},\quad \theta = \tan^{-1}\frac{A_y}{A_x} \quad (4.15)$$

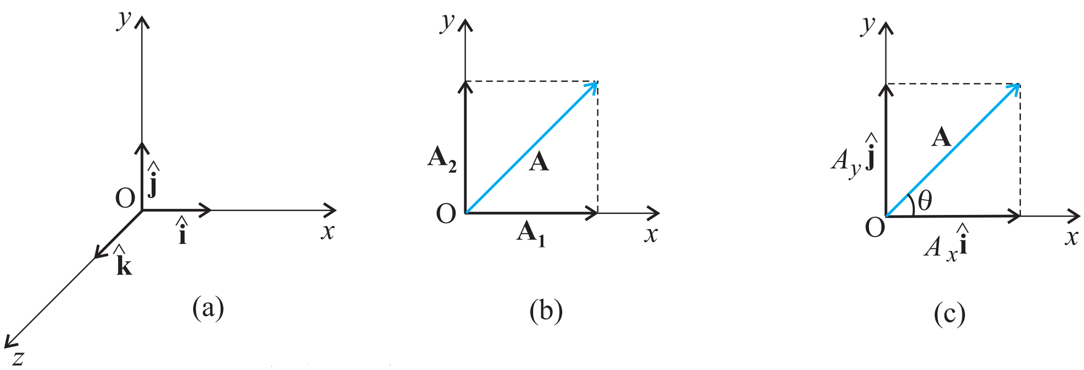
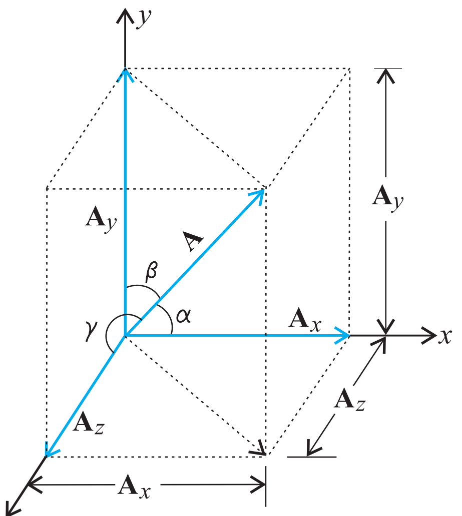


Fig. 4.9 (a) *Unit vectors* $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ *lie along the x-, y-, and z-axes.* (b) *A vector* **A** *is resolved into its components* $\mathbf{A_1}$ *and* $\mathbf{A_2}$ *along x-, and y- axes.* (c) $\mathbf{A_1}$ *and* $\mathbf{A_2}$ *expressed in terms of* $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$.

So far we have considered a vector lying in an x-y plane. The same procedure can be used to resolve a general vector **A** into three components along x-, y-, and z-axes in three dimensions. If α, β, and γ are the angles* between **A** and the x-, y-, and z-axes, respectively Fig. 4.9(d), we have



(d)

Fig. 4.9 (d) *A vector* **A** *resolved into components along x-, y-, and z-axes*

$$A_x = A\cos\alpha,\ A_y = A\cos\beta,\ A_z = A\cos\gamma \quad (4.16a)$$

In general, we have

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \quad (4.16b)$$

The magnitude of vector **A** is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (4.16c)$$

A position vector **r** can be expressed as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \quad (4.17)$$

where x, y, and z are the components of **r** along x-, y-, z-axes, respectively.

4.6 VECTOR ADDITION – ANALYTICAL METHOD

Although the graphical method of adding vectors helps us in visualising the vectors and the resultant vector, it is sometimes tedious and has limited accuracy. It is much easier to add vectors by combining their respective components. Consider two vectors **A** and **B** in x-y plane with components A_x, A_y and B_x, B_y :

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \quad (4.18)$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}$$

Let **R** be their sum. We have

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

$$= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}\right) \quad (4.19a)$$

Since vectors obey the commutative and associative laws, we can arrange and regroup the vectors in Eq. (4.19a) as convenient to us :

$$\mathbf{R} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} \quad (4.19b)$$

$$\text{Since}\ \mathbf{R} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} \quad (4.20)$$

$$\text{we have,}\ R_x = A_x + B_x,\ R_y = A_y + B_y \quad (4.21)$$

Thus, each component of the resultant vector **R** is the sum of the corresponding components of **A** and **B**.

In three dimensions, we have

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$$

with

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$
$$R_z = A_z + B_z \quad (4.22)$$

This method can be extended to addition and subtraction of any number of vectors. For example, if vectors **a**, **b** and **c** are given as

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$

$$\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$$

$$\mathbf{c} = c_x\hat{\mathbf{i}} + c_y\hat{\mathbf{j}} + c_z\hat{\mathbf{k}} \quad (4.23a)$$

then, a vector **T** = **a** + **b** – **c** has components :

$$T_x = a_x + b_x - c_x$$
$$T_y = a_y + b_y - c_y \quad (4.23b)$$
$$T_z = a_z + b_z - c_z.$$

▶ ***Example 4.2*** Find the magnitude and direction of the resultant of two vectors **A** and **B** in terms of their magnitudes and angle θ between them.

* *Note that angles α, β, and γ are angles in space. They are between pairs of lines, which are not coplanar.*

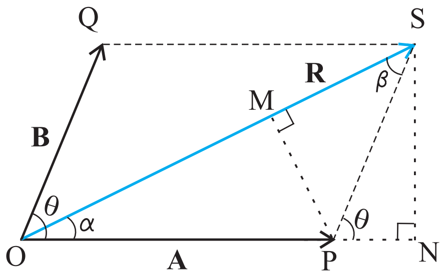


Fig. 4.10

Answer Let **OP** and **OQ** represent the two vectors **A** and **B** making an angle θ (Fig. 4.10). Then, using the parallelogram method of vector addition, **OS** represents the resultant vector **R** :

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

SN is normal to OP and PM is normal to OS.

From the geometry of the figure,

$$OS^2 = ON^2 + SN^2$$

but $ON = OP + PN = A + B\cos\theta$

$SN = B\sin\theta$

$$OS^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

or, $R^2 = A^2 + B^2 + 2AB\cos\theta$

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \qquad (4.24a)$$

In Δ OSN, $SN = OS\sin\alpha = R\sin\alpha$, and

in Δ PSN, $SN = PS\sin\theta = B\sin\theta$

Therefore, $R\sin\alpha = B\sin\theta$

$$\text{or,} \quad \frac{R}{\sin\theta} = \frac{B}{\sin\alpha} \qquad (4.24b)$$

Similarly,

$$PM = A\sin\alpha = B\sin\beta$$

$$\text{or,} \quad \frac{A}{\sin\beta} = \frac{B}{\sin\alpha} \qquad (4.24c)$$

Combining Eqs. (4.24b) and (4.24c), we get

$$\frac{R}{\sin\theta} = \frac{A}{\sin\beta} = \frac{B}{\sin\alpha} \qquad (4.24d)$$

Using Eq. (4.24d), we get:

$$\sin\alpha = \frac{B}{R}\sin\theta \qquad (4.24e)$$

where R is given by Eq. (4.24a).

$$\text{or,} \quad \tan\alpha = \frac{SN}{OP + PN} = \frac{B\sin\theta}{A + B\cos\theta} \qquad (4.24f)$$

Equation (4.24a) gives the magnitude of the resultant and Eqs. (4.24e) and (4.24f) its direction. Equation (4.24a) is known as the **Law of cosines** and Eq. (4.24d) as the **Law of sines**. ◀

▶ ***Example 4.3*** A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

Answer The vector $\mathbf{v}_b$ representing the velocity of the motorboat and the vector $\mathbf{v}_c$ representing the water current are shown in Fig. 4.11 in directions specified by the problem. Using the parallelogram method of addition, the resultant **R** is obtained in the direction shown in the figure.

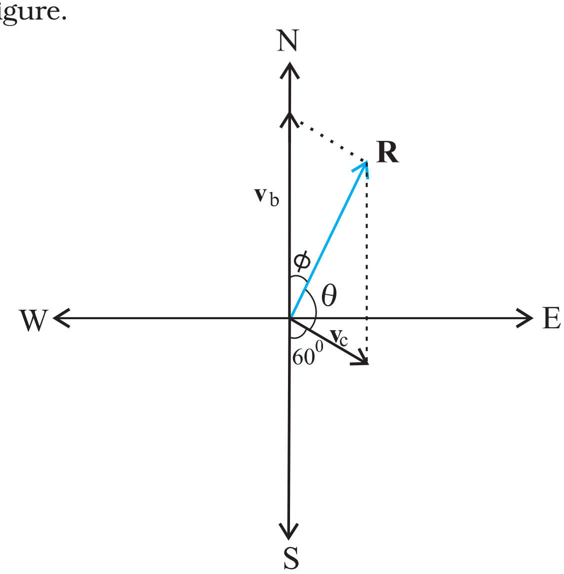


Fig. 4.11

We can obtain the magnitude of **R** using the Law of cosine :

$$R = \sqrt{v_b^2 + v_c^2 + 2v_b v_c \cos 120^\circ}$$

$$= \sqrt{25^2 + 10^2 + 2\times 25\times 10(-1/2)} \cong 22 \text{ km/h}$$

To obtain the direction, we apply the Law of sines

$$\frac{R}{\sin\theta} = \frac{v_c}{\sin\phi} \quad \text{or,} \quad \sin\phi = \frac{v_c}{R}\sin\theta$$

$$= \frac{10\times\sin 120^\circ}{21.8} = \frac{10\sqrt{3}}{2\times 21.8} \cong 0.397$$

$$\phi \cong 23.4^\circ$$ ◀

4.7 MOTION IN A PLANE

In this section we shall see how to describe motion in two dimensions using vectors.

4.7.1 Position Vector and Displacement

The position vector **r** of a particle P located in a plane with reference to the origin of an x-y reference frame (Fig. 4.12) is given by

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$$

where x and y are components of **r** along x-, and y- axes or simply they are the coordinates of the object.

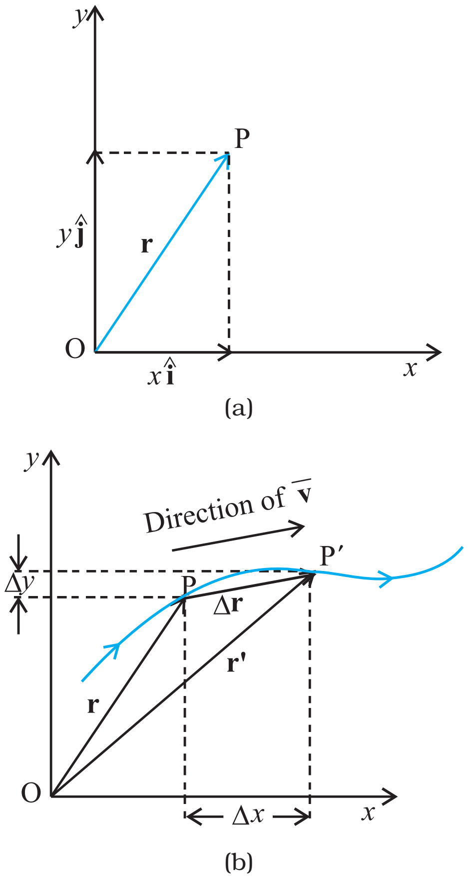


Fig. 4.12 *(a) Position vector* **r**. *(b) Displacement* $\Delta\mathbf{r}$ *and average velocity* $\bar{\mathbf{v}}$ *of a particle.*

Suppose a particle moves along the curve shown by the thick line and is at P at time t and P′ at time t' [Fig. 4.12(b)]. Then, the displacement is :

$$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r} \qquad (4.25)$$

and is directed from P to P′ .

We can write Eq. (4.25) in a component form:

$$\Delta\mathbf{r} = \left(x'\,\hat{\mathbf{i}} + y'\,\hat{\mathbf{j}}\right) - \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}\right)$$

$$= \hat{\mathbf{i}}\Delta x + \hat{\mathbf{j}}\Delta y$$

where $\quad \Delta x = x' - x,\ \Delta y = y' - y \qquad (4.26)$

Velocity

The average velocity $(\bar{\mathbf{v}})$ of an object is the ratio of the displacement and the corresponding time interval :

$$\bar{\mathbf{v}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}}}{\Delta t} = \hat{\mathbf{i}}\frac{\Delta x}{\Delta t} + \hat{\mathbf{j}}\frac{\Delta y}{\Delta t} \qquad (4.27)$$

Or, $\quad \bar{\mathbf{v}} = \bar{v}_x\,\hat{\mathbf{i}} + \bar{v}_y\,\hat{\mathbf{j}}$

Since $\bar{\mathbf{v}} = \dfrac{\Delta\mathbf{r}}{\Delta t}$, the direction of the average velocity is the same as that of $\Delta\mathbf{r}$ (Fig. 4.12). The **velocity** (instantaneous velocity) is given by the limiting value of the average velocity as the time interval approaches zero :

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \qquad (4.28)$$

The meaning of the limiting process can be easily understood with the help of Fig 4.13(a) to (d). In these figures, the thick line represents the path of an object, which is at P at time t. P_1, P_2 and P_3 represent the positions of the object after times $\Delta t_1, \Delta t_2$, and Δt_3. $\Delta\mathbf{r}_1$, $\Delta\mathbf{r}_2$, and $\Delta\mathbf{r}_3$ are the displacements of the object in times Δt_1, Δt_2, and

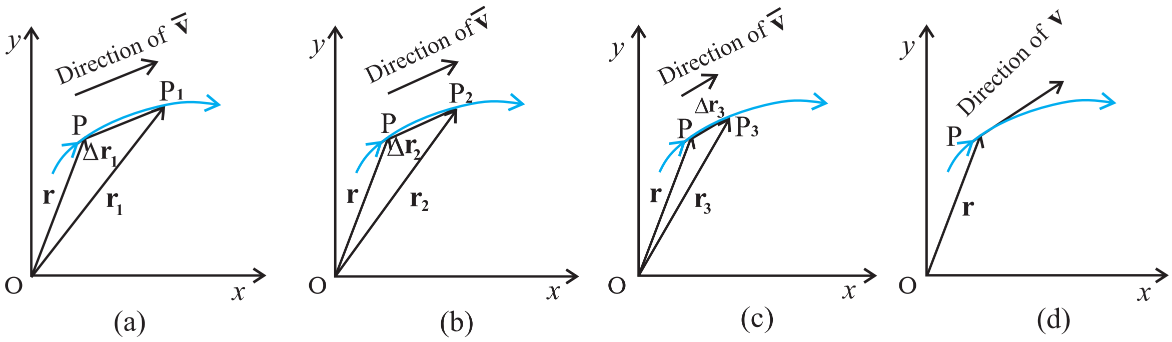


Fig. 4.13 *As the time interval* Δt *approaches zero, the average velocity approaches the velocity* **v**. *The direction of* $\bar{\mathbf{v}}$ *is parallel to the line tangent to the path.*

Δt_3, respectively. The direction of the average velocity $\overline{\mathbf{v}}$ is shown in figures (a), (b) and (c) for three decreasing values of Δt, i.e. Δt_1,Δt_2, and Δt_3, ($\Delta t_1 > \Delta t_2 > \Delta t_3$). As $\Delta t \to 0$, $\Delta \mathbf{r} \to 0$ and is along the tangent to the path [Fig. 4.13(d)]. Therefore, **the direction of velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion**.

We can express **v** in a component form :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$= \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta y}{\Delta t}\hat{\mathbf{j}}\right) \quad (4.29)$$

$$= \hat{\mathbf{i}} \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \hat{\mathbf{j}} \lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t}$$

Or, $$\mathbf{v} = \hat{\mathbf{i}}\frac{dx}{dt} + \hat{\mathbf{j}}\frac{dy}{dt} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}.$$

where $$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt} \quad (4.30a)$$

So, if the expressions for the coordinates x and y are known as functions of time, we can use these equations to find v_x and v_y.

The magnitude of **v** is then

$$v = \sqrt{v_x^2 + v_y^2} \quad (4.30b)$$

and the direction of **v** is given by the angle θ :

$$\tan\theta = \frac{v_y}{v_x}, \quad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \quad (4.30c)$$

v_x, v_y and angle θ are shown in Fig. 4.14 for a velocity vector **v**.

Acceleration

The **average acceleration a** of an object for a time interval Δt moving in x-y plane is the change in velocity divided by the time interval :

$$\overline{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\Delta\left(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}\right)}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}} \quad (4.31a)$$

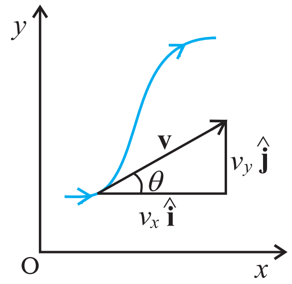


Fig. 4.14 *The components v_x and v_y of velocity **v** and the angle θ it makes with x-axis. Note that $v_x = v\cos\theta$, $v_y = v\sin\theta$.*

Or, $$\overline{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}. \quad (4.31b)$$

The **acceleration** (instantaneous acceleration) is the limiting value of the average acceleration as the time interval approaches zero :

$$\mathbf{a} = \lim_{\Delta t \to 0}\frac{\Delta \mathbf{v}}{\Delta t} \quad (4.32a)$$

Since $\Delta \mathbf{v} = \Delta v_x\hat{\mathbf{i}} + \Delta v_y\hat{\mathbf{j}}$, we have

$$\mathbf{a} = \hat{\mathbf{i}} \lim_{\Delta t \to 0}\frac{\Delta v_x}{\Delta t} + \hat{\mathbf{j}} \lim_{\Delta t \to 0}\frac{\Delta v_y}{\Delta t}$$

Or, $$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} \quad (4.32b)$$

where, $$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt} \quad (4.32c)^*$$

As in the case of velocity, we can understand graphically the limiting process used in defining acceleration on a graph showing the path of the object's motion. This is shown in Figs. 4.15(a) to (d). P represents the position of the object at time t and P_1, P_2, P_3 positions after time Δt_1, Δt_2, Δt_3, respectively ($\Delta t_1 > \Delta t_2 > \Delta t_3$). The velocity vectors at points P, P_1, P_2, P_3 are also shown in Figs. 4.15 (a), (b) and (c). In each case of Δt, $\Delta \mathbf{v}$ is obtained using the triangle law of vector addition. By definition, the direction of average acceleration is the same as that of $\Delta \mathbf{v}$. We see that as Δt decreases, the direction of $\Delta \mathbf{v}$ changes and consequently, the direction of the acceleration changes. Finally, in the limit $\Delta t \to 0$

* *In terms of x and y, a_x and a_y can be expressed as*

$$a_x = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}, \quad a_y = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

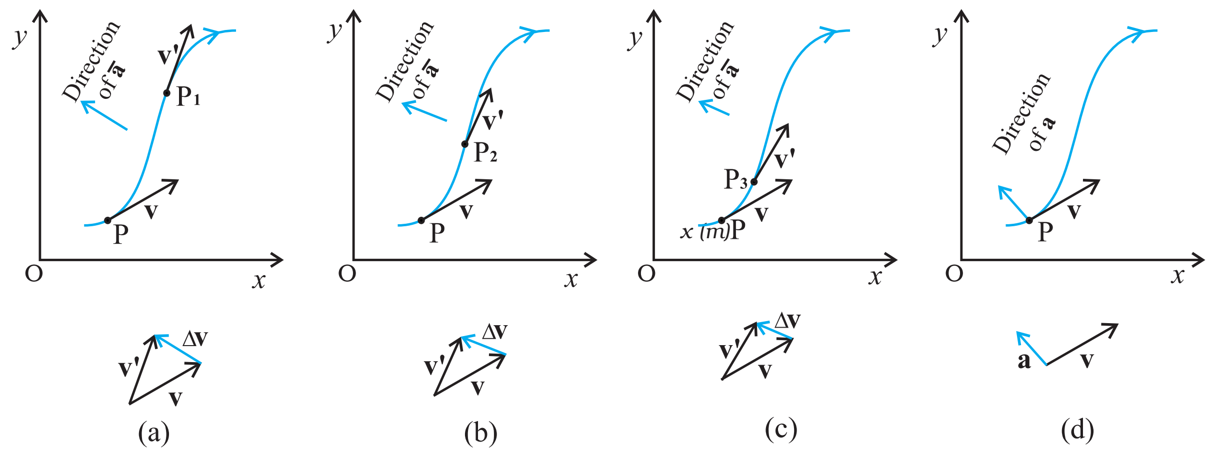


Fig. 4.15 *The average acceleration for three time intervals* (a) Δt_1, (b) Δt_2, *and* (c) Δt_3, *(*$\Delta t_1 > \Delta t_2 > \Delta t_3$*).* (d) *In the limit* $\Delta t \to 0$, *the average acceleration becomes the acceleration.*

Fig. 4.15(d), the average acceleration becomes the instantaneous acceleration and has the direction as shown.

Note that in one dimension, the velocity and the acceleration of an object are always along the same straight line (either in the same direction or in the opposite direction). However, for motion in two or three dimensions, velocity and acceleration vectors may have any angle between 0° and 180° between them.

▶ ***Example 4.4*** The position of a particle is given by

$$\mathbf{r} = 3.0t\,\hat{\mathbf{i}} + 2.0t^2\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}$$

where t is in seconds and the coefficients have the proper units for **r** to be in metres. (a) Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ of the particle. (b) Find the magnitude and direction of $\mathbf{v}(t)$ at $t = 1.0$ s.

Answer

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(3.0\,t\,\hat{\mathbf{i}} + 2.0t^2\,\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}\right)$$

$$= 3.0\hat{\mathbf{i}} + 4.0t\hat{\mathbf{j}}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = +4.0\hat{\mathbf{j}}$$

$a = 4.0$ m s^{-2} along y- direction

At $t = 1.0$ s, $\mathbf{v} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$

It's magnitude is $v = \sqrt{3^2 + 4^2} = 5.0 \text{ m s}^{-1}$

and direction is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) \cong 53^\circ \text{ with } x\text{-axis.}$$

◀

4.8 MOTION IN A PLANE WITH CONSTANT ACCELERATION

Suppose that an object is moving in x-y plane and its acceleration **a** is constant. Over an interval of time, the average acceleration will equal this constant value. Now, let the velocity of the object be $\mathbf{v}_0$ at time $t = 0$ and **v** at time t.

Then, by definition

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v_0}}{t - 0} = \frac{\mathbf{v} - \mathbf{v_0}}{t}$$

Or, $\quad \mathbf{v} = \mathbf{v_0} + \mathbf{a}t \quad$ (4.33a)

In terms of components :

$$v_x = v_{ox} + a_x t$$

$$v_y = v_{oy} + a_y t \quad (4.33b)$$

Let us now find how the position **r** changes with time. We follow the method used in the one-dimensional case. Let $\mathbf{r}_o$ and **r** be the position vectors of the particle at time 0 and t and let the velocities at these instants be $\mathbf{v}_o$ and **v**. Then, over this time interval t, the average velocity is $(\mathbf{v}_o + \mathbf{v})/2$. The displacement is the average velocity multiplied by the time interval :

$$\mathbf{r} - \mathbf{r_0} = \left(\frac{\mathbf{v} + \mathbf{v_0}}{2}\right)t = \left(\frac{(\mathbf{v_0} + \mathbf{a}t) + \mathbf{v_0}}{2}\right)t$$

$$= \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

Or, $$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2 \qquad (4.34a)$$

It can be easily verified that the derivative of Eq. (4.34a), i.e. $\frac{d\mathbf{r}}{dt}$ gives Eq.(4.33a) and it also satisfies the condition that at t=0, $\mathbf{r} = \mathbf{r}_o$. Equation (4.34a) can be written in component form as

$$x = x_0 + v_{ox}t + \frac{1}{2}a_x t^2$$

$$y = y_0 + v_{oy}t + \frac{1}{2}a_y t^2 \qquad (4.34b)$$

One immediate interpretation of Eq.(4.34b) is that the motions in x- and y-directions can be treated independently of each other. That is, **motion in a plane (two-dimensions) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions**. This is an important result and is useful in analysing motion of objects in two dimensions. A similar result holds for three dimensions. The choice of perpendicular directions is convenient in many physical situations, as we shall see in section 4.10 for projectile motion.

▶ ***Example 4.5*** A particle starts from origin at t = 0 with a velocity 5.0 $\hat{\mathbf{i}}$ m/s and moves in x-y plane under action of a force which produces a constant acceleration of $(3.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})$ m/s². (a) What is the y-coordinate of the particle at the instant its x-coordinate is 84 m ? (b) What is the speed of the particle at this time ?

Answer The position of the particle is given by

$$\mathbf{r}(t) = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

$$= 5.0\hat{\mathbf{i}}t + (1/2)(3.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\,t^2$$

$$= (5.0t + 1.5t^2)\hat{\mathbf{i}} + 1.0t^2\hat{\mathbf{j}}$$

Therefore, $x(t) = 5.0t + 1.5t^2$

$y(t) = +1.0t^2$

Given x (t) = 84 m, t = ?

$5.0\ t + 1.5\ t^2 = 84 \Rightarrow t = 6$ s

At t = 6 s, $y = 1.0\ (6)^2 = 36.0$ m

Now, the velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (5.0 + 3.0t)\hat{\mathbf{i}} + 2.0t\ \hat{\mathbf{j}}$

At t = 6 s, $\mathbf{v} = 23.0\hat{\mathbf{i}} + 12.0\hat{\mathbf{j}}$

speed $= |\mathbf{v}| = \sqrt{23^2 + 12^2} \cong 26 \text{ m s}^{-1}$. ◀

4.9 RELATIVE VELOCITY IN TWO DIMENSIONS

The concept of relative velocity, introduced in section 3.7 for motion along a straight line, can be easily extended to include motion in a plane or in three dimensions. Suppose that two objects A and B are moving with velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ (each with respect to some common frame of reference, say ground.). Then, velocity of object A **relative to that of B is** :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B \qquad (4.35a)$$

and similarly, the velocity of object B *relative to that of* A is :

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$$

Therefore, $\mathbf{v}_{AB} = -\mathbf{v}_{BA}$ (4.35b)

and, $|\mathbf{v}_{AB}| = |\mathbf{v}_{BA}|$ (4.35c)

▶ ***Example 4.6*** Rain is falling vertically with a speed of 35 m s⁻¹. A woman rides a bicycle with a speed of 12 m s⁻¹ in east to west direction. What is the direction in which she should hold her umbrella ?

Answer In Fig. 4.16 $\mathbf{v}_r$ represents the velocity of rain and $\mathbf{v}_b$, the velocity of the bicycle, the woman is riding. Both these velocities are with respect to the ground. Since the woman is riding a bicycle, the velocity of rain as experienced by her is the velocity of rain relative to the velocity of the bicycle she is riding. That is $\mathbf{v}_{rb} = \mathbf{v}_r - \mathbf{v}_b$

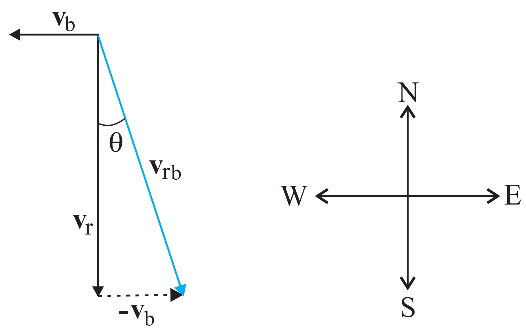


Fig. 4.16

This relative velocity vector as shown in Fig. 4.16 makes an angle θ with the vertical. It is given by

$$\tan\theta = \frac{v_b}{v_r} = \frac{12}{35} = 0.343$$

Or, $\theta \cong 19°$

Therefore, the woman should hold her umbrella at an angle of about 19° with the vertical towards the west.

Note carefully the difference between this Example and the Example 4.1. In Example 4.1, the boy experiences the resultant (vector sum) of two velocities while in this example, the woman experiences the velocity of rain relative to the bicycle (the vector difference of the two velocities). ◀

4.10 PROJECTILE MOTION

As an application of the ideas developed in the previous sections, we consider the motion of a projectile. An object that is in flight after being thrown or projected is called a **projectile**. Such a projectile might be a football, a cricket ball, a baseball or any other object. The motion of a projectile may be thought of as the result of two separate, simultaneously occurring components of motions. One component is along a horizontal direction without any acceleration and the other along the vertical direction with constant acceleration due to the force of gravity. It was Galileo who first stated this independency of the horizontal and the vertical components of projectile motion in his **Dialogue on the great world systems** (1632).

In our discussion, we shall assume that the air resistance has negligible effect on the motion of the projectile. Suppose that the projectile is launched with velocity $\mathbf{v}_o$ that makes an angle θ_o with the x-axis as shown in Fig. 4.17.

After the object has been projected, the acceleration acting on it is that due to gravity which is directed vertically downward:

$$\mathbf{a} = -g\,\hat{\mathbf{j}}$$

Or, $a_x = 0,\ a_y = -g$ (4.36)

The components of initial velocity $\mathbf{v}_o$ are :

$$v_{ox} = v_o \cos\theta_o$$
$$v_{oy} = v_o \sin\theta_o \qquad (4.37)$$

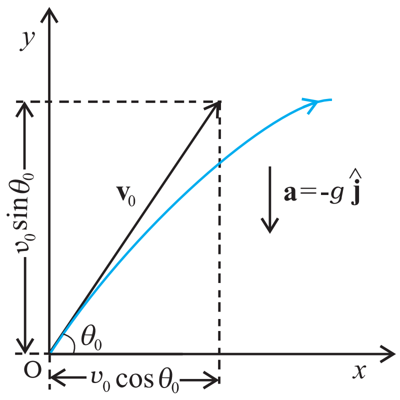


Fig 4.17 *Motion of an object projected with velocity $\mathbf{v}_o$ at angle θ_0.*

If we take the initial position to be the origin of the reference frame as shown in Fig. 4.17, we have :

$$x_o = 0,\ y_o = 0$$

Then, Eq.(4.47b) becomes :

$$x = v_{ox}\,t = (v_o \cos\theta_o)\,t$$

and $\quad y = (v_o \sin\theta_o)\,t - (\tfrac{1}{2})g\,t^2 \qquad (4.38)$

The components of velocity at time t can be obtained using Eq.(4.33b) :

$$v_x = v_{ox} = v_o \cos\theta_o$$
$$v_y = v_o \sin\theta_o - g\,t \qquad (4.39)$$

Equation (4.38) gives the x-, and y-coordinates of the position of a projectile at time t in terms of two parameters — initial speed v_o and projection angle θ_o. Notice that the choice of mutually perpendicular x-, and y-directions for the analysis of the projectile motion has resulted in a simplification. One of the components of velocity, i.e. x-component remains constant throughout the motion and only the y- component changes, like an object in free fall in vertical direction. This is shown graphically at few instants in Fig. 4.18. Note that at the point of maximum height, $v_y = 0$ and therefore,

$$\theta = \tan^{-1}\frac{v_y}{v_x} = 0$$

Equation of path of a projectile

What is the shape of the path followed by the projectile? This can be seen by eliminating the time between the expressions for x and y as given in Eq. (4.38). We obtain:

$$y = (\tan\theta_o)\,x - \frac{g}{2\,(v_o\cos\theta_o)^2}\,x^2 \qquad (4.40)$$

Now, since g, θ_o and v_o are constants, Eq. (4.40) is of the form $y = a\,x + b\,x^2$, in which a and b are constants. This is the equation of a parabola, i.e. the path of the projectile is a parabola (Fig. 4.18).

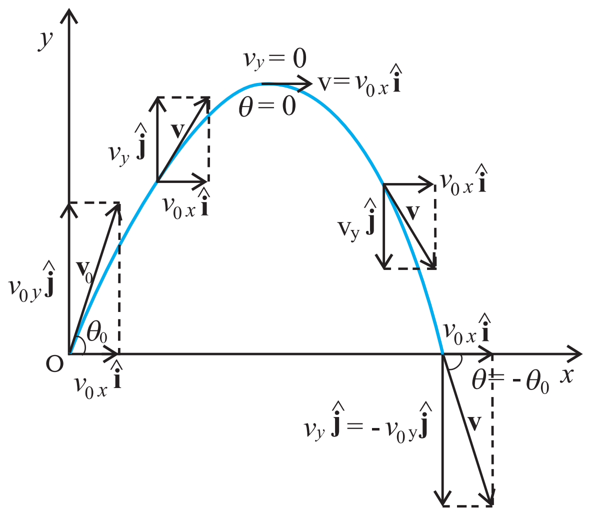


Fig. 4.18 *The path of a projectile is a parabola.*

Time of maximum height

How much time does the projectile take to reach the maximum height ? Let this time be denoted by t_m. Since at this point, $v_y = 0$, we have from Eq. (4.39):

$$v_y = v_o\sin\theta_o - g\,t_m = 0$$

Or, $$t_m = v_o\sin\theta_o/g \qquad (4.41a)$$

The total time T_f during which the projectile is in flight can be obtained by putting $y = 0$ in Eq. (4.38). We get :

$$T_f = 2\,(v_o\sin\theta_o)/g \qquad (4.41b)$$

T_f is known as the **time of flight** of the projectile. We note that $T_f = 2\,t_m$, which is expected because of the symmetry of the parabolic path.

Maximum height of a projectile

The maximum height h_m reached by the projectile can be calculated by substituting $t = t_m$ in Eq. (4.38) :

$$y = h_m = (v_0\sin\theta_0)\left(\frac{v_0\sin\theta_0}{g}\right) - \frac{g}{2}\left(\frac{v_0\sin\theta_0}{g}\right)^2$$

Or, $$h_m = \frac{(v_0\sin\theta_0)^2}{2g} \qquad (4.42)$$

Horizontal range of a projectile

The horizontal distance travelled by a projectile from its initial position ($x = y = 0$) to the position where it passes $y = 0$ during its fall is called the **horizontal range**, R. It is the distance travelled during the time of flight T_f. Therefore, the range R is

$$R = (v_o\cos\theta_o)\,(T_f)$$

$$=(v_o\cos\theta_o)\,(2\,v_o\sin\theta_o)/g$$

Or, $$R = \frac{v_0^2\sin 2\theta_0}{g} \qquad (4.43a)$$

Equation (4.43a) shows that for a given projection velocity v_o, R is maximum when $\sin 2\theta_0$ is maximum, i.e., when $\theta_0 = 45^0$.

The maximum horizontal range is, therefore,

$$R_m = \frac{v_0^2}{g} \qquad (4.43b)$$

▶ ***Example 4.7*** Galileo, in his book **Two new sciences,** stated that "for elevations which exceed or fall short of 45° by equal amounts, the ranges are equal". Prove this statement.

Answer For a projectile launched with velocity $\mathbf{v_o}$ at an angle θ_o, the range is given by

$$R = \frac{v_0^2\sin 2\theta_0}{g}$$

Now, for angles, (45° + α) and (45° – α), $2\theta_o$ is (90° + 2α) and (90° – 2α) , respectively. The values of sin (90° + 2α) and sin (90° – 2α) are the same, equal to that of cos 2α. Therefore, ranges are equal for elevations which exceed or fall short of 45° by equal amounts α. ◀

▶ ***Example 4.8*** A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 m s^{-1}. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take g = 9.8 m s^{-2}).

Answer We choose the origin of the x-,and y-axis at the edge of the cliff and t = 0 s at the instant the stone is thrown. Choose the positive direction of x-axis to be along the initial velocity and the positive direction of y-axis to be the vertically upward direction. The x-, and y-components of the motion can be treated independently. The equations of motion are :

$$x(t) = x_o + v_{ox}\, t$$
$$y(t) = y_o + v_{oy}\, t + (1/2)\, a_y\, t^2$$

Here, $x_o = y_o = 0$, $v_{oy} = 0$, $a_y = -g = -9.8 \text{ m s}^{-2}$, $v_{ox} = 15 \text{ m s}^{-1}$.

The stone hits the ground when $y(t) = -490$ m.

$$-490 \text{ m} = -(1/2)(9.8)\, t^2.$$

This gives $t = 10$ s.

The velocity components are $v_x = v_{ox}$ and $v_y = v_{oy} - g\,t$

so that when the stone hits the ground :

$$v_{ox} = 15 \text{ m s}^{-1}$$
$$v_{oy} = 0 - 9.8 \times 10 = -98 \text{ m s}^{-1}$$

Therefore, the speed of the stone is

$$\sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 98^2} = 99 \text{ m s}^{-1}$$ ◀

▶ **Example 4.9** A cricket ball is thrown at a speed of 28 m s^{-1} in a direction 30° above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball returns to the same level.

Answer (a) The maximum height is given by

$$h_m = \frac{(v_o \sin\theta_o)^2}{2g} = \frac{(28 \sin 30°)^2}{2\ (9.8)} \text{ m}$$

$$= \frac{14 \times 14}{2 \times 9.8} = 10.0 \text{ m}$$

(b) The time taken to return to the same level is

$$T_f = (2\, v_o \sin\theta_o)/g = (2 \times 28 \times \sin 30°)/9.8$$
$$= 28/9.8 \text{ s} = 2.9 \text{ s}$$

(c) The distance from the thrower to the point where the ball returns to the same level is

$$R = \frac{(v_o^2 \sin 2\theta_o)}{g} = \frac{28 \times 28 \times \sin 60^o}{9.8} = 69 \text{ m}$$ ◀

Neglecting air resistance - what does the assumption really mean?

While treating the topic of projectile motion, we have stated that we assume that the air resistance has no effect on the motion of the projectile. You must understand what the statement really means. Friction, force due to viscosity, air resistance are all dissipative forces. In the presence of any of such forces opposing motion, any object will lose some part of its initial energy and consequently, momentum too. Thus, a projectile that traverses a parabolic path would certainly show deviation from its idealised trajectory in the presence of air resistance. It will not hit the ground with the same speed with which it was projected from it. In the absence of air resistance, the x-component of the velocity remains constant and it is only the y-component that undergoes a continuous change. However, in the presence of air resistance, both of these would get affected. That would mean that the range would be less than the one given by Eq. (4.43). Maximum height attained would also be less than that predicted by Eq. (4.42). Can you then, anticipate the change in the time of flight?

In order to avoid air resistance, we will have to perform the experiment in vacuum or under low pressure, which is not easy. When we use a phrase like 'neglect air resistance', we imply that the change in parameters such as range, height etc. is much smaller than their values without air resistance. The calculation without air resistance is much simpler than that with air resistance.

4.11 UNIFORM CIRCULAR MOTION

When an object follows a circular path at a constant speed, the motion of the object is called **uniform circular motion**. The word "uniform" refers to the speed, which is uniform (constant) throughout the motion. Suppose an object is moving with uniform speed v in a circle of radius R as shown in Fig. 4.19. Since the velocity of the object is changing continuously in direction, the object undergoes acceleration. Let us find the magnitude and the direction of this acceleration.

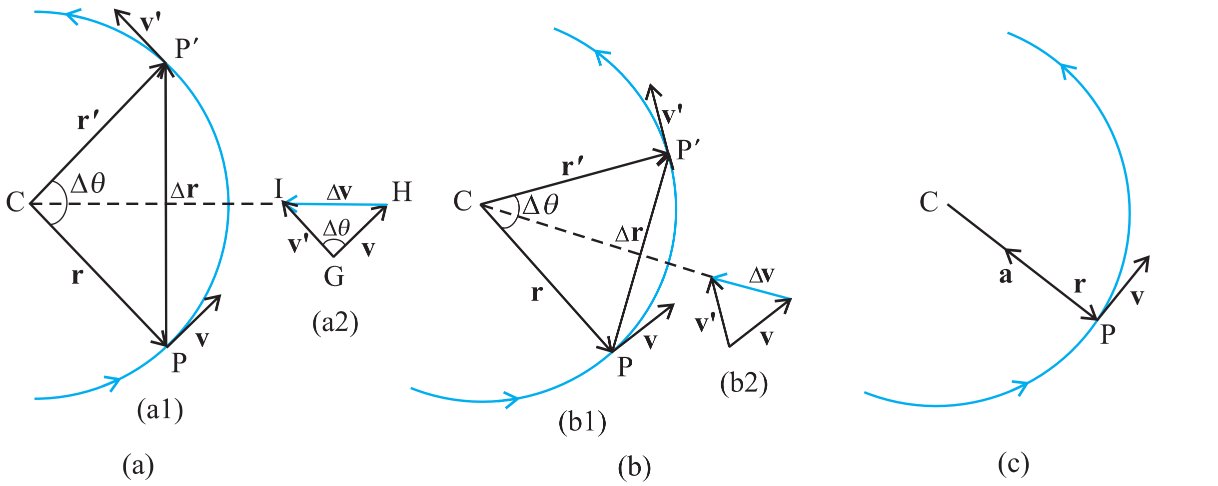


Fig. 4.19 *Velocity and acceleration of an object in uniform circular motion. The time interval Δt decreases from (a) to (c) where it is zero. The acceleration is directed, at each point of the path, towards the centre of the circle.*

Let $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors and $\mathbf{v}$ and $\mathbf{v}'$ the velocities of the object when it is at point P and P' as shown in Fig. 4.19(a). By definition, velocity at a point is along the tangent at that point in the direction of motion. The velocity vectors $\mathbf{v}$ and $\mathbf{v}'$ are as shown in Fig. 4.19(a1). $\Delta\mathbf{v}$ is obtained in Fig. 4.19 (a2) using the triangle law of vector addition. Since the path is circular, $\mathbf{v}$ is perpendicular to $\mathbf{r}$ and so is $\mathbf{v}'$ to $\mathbf{r}'$. Therefore, $\Delta\mathbf{v}$ is perpendicular to $\Delta\mathbf{r}$. Since average acceleration is along $\Delta\mathbf{v}$ $\left(\bar{\mathbf{a}} = \dfrac{\Delta\mathbf{v}}{\Delta t}\right)$, the average acceleration $\bar{\mathbf{a}}$ is perpendicular to $\Delta\mathbf{r}$. If we place $\Delta\mathbf{v}$ on the line that bisects the angle between $\mathbf{r}$ and $\mathbf{r}'$, we see that it is directed towards the centre of the circle. Figure 4.19(b) shows the same quantities for smaller time interval. $\Delta\mathbf{v}$ and hence $\bar{\mathbf{a}}$ is again directed towards the centre. In Fig. 4.19(c), $\Delta t \to 0$ and the average acceleration becomes the instantaneous acceleration. It is directed towards the centre*. Thus, we find that the acceleration of an object in uniform circular motion is always directed towards the centre of the circle. Let us now find the magnitude of the acceleration.

The magnitude of $\mathbf{a}$ is, by definition, given by

$$|\mathbf{a}| = \lim_{\Delta t \to 0} \frac{|\Delta\mathbf{v}|}{\Delta t}$$

Let the angle between position vectors $\mathbf{r}$ and $\mathbf{r}'$ be $\Delta\theta$. Since the velocity vectors $\mathbf{v}$ and $\mathbf{v}'$ are always perpendicular to the position vectors, the angle between them is also $\Delta\theta$. Therefore, the triangle CPP′ formed by the position vectors and the triangle GHI formed by the velocity vectors $\mathbf{v}$, $\mathbf{v}'$ and $\Delta\mathbf{v}$ are similar (Fig. 4.19a). Therefore, the ratio of the base-length to side-length for one of the triangles is equal to that of the other triangle. That is :

$$\frac{|\Delta\mathbf{v}|}{v} = \frac{|\Delta\mathbf{r}|}{R}$$

Or, $$|\Delta\mathbf{v}| = v\frac{|\Delta\mathbf{r}|}{R}$$

Therefore,

$$|\mathbf{a}| = \lim_{\Delta t \to 0} \frac{|\Delta\mathbf{v}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{v|\Delta\mathbf{r}|}{R\Delta t} = \frac{v}{R} \lim_{\Delta t \to 0} \frac{|\Delta\mathbf{r}|}{\Delta t}$$

If Δt is small, $\Delta\theta$ will also be small and then arc PP' can be approximately taken to be $|\Delta\mathbf{r}|$:

$$|\Delta\mathbf{r}| \cong v\Delta t$$

$$\frac{|\Delta\mathbf{r}|}{\Delta t} \cong v$$

Or, $$\lim_{\Delta t \to 0} \frac{|\Delta\mathbf{r}|}{\Delta t} = v$$

Therefore, the centripetal acceleration a_c is :

* *In the limit $\Delta t \to 0$, $\Delta\mathbf{r}$ becomes perpendicular to $\mathbf{r}$. In this limit $\Delta\mathbf{v} \to \mathbf{0}$ and is consequently also perpendicular to $\mathbf{V}$. Therefore, the acceleration is directed towards the centre, at each point of the circular path.*

$$a_c = \left(\frac{v}{R}\right) v = v^2/R \qquad (4.44)$$

Thus, the acceleration of an object moving with speed v in a circle of radius R has a magnitude v^2/R and is always **directed towards the centre**. This is why this acceleration is called **centripetal acceleration** (a term proposed by Newton). A thorough analysis of centripetal acceleration was first published in 1673 by the Dutch scientist Christiaan Huygens (1629-1695) but it was probably known to Newton also some years earlier. "Centripetal" comes from a Greek term which means 'centre-seeking'. Since v and R are constant, the magnitude of the centripetal acceleration is also constant. However, the direction changes — pointing always towards the centre. Therefore, a centripetal acceleration is not a constant vector.

We have another way of describing the velocity and the acceleration of an object in uniform circular motion. As the object moves from P to P′ in time Δt (= $t' - t$), the line CP (Fig. 4.19) turns through an angle $\Delta\theta$ as shown in the figure. $\Delta\theta$ is called angular distance. We define the angular speed ω (Greek letter omega) as the time rate of change of angular displacement :

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad (4.45)$$

Now, if the distance travelled by the object during the time Δt is Δs, i.e. PP' is Δs, then :

$$v = \frac{\Delta s}{\Delta t}$$

but $\Delta s = R\,\Delta\theta$. Therefore :

$$v = R\frac{\Delta\theta}{\Delta t} = R\,\omega$$

$$v = R\,\omega \qquad (4.46)$$

We can express centripetal acceleration a_c in terms of angular speed :

$$a_c = \frac{v^2}{R} = \frac{\omega^2 R^2}{R} = \omega^2 R$$

$$a_c = \omega^2 R \qquad (4.47)$$

The time taken by an object to make one revolution is known as its time period T and the number of revolution made in one second is called its frequency ν (=$1/T$). However, during this time the distance moved by the object is $s = 2\pi R$.

Therefore, $v = 2\pi R/T = 2\pi R\nu$ (4.48)

In terms of frequency ν, we have

$$\omega = 2\pi\nu$$

$$v = 2\pi R\nu$$

$$a_c = 4\pi^2 \nu^2 R \qquad (4.49)$$

▶ ***Example 4.10*** An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s. (a) What is the angular speed, and the linear speed of the motion? (b) Is the acceleration vector a constant vector ? What is its magnitude ?

Answer This is an example of uniform circular motion. Here R = 12 cm. The angular speed ω is given by

$$\omega = 2\pi/T = 2\pi \times 7/100 = 0.44 \text{ rad/s}$$

The linear speed v is :

$$v = \omega R = 0.44 \text{ s}^{-1} \times 12 \text{ cm} = 5.3 \text{ cm s}^{-1}$$

The direction of velocity $\boldsymbol{v}$ is along the tangent to the circle at every point. The acceleration is directed towards the centre of the circle. Since this direction changes continuously, acceleration here is *not* a constant vector. However, the magnitude of acceleration is constant:

$$a = \omega^2 R = (0.44 \text{ s}^{-1})^2 (12 \text{ cm})$$

$$= 2.3 \text{ cm s}^{-2}$$ ◀

SUMMARY

1. *Scalar quantities* are quantities with magnitudes only. Examples are distance, speed, mass and temperature.
2. *Vector quantities* are quantities with magnitude and direction both. Examples are displacement, velocity and acceleration. They obey special rules of vector algebra.
3. A vector **A** multiplied by a real number λ is also a vector, whose magnitude is λ times the magnitude of the vector **A** and whose direction is the same or opposite depending upon whether λ is positive or negative.
4. Two vectors **A** and **B** may be *added graphically* using *head-to-tail method* or *parallelogram method.*
5. Vector addition is *commutative* :

   $$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

   It also obeys the *associative law* :

   $$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$
6. A *null* or *zero vector* is a vector with zero magnitude. Since the magnitude is zero, we don't have to specify its direction. It has the properties :

   $$\mathbf{A} + \mathbf{0} = \mathbf{A}$$
   $$\lambda\mathbf{0} = \mathbf{0}$$
   $$0\,\mathbf{A} = \mathbf{0}$$
7. The *subtraction* of vector **B** from **A** is defined as the sum of **A** and –**B** :

   $$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$
8. A vector **A** can be *resolved* into component along two given vectors **a** and **b** lying in the same plane :

   $$\mathbf{A} = \lambda\,\mathbf{a} + \mu\,\mathbf{b}$$

   where λ and μ are real numbers.
9. A *unit vector* associated with a vector **A** has magnitude one and is along the vector **A**:

   $$\hat{\mathbf{n}} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

   The unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are vectors of unit magnitude and point in the direction of the *x*-, *y*-, and *z*-axes, respectively in a right-handed coordinate system.
10. A vector **A** can be expressed as

    $$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$

    where A_x, A_y are its components along *x*-, and *y* -axes. If vector **A** makes an angle θ with the *x*-axis, then $A_x = A\cos\theta$, $A_y = A\sin\theta$ and $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2}$, $\tan\theta = \dfrac{A_y}{A_x}$.
11. Vectors can be conveniently added using *analytical method.* If sum of two vectors **A** and **B**, that lie in *x-y* plane, is **R**, then :

    $\mathbf{R} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$, where, $R_x = A_x + B_x$, and $R_y = A_y + B_y$
12. The *position vector* of an object in *x-y* plane is given by $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ and the *displacement* from position **r** to position **r'** is given by

    $$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$$
    $$= (x' - x)\,\hat{\mathbf{i}} + (y' - y)\,\hat{\mathbf{j}}$$
    $$= \Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}}$$
13. If an object undergoes a displacement $\Delta\mathbf{r}$ in time Δt, its *average velocity* is given by $\mathbf{v} = \dfrac{\Delta\mathbf{r}}{\Delta t}$. The *velocity* of an object at time *t* is the limiting value of the average velocity

as Δt tends to zero :

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}.$$ It can be written in unit vector notation as :

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \quad \text{where} \quad v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}, v_z = \frac{dz}{dt}$$

When position of an object is plotted on a coordinate system, $\mathbf{v}$ is always tangent to the curve representing the path of the object.

14. If the velocity of an object changes from $\mathbf{v}$ to $\mathbf{v}'$ in time Δt, then its *average acceleration* is given by: $\bar{\mathbf{a}} = \dfrac{\mathbf{v} - \mathbf{v}'}{\Delta t} = \dfrac{\Delta \mathbf{v}}{\Delta t}$

The *acceleration* $\mathbf{a}$ at any time t is the limiting value of $\bar{\mathbf{a}}$ as $\Delta t \to 0$:

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

In component form, we have : $\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$

where, $a_x = \dfrac{dv_x}{dt},\ a_y = \dfrac{dv_y}{dt},\ a_z = \dfrac{dv_z}{dt}$

15. If an object is moving in a plane with constant acceleration $a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2}$ and its position vector at time $t = 0$ is $\mathbf{r}_o$, then at any other time t, it will be at a point given by:

$$\mathbf{r} = \mathbf{r}_o + \mathbf{v}_o t + \frac{1}{2}\mathbf{a}t^2$$

and its velocity is given by :

$$\mathbf{v} = \mathbf{v}_o + \mathbf{a}\, t$$

where $\mathbf{v}_o$ is the velocity at time $t = 0$

In component form :

$$x = x_o + v_{ox}t + \frac{1}{2}a_x t^2$$

$$y = y_o + v_{oy}\, t + \frac{1}{2}a_y t^2$$

$$v_x = v_{ox} + a_x t$$

$$v_y = v_{oy} + a_y t$$

Motion in a plane can be treated as superposition of two separate simultaneous one-dimensional motions along two perpendicular directions

16. An object that is in flight after being projected is called a *projectile*. If an object is projected with initial velocity $\mathbf{v}_o$ making an angle θ_o with x-axis and if we assume its initial position to coincide with the origin of the coordinate system, then the position and velocity of the projectile at time t are given by :

$$x = (v_o \cos \theta_o)\, t$$
$$y = (v_o \sin \theta_o)\, t - (1/2)\, g\, t^2$$
$$v_x = v_{ox} = v_o \cos \theta_o$$
$$v_y = v_o \sin \theta_o - g\, t$$

The path of a projectile is *parabolic* and is given by :

$$y = (\tan\theta_0)\, x - \frac{gx^2}{2\,(v_o \cos\theta_o)^2}$$

The *maximum height* that a projectile attains is :

$$h_m = \frac{(v_o \ \text{sin}q_o)^2}{2g}$$

The *time* taken to reach this height is :

$$t_m = \frac{v_o \sin\theta_o}{g}$$

The horizontal distance travelled by a projectile from its initial position to the position it passes $y = 0$ during its fall is called the *range*, R of the projectile. It is :

$$R = \frac{v_o^2}{g} \sin 2\theta_o$$

17. When an object follows a circular path at constant speed, the motion of the object is called *uniform circular motion*. The magnitude of its acceleration is $a_c = v^2/R$. The direction of a_c is always towards the centre of the circle.
The angular speed ω is the rate of change of angular distance. It is related to velocity v by $v = \omega R$. The acceleration is $a_c = \omega^2 R$.
If T is the time period of revolution of the object in circular motion and ν is its frequency, we have $\omega = 2\pi\nu$, $v = 2\pi\nu R$, $a_c = 4\pi^2\nu^2 R$

Physical Quantity	**Symbol**	**Dimensions**	**Unit**	**Remark**
Position vector	$\mathbf{r}$	[L]	m	Vector. It may be denoted by any other symbol as well.
Displacement	$\Delta\mathbf{r}$	[L]	m	- do -
Velocity		$[LT^{-1}]$	m s^{-1}	
(a) Average	$\bar{\mathbf{v}}$			$= \frac{\Delta\mathbf{r}}{\Delta t}$, vector
(b) Instantaneous	$\mathbf{v}$			$= \frac{d\mathbf{r}}{dt}$, vector
Acceleration		$[LT^{-2}]$	m s^{-2}	
(a) Average	$\bar{\mathbf{a}}$			$= \frac{\Delta\mathbf{v}}{\Delta t}$, vector
(b) Instantaneous	$\mathbf{a}$			$= \frac{d\mathbf{v}}{dt}$, vector
Projectile motion				
(a) Time of max. height	t_m	[T]	s	$= \frac{v_0 \sin\theta_0}{g}$
(b) Max. height	h_m	[L]	m	$= \frac{(v_0 \sin\theta_0)^2}{2g}$
(c) Horizontal range	R	[L]	m	$= \frac{v_0^2 \sin 2\theta_0}{g}$
Circular motion				
(a) Angular speed	ω	$[T^{-1}]$	rad/s	$= \frac{\Delta\theta}{\Delta t} = \frac{v}{r}$
(b) Centripetal acceleration	a_c	$[LT^{-2}]$	m s^{-2}	$= \frac{v^2}{r}$

POINTS TO PONDER

1. The path length traversed by an object between two points is, in general, not the same as the magnitude of displacement. The displacement depends only on the end points; the path length (as the name implies) depends on the actual path. The two quantities are equal only if the object does not change its direction during the course of motion. In all other cases, the path length is greater than the magnitude of displacement.
2. In view of point 1 above, the average speed of an object is greater than or equal to the magnitude of the average velocity over a given time interval. The two are equal only if the path length is equal to the magnitude of displacement.
3. The vector equations (4.33a) and (4.34a) do not involve any choice of axes. Of course, you can always resolve them along any two independent axes.
4. The kinematic equations for uniform acceleration do not apply to the case of uniform circular motion since in this case the magnitude of acceleration is constant but its direction is changing.
5. An object subjected to two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ has a resultant velocity $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Take care to distinguish it from velocity of object 1 relative to velocity of object 2 : $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$. Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are velocities with reference to some common reference frame.
6. The resultant acceleration of an object in circular motion is towards the centre only if the speed is constant.
7. The shape of the trajectory of the motion of an object is not determined by the acceleration alone but also depends on the initial conditions of motion (initial position and initial velocity). For example, the trajectory of an object moving under the same acceleration due to gravity can be a straight line or a parabola depending on the initial conditions.

EXERCISES

4.1 State, for each of the following physical quantities, if it is a scalar or a vector : volume, mass, speed, acceleration, density, number of moles, velocity, angular frequency, displacement, angular velocity.

4.2 Pick out the two scalar quantities in the following list : force, angular momentum, work, current, linear momentum, electric field, average velocity, magnetic moment, relative velocity.

4.3 Pick out the only vector quantity in the following list : Temperature, pressure, impulse, time, power, total path length, energy, gravitational potential, coefficient of friction, charge.

4.4 State with reasons, whether the following algebraic operations with scalar and vector physical quantities are meaningful : (a) adding any two scalars, (b) adding a scalar to a vector of the same dimensions , (c) multiplying any vector by any scalar, (d) multiplying any two scalars, (e) adding any two vectors, (f) adding a component of a vector to the same vector.

4.5 Read each statement below carefully and state with reasons, if it is true or false : (a) The magnitude of a vector is always a scalar, (b) each component of a vector is always a scalar, (c) the total path length is always equal to the magnitude of the displacement vector of a particle. (d) the average speed of a particle (defined as total path length divided by the time taken to cover the path) is either greater or equal to the magnitude of average velocity of the particle over the same interval of time, (e) Three vectors not lying in a plane can never add up to give a null vector.

4.6 Establish the following vector inequalities geometrically or otherwise :

(a) $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(b) $|\mathbf{a}+\mathbf{b}| \geq ||\mathbf{a}| - |\mathbf{b}||$

(c) $|\mathbf{a}-\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(d) $|\mathbf{a}-\mathbf{b}| \geq ||\mathbf{a}| - |\mathbf{b}||$

When does the equality sign above apply?

4.7 Given $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$, which of the following statements are correct :

(a) **a**, **b**, **c**, and **d** must each be a null vector,

(b) The magnitude of $(\mathbf{a} + \mathbf{c})$ equals the magnitude of $(\mathbf{b} + \mathbf{d})$,

(c) The magnitude of **a** can never be greater than the sum of the magnitudes of **b**, **c**, and **d**,

(d) **b** + **c** must lie in the plane of **a** and **d** if **a** and **d** are not collinear, and in the line of **a** and **d**, if they are collinear ?

4.8 Three girls skating on a circular ice ground of radius 200 m start from a point *P* on the edge of the ground and reach a point **Q** diametrically opposite to *P* following different paths as shown in Fig. 4.20. What is the magnitude of the displacement vector for each ? For which girl is this equal to the actual length of path skate ?

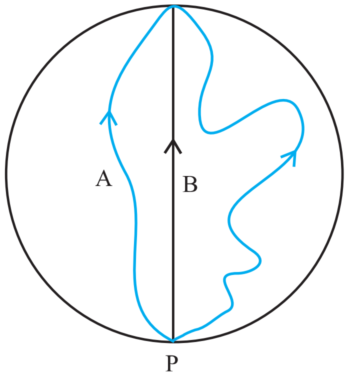


Fig. 4.20

4.9 A cyclist starts from the centre *O* of a circular park of radius 1 km, reaches the edge *P* of the park, then cycles along the circumference, and returns to the centre along **QO** as shown in Fig. 4.21. If the round trip takes 10 min, what is the (a) net displacement, (b) average velocity, and (c) average speed of the cyclist ?

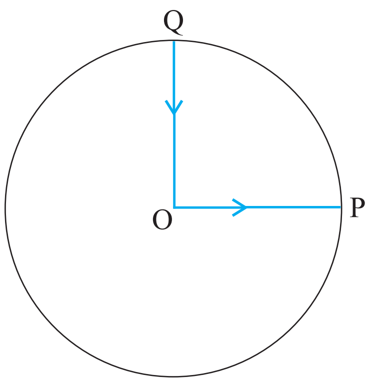


Fig. 4.21

4.10 On an open ground, a motorist follows a track that turns to his left by an angle of 60^0 after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of the displacement with the total path length covered by the motorist in each case.

4.11 A passenger arriving in a new town wishes to go from the station to a hotel located 10 km away on a straight road from the station. A dishonest cabman takes him along a circuitous path 23 km long and reaches the hotel in 28 min. What is (a) the average speed of the taxi, (b) the magnitude of average velocity ? Are the two equal ?

4.12 Rain is falling vertically with a speed of 30 m s^{-1}. A woman rides a bicycle with a speed of 10 m s^{-1} in the north to south direction. What is the direction in which she should hold her umbrella ?

4.13 A man can swim with a speed of 4.0 km/h in still water. How long does he take to cross a river 1.0 km wide if the river flows steadily at 3.0 km/h and he makes his

strokes normal to the river current? How far down the river does he go when he reaches the other bank ?

4.14 In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?

4.15 The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 m s^{-1} can go without hitting the ceiling of the hall ?

4.16 A cricketer can throw a ball to a maximum horizontal distance of 100 m. How much high above the ground can the cricketer throw the same ball ?

4.17 A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, what is the magnitude and direction of acceleration of the stone ?

4.18 An aircraft executes a horizontal loop of radius 1.00 km with a steady speed of 900 km/h. Compare its centripetal acceleration with the acceleration due to gravity.

4.19 Read each statement below carefully and state, with reasons, if it is true or false :

(a) The net acceleration of a particle in circular motion is *always* along the radius of the circle towards the centre

(b) The velocity vector of a particle at a point is *always* along the tangent to the path of the particle at that point

(c) The acceleration vector of a particle in *uniform* circular motion averaged over one cycle is a null vector

4.20 The position of a particle is given by

$$\mathbf{r} = 3.0t\,\hat{\mathbf{i}} - 2.0t^2\,\hat{\mathbf{j}} + 4.0\,\hat{\mathbf{k}}\ \text{m}$$

where t is in seconds and the coefficients have the proper units for **r** to be in metres.

(a) Find the **v** and **a** of the particle? (b) What is the magnitude and direction of velocity of the particle at $t = 2.0$ s ?

4.21 A particle starts from the origin at $t = 0$ s with a velocity of $10.0\,\hat{\mathbf{j}}$ m/s and moves in the x-y plane with a constant acceleration of $(8.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})$ m s^{-2}. (a) At what time is the x- coordinate of the particle 16 m? What is the y-coordinate of the particle at that time? (b) What is the speed of the particle at the time ?

4.22 $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along x- and y- axis respectively. What is the magnitude and direction of the vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, and $\hat{\mathbf{i}} - \hat{\mathbf{j}}$? What are the components of a vector **A**= $2\,\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ along the directions of $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} - \hat{\mathbf{j}}$? [You may use graphical method]

4.23 For any arbitrary motion in space, which of the following relations are true :

(a) $\mathbf{v}_{average} = (1/2)\,(\mathbf{v}\,(t_1) + \mathbf{v}\,(t_2))$

(b) $\mathbf{v}_{average} = [\mathbf{r}(t_2) - \mathbf{r}(t_1)\,] \,/(t_2 - t_1)$

(c) $\mathbf{v}\,(t) = \mathbf{v}\,(0) + \mathbf{a}\,t$

(d) $\mathbf{r}\,(t) = \mathbf{r}\,(0) + \mathbf{v}\,(0)\,t + (1/2)\,\mathbf{a}\,t^2$

(e) $\mathbf{a}_{average} = [\,\mathbf{v}\,(t_2) - \mathbf{v}\,(t_1\,)] \,/(\,t_2 - t_1)$

(The 'average' stands for average of the quantity over the time interval t_1 to t_2)

4.24 Read each statement below carefully and state, with reasons and examples, if it is true or false :

A scalar quantity is one that

(a) is conserved in a process

(b) can never take negative values

(c) must be dimensionless

(d) does not vary from one point to another in space

(e) has the same value for observers with different orientations of axes.

4.25 An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10.0 s apart is 30°, what is the speed of the aircraft ?

Additional Exercises

4.26 A vector has magnitude and direction. Does it have a location in space ? Can it vary with time ? Will two equal vectors **a** and **b** at different locations in space necessarily have identical physical effects ? Give examples in support of your answer.

4.27 A vector has both magnitude and direction. Does it mean that anything that has magnitude and direction is necessarily a vector ? The rotation of a body can be specified by the direction of the axis of rotation, and the angle of rotation about the axis. Does that make any rotation a vector ?

4.28 Can you associate vectors with (a) the length of a wire bent into a loop, (b) a plane area, (c) a sphere ? Explain.

4.29 A bullet fired at an angle of 30° with the horizontal hits the ground 3.0 km away. By adjusting its angle of projection, can one hope to hit a target 5.0 km away ? Assume the *muzzle* speed to the fixed, and neglect air resistance.

4.30 A fighter plane flying horizontally at an altitude of 1.5 km with speed 720 km/h passes directly overhead an anti-aircraft gun. At what angle from the vertical should the gun be fired for the shell with muzzle speed 600 m s^{-1} to hit the plane ? At what minimum altitude should the pilot fly the plane to avoid being hit ? (Take g = 10 m s^{-2}).

4.31 A cyclist is riding with a speed of 27 km/h. As he approaches a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at the constant rate of 0.50 m/s every second. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn ?

4.32 (a) Show that for a projectile the angle between the velocity and the x-axis as a function of time is given by

$$\boldsymbol{\theta}(t) = \tan^{-1}\left(\frac{v_{0y} - gt}{v_{ox}}\right)$$

(b) Shows that the projection angle θ_0 for a projectile launched from the origin is given by

$$\boldsymbol{\theta}_0 = \tan^{-1}\left(\frac{4h_m}{R}\right)$$

where the symbols have their usual meaning.

CHAPTER FIVE

LAWS OF MOTION

5.1 INTRODUCTION

In the preceding Chapter, our concern was to describe the motion of a particle in space quantitatively. We saw that uniform motion needs the concept of velocity alone whereas non-uniform motion requires the concept of acceleration in addition. So far, we have not asked the question as to what governs the motion of bodies. In this chapter, we turn to this basic question.

Let us first guess the answer based on our common experience. To move a football at rest, someone must kick it. To throw a stone upwards, one has to give it an upward push. A breeze causes the branches of a tree to swing; a strong wind can even move heavy objects. A boat moves in a flowing river without anyone rowing it. Clearly, some external agency is needed to provide force to move a body from rest. Likewise, an external force is needed also to retard or stop motion. You can stop a ball rolling down an inclined plane by applying a force against the direction of its motion.

In these examples, the external agency of force (hands, wind, stream, etc) is in contact with the object. This is not always necessary. A stone released from the top of a building accelerates downward due to the gravitational pull of the earth. A bar magnet can attract an iron nail from a distance. **This shows that external agencies (e.g. gravitational and magnetic forces) can exert force on a body even from a distance.**

In short, a force is required to put a stationary body in motion or stop a moving body, and some external agency is needed to provide this force. The external agency may or may not be in contact with the body.

So far so good. But what if a body is moving uniformly (e.g. a skater moving straight with constant speed on a horizontal ice slab) ? **Is an external force required to keep a body in uniform motion?**

5.2 ARISTOTLE'S FALLACY

The question posed above appears to be simple. However, it took ages to answer it. Indeed, the correct answer to this question given by Galileo in the seventeenth century was the foundation of Newtonian mechanics, which signalled the birth of modern science.

The Greek thinker, Aristotle (384 B.C– 322 B.C.), held the view that if a body is moving, something external is required to keep it moving. According to this view, for example, an arrow shot from a bow keeps flying since the air behind the arrow keeps pushing it. The view was part of an elaborate framework of ideas developed by Aristotle on the motion of bodies in the universe. Most of the Aristotelian ideas on motion are now known to be wrong and need not concern us. For our purpose here, the Aristotelian law of motion may be phrased thus: **An external force is required to keep a body in motion**.

Aristotelian law of motion is flawed, as we shall see. However, it is a natural view that anyone would hold from common experience. Even a small child playing with a simple (non-electric) toy-car on a floor knows intuitively that it needs to constantly drag the string attached to the toy-car with some force to keep it going. If it releases the string, it comes to rest. This experience is common to most terrestrial motion. External forces seem to be needed to keep bodies in motion. Left to themselves, all bodies eventually come to rest.

What is the flaw in Aristotle's argument? The answer is: a moving toy car comes to rest because the external force of friction on the car by the floor opposes its motion. To counter this force, the child has to apply an external force on the car in the direction of motion. When the car is in uniform motion, there is no net external force acting on it: the force by the child cancels the force (friction) by the floor. The corollary is: if there were no friction, the child would not be required to apply any force to keep the toy car in uniform motion.

The opposing forces such as friction (solids) and viscous forces (for fluids) are always present in the natural world. This explains why forces by external agencies are necessary to overcome the frictional forces to keep bodies in uniform motion. Now we understand where Aristotle went wrong. He coded this practical experience in the form of a basic argument. To get at the true law of nature for forces and motion, one has to imagine a world in which uniform motion is possible with no frictional forces opposing. This is what Galileo did.

5.3 THE LAW OF INERTIA

Galileo studied motion of objects on an inclined plane. Objects (i) moving down an inclined plane accelerate, while those (ii) moving up retard. (iii) Motion on a horizontal plane is an intermediate situation. Galileo concluded that an object moving on a frictionless horizontal plane must neither have acceleration nor retardation, i.e. it should move with constant velocity (Fig. 5.1(a)).

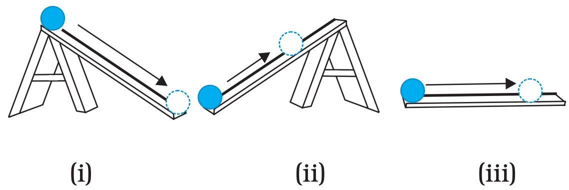


Fig. 5.1(a)

Another experiment by Galileo leading to the same conclusion involves a double inclined plane. A ball released from rest on one of the planes rolls down and climbs up the other. If the planes are smooth, the final height of the ball is nearly the same as the initial height (a little less but never greater). In the ideal situation, when friction is absent, the final height of the ball is the same as its initial height.

If the slope of the second plane is decreased and the experiment repeated, the ball will still reach the same height, but in doing so, it will travel a longer distance. In the limiting case, when the slope of the second plane is zero (i.e. is a horizontal) the ball travels an infinite distance. In other words, its motion never ceases. This is, of course, an idealised situation (Fig. 5.1(b)).

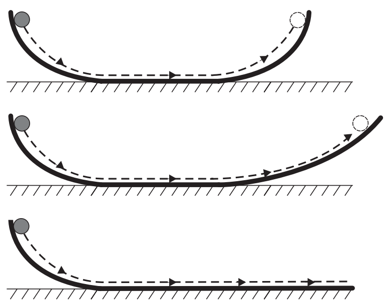


Fig. 5.1(b) *The law of inertia was inferred by Galileo from observations of motion of a ball on a double inclined plane.*

In practice, the ball does come to a stop after moving a finite distance on the horizontal plane, because of the opposing force of friction which can never be totally eliminated. However, if there were no friction, the ball would continue to move with a constant velocity on the horizontal plane.

Galileo thus, arrived at a new insight on motion that had eluded Aristotle and those who followed him. The state of rest and the state of uniform linear motion (motion with constant velocity) are equivalent. In both cases, there is no net force acting on the body. It is incorrect to assume that a net force is needed to keep a body in uniform motion. To maintain a body in uniform motion, we need to apply an external force to ecounter the frictional force, so that the two forces sum up to zero net external force.

To summarise, if the net external force is zero, a body at rest continues to remain at rest and a body in motion continues to move with a uniform velocity. This property of the body is called inertia. Inertia means **'resistance to change'**. A body does not change its state of rest or uniform motion, unless an external force compels it to change that state.

5.4 NEWTON'S FIRST LAW OF MOTION

Galileo's simple, but revolutionary ideas dethroned Aristotelian mechanics. A new mechanics had to be developed. This task was accomplished almost single-handedly by Isaac Newton, one of the greatest scientists of all times.

Newton built on Galileo's ideas and laid the foundation of mechanics in terms of three laws of motion that go by his name. Galileo's law of inertia was his starting point which he formulated as the **First Law of motion***:*

Every body continues to be in its state of rest or of uniform motion in a straight line unless compelled by some external force to act otherwise.

Ideas on Motion in Ancient Indian Science

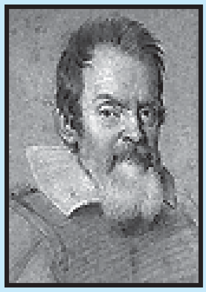
Ancient Indian thinkers had arrived at an elaborate system of ideas on motion. Force, the cause of motion, was thought to be of different kinds : force due to continuous pressure (nodan), as the force of wind on a sailing vessel; impact (abhighat), as when a potter's rod strikes the wheel; persistent tendency (sanskara) to move in a straight line(vega) or restoration of shape in an elastic body; transmitted force by a string, rod, etc. The notion of (vega) in the Vaisesika theory of motion perhaps comes closest to the concept of inertia. Vega, the tendency to move in a straight line, was thought to be opposed by contact with objects including atmosphere, a parallel to the ideas of friction and air resistance. It was correctly summarised that the different kinds of motion (translational, rotational and vibrational) of an extended body arise from only the translational motion of its constituent particles. A falling leaf in the wind may have downward motion as a whole (patan) and also rotational and vibrational motion (bhraman, spandan), but each particle of the leaf at an instant only has a definite (small) displacement. There was considerable focus in Indian thought on measurement of motion and units of length and time. It was known that the position of a particle in space can be indicated by distance measured along three axes. Bhaskara (1150 A.D.) had introduced the concept of 'instantaneous motion' (*tatkaliki gati*), which anticipated the modern notion of instantaneous velocity using Differential Calculus. The difference between a wave and a current (of water) was clearly understood; a current is a motion of particles of water under gravity and fluidity while a wave results from the transmission of vibrations of water particles.

The state of rest or uniform linear motion both imply zero acceleration. The first law of motion can, therefore, be simply expressed as:

If the net external force on a body is zero, its acceleration is zero. Acceleration can be non zero only if there is a net external force on the body.

Two kinds of situations are encountered in the application of this law in practice. In some examples, we know that the net external force on the object is zero. In that case we can conclude that the acceleration of the object is zero. For example, a spaceship out in interstellar space, far from all other objects and with all its rockets turned off, has no net external force acting on it. Its acceleration, according to the First Law, must be zero. If it is in motion, it must continue to move with a uniform velocity.

Galileo Galilei (1564 - 1642)



Galileo Galilei, born in Pisa, Italy in 1564 was a key figure in the scientific revolution in Europe about four centuries ago. Galileo proposed the concept of acceleration. From experiments on motion of bodies on inclined planes or falling freely, he contradicted the Aristotelian notion that a force was required to keep a body in motion, and that heavier bodies fall faster than lighter bodies under gravity. He thus arrived at the law of inertia that was the starting point of the subsequent epochal work of Isaac Newton.

Galileo's discoveries in astronomy were equally revolutionary. In 1609, he designed his own telescope (invented earlier in Holland) and used it to make a number of startling observations : mountains and depressions on the surface of the moon; dark spots on the sun; the moons of Jupiter and the phases of Venus. He concluded that the Milky Way derived its luminosity because of a large number of stars not visible to the naked eye. In his masterpiece of scientific reasoning : Dialogue on the Two Chief World Systems, Galileo advocated the heliocentric theory of the solar system proposed by Copernicus, which eventually got universal acceptance.

With Galileo came a turning point in the very method of scientific inquiry. Science was no longer merely observations of nature and inferences from them. Science meant devising and doing experiments to verify or refute theories. Science meant measurement of quantities and a search for mathematical relations between them. Not undeservedly, many regard Galileo as the father of modern science.

More often, however, we do not know all the forces to begin with. In that case, if we know that an object is unaccelerated (i.e. it is either at rest or in uniform linear motion), we can infer from the first law that the net external force on the object must be zero. Gravity is everywhere. For terrestrial phenomena, in particular, every object experiences gravitational force due to the earth. Also objects in motion generally experience friction, viscous drag, etc. If then, on earth, an object is at rest or in uniform linear motion, it is not because there are no forces acting on it, but because the various external forces cancel out i.e. add up to zero net external force.

Consider a book at rest on a horizontal surface Fig. (5.2(a)). It is subject to two external forces : the force due to gravity (i.e. its weight W) acting downward and the upward force on the book by the table, the normal force R . R is a self-adjusting force. This is an example of the kind of situation mentioned above. The forces are not quite known fully but the state of motion is known. We observe the book to be at rest. Therefore, we conclude from the first law that the magnitude of R equals that of W. A statement often encountered is : "Since $W = R$, forces cancel and, therefore, the book is at rest". This is incorrect reasoning. The correct statement is : "Since the book is observed to be at rest, the net external force on it must be zero, according to the first law. This implies that the normal force R must be equal and opposite to the weight W ".

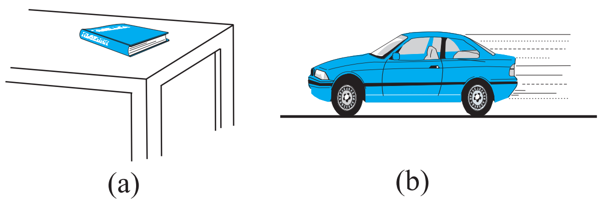


Fig. 5.2 (a) *a book at rest on the table, and* (b) *a car moving with uniform velocity. The net force is zero in each case.*

Consider the motion of a car starting from rest, picking up speed and then moving on a smooth straight road with uniform speed (Fig. (5.2(b)). When the car is stationary, there is no net force acting on it. During pick-up, it accelerates. This must happen due to a net external force. Note, it has to be an external force. The acceleration of the car cannot be accounted for by any internal force. This might sound surprising, but it is true. The only conceivable external force along the road is the force of friction. It is the frictional force that accelerates the car as a whole. (You will learn about friction in section 5.9). When the car moves with constant velocity, there is no net external force.

The property of inertia contained in the First law is evident in many situations. Suppose we are standing in a stationary bus and the driver starts the bus suddenly. We get thrown backward with a jerk. Why ? Our feet are in touch with the floor. If there were no friction, we would remain where we were, while the floor of the bus would simply slip forward under our feet and the back of the bus would hit us. However, fortunately, there is some friction between the feet and the floor. If the start is not too sudden, i.e. if the acceleration is moderate, the frictional force would be enough to accelerate our feet along with the bus. But our body is not strictly a rigid body. It is deformable, i.e. it allows some relative displacement between different parts. What this means is that while our feet go with the bus, the rest of the body remains where it is due to inertia. Relative to the bus, therefore, we are thrown backward. As soon as that happens, however, the muscular forces on the rest of the body (by the feet) come into play to move the body along with the bus. A similar thing happens when the bus suddenly stops. Our feet stop due to the friction which does not allow relative motion between the feet and the floor of the bus. But the rest of the body continues to move forward due to inertia. We are thrown forward. The restoring muscular forces again come into play and bring the body to rest.

Example 5.1 An astronaut accidentally gets separated out of his small spaceship accelerating in inter stellar space at a constant rate of 100 m s^{-2}. What is the acceleration of the astronaut the instant after he is outside the spaceship ? (Assume that there are no nearby stars to exert gravitational force on him.)

Answer Since there are no nearby stars to exert gravitational force on him and the small spaceship exerts negligible gravitational attraction on him, the net force acting on the astronaut, once he is out of the spaceship, is zero. By the first law of motion the acceleration of the astronaut is zero.

5.5 NEWTON'S SECOND LAW OF MOTION

The first law refers to the simple case when the net external force on a body is zero. The second law of motion refers to the general situation when there is a net external force acting on the body. It relates the net external force to the acceleration of the body.

Momentum

Momentum, **P** of a body is defined to be the product of its mass m and velocity $\mathbf{v}$, and is denoted by $\mathbf{p}$:

$$\mathbf{p} = m\,\mathbf{v} \qquad (5.1)$$

Momentum is clearly a vector quantity. The following common experiences indicate the importance of this quantity for considering the effect of force on motion.

- Suppose a light-weight vehicle (say a small car) and a heavy weight vehicle (say a loaded truck) are parked on a horizontal road. We all know that a much greater force is needed to push the truck than the car to bring them to the same speed in same time. Similarly, a greater opposing force is needed to stop a heavy body than a light body in the same time, if they are moving with the same speed.
- If two stones, one light and the other heavy, are dropped from the top of a building, a person on the ground will find it easier to catch the light stone than the heavy stone. The mass of a body is thus an important parameter that determines the effect of force on its motion.
- Speed is another important parameter to consider. A bullet fired by a gun can easily pierce human tissue before it stops, resulting in casualty. The same bullet fired with moderate speed will not cause much damage. Thus for a given mass, the greater the speed, the greater is the opposing force needed to stop the body in a certain time. Taken together, the product of mass and velocity, that is momentum, is evidently a relevant variable of motion. The greater the change in the momentum in a given time, the greater is the force that needs to be applied.
- A seasoned cricketer catches a cricket ball coming in with great speed far more easily than a novice, who can hurt his hands in the act. One reason is that the cricketer allows a longer time for his hands to stop the ball. As you may have noticed, he draws in the hands backward in the act of catching the ball (Fig. 5.3). The novice, on the other hand, keeps his hands fixed and tries to catch the ball almost instantly. He needs to provide a much greater force to stop the ball instantly, and

this hurts.The conclusion is clear: force not only depends on the change in momentum, but also on how fast the change is brought about. The same change in momentum brought about in a shorter time needs a greater applied force. In short, the greater the rate of change of momentum, the greater is the force.

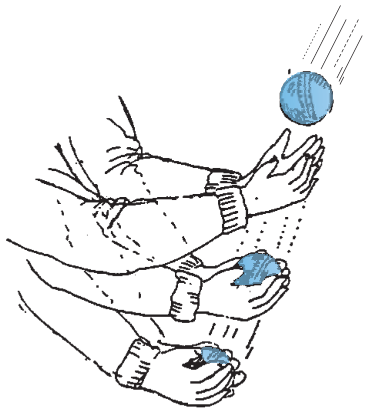


Fig. 5.3 *Force not only depends on the change in momentum but also on how fast the change is brought about. A seasoned cricketer draws in his hands during a catch, allowing greater time for the ball to stop and hence requires a smaller force.*

- Observations confirm that the product of mass and velocity (i.e. momentum) is basic to the effect of force on motion. Suppose a fixed force is applied for a certain interval of time on two bodies of different masses, initially at rest, the lighter body picks up a greater speed than the heavier body. However, at the end of the time interval, observations show that each body acquires the same momentum. **Thus the same force for the same time causes the same change in momentum for different bodies**. This is a crucial clue to the second law of motion.
- In the preceding observations, the vector character of momentum has not been evident. In the examples so far, momentum and change in momentum both have the same direction. But this is not always the case. Suppose a stone is rotated with uniform speed in a horizontal plane by means of a string, the magnitude of momentum is fixed, but its direction changes (Fig. 5.4). A force is needed to cause this change in momentum vector. This force is provided by our hand through the string. Experience suggests that our hand needs to exert a greater force if the stone is rotated at greater speed or in a circle of smaller radius, or both. This corresponds to greater acceleration or equivalently a greater rate of change in momentum vector. This suggests that the greater the rate of change in momentum vector the greater is the force applied.

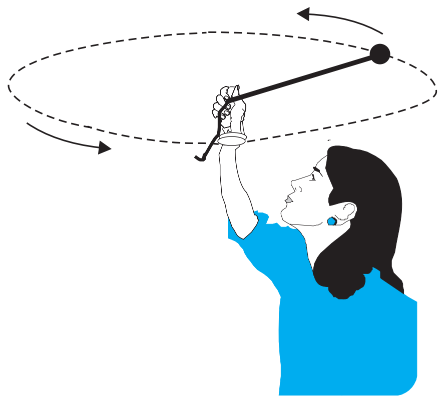


Fig. 5.4 *Force is necessary for changing the direction of momentum, even if its magnitude is constant. We can feel this while rotating a stone in a horizontal circle with uniform speed by means of a string.*

These qualitative observations lead to the **Second Law of motion** expressed by Newton as follows :

The rate of change of momentum of a body is directly proportional to the applied force and takes place in the direction in which the force acts.

Thus, if under the action of a force **F** for time interval Δt, the velocity of a body of mass m changes from $\mathbf{v}$ to $\mathbf{v} + \Delta\mathbf{v}$ i.e. its initial momentum $\boldsymbol{p} = m\,\mathbf{v}$ changes by $\Delta\boldsymbol{p} = m\Delta\boldsymbol{v}$. According to the Second Law,

$$\mathbf{F} \propto \frac{\Delta \mathbf{p}}{\Delta t} \quad \text{or} \quad \mathbf{F} = k\,\frac{\Delta \mathbf{p}}{\Delta t}$$

where k is a constant of proportionality. Taking the limit $\Delta t \to 0$, the term $\frac{\Delta \mathbf{p}}{\Delta t}$ becomes the derivative or differential co-efficient of **p** with respect to t, denoted by $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$. Thus

$$\mathbf{F} = k\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \quad (5.2)$$

For a body of fixed mass m,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\,\mathbf{v}) = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\,\mathbf{a} \quad (5.3)$$

i.e the Second Law can also be written as

$$\mathbf{F} = k\,m\,\mathbf{a} \quad (5.4)$$

which shows that force is proportional to the product of mass m and acceleration **a**.

The unit of force has not been defined so far. In fact, we use Eq. (5.4) to define the unit of force. We, therefore, have the liberty to choose any constant value for k. For simplicity, we choose $k = 1$. The Second Law then is

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\,\mathbf{a} \quad (5.5)$$

In SI unit force is one that causes an acceleration of 1 m s^{-2} to a mass of 1 kg. This unit is known as **newton** : 1 N = 1 kg m s^{-2}.

Let us note at this stage some important points about the second law :

1. In the second law, $\mathbf{F} = 0$ implies $\mathbf{a} = 0$. The second Law is obviously consistent with the first law.
2. The second law of motion is a vector law. It is equivalent to three equations, one for each component of the vectors :

$$F_x = \frac{\mathrm{d}p_x}{\mathrm{d}t} = ma_x$$

$$F_y = \frac{\mathrm{d}p_y}{\mathrm{d}t} = ma_y$$

$$F_z = \frac{\mathrm{d}p_z}{\mathrm{d}t} = m\,a_z \quad (5.6)$$

This means that if a force is not parallel to the velocity of the body, but makes some angle with it, it changes only the component of velocity along the direction of force. The component of velocity normal to the force remains unchanged. For example, in the motion of a projectile under the vertical gravitational force, the horizontal component of velocity remains unchanged (Fig. 5.5).

3. The second law of motion given by Eq. (5.5) is applicable to a single point particle. The force **F** in the law stands for the net external force on the particle and **a** stands for acceleration of the particle. It turns out, however, that the law in the same form applies to a rigid body or, even more generally, to a system of particles. In that case, **F** refers to the total external force on the system and **a** refers to the acceleration of the system as a whole. More precisely, **a** is the acceleration of the centre of mass of the system. **Any internal forces in the system are not to be included in F.**

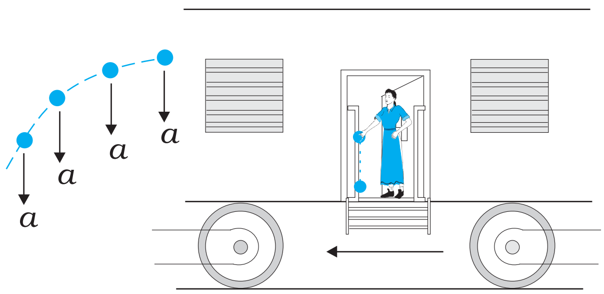


Fig. 5.5 *Acceleration at an instant is determined by the force at that instant. The moment after a stone is dropped out of an accelerated train, it has no horizontal acceleration or force, if air resistance is neglected. The stone carries no memory of its acceleration with the train a moment ago.*

4. The second law of motion is a local relation which means that force **F** at a point in space (location of the particle) at a certain instant of time is related to **a** at that point at that instant. Acceleration here and now is determined by the force here and now, **not by any history of the motion of the particle (See Fig. 5.5).**

> ▶ ***Example 5.2*** A bullet of mass 0.04 kg moving with a speed of 90 m s^{-1} enters a heavy wooden block and is stopped after a distance of 60 cm. What is the average resistive force exerted by the block on the bullet?

Answer The retardation 'a' of the bullet (assumed constant) is given by

$$a = \frac{-u^2}{2s} = \frac{-90 \times 90}{2 \times 0.6}\,\mathrm{m\,s^{-2}} = -6750\ \mathrm{m\,s^{-2}}$$

The retarding force, by the Second Law of motion, is

= 0.04 kg x 6750 m s^{-2} = 270 N

The actual resistive force, and therefore, retardation of the bullet may not be uniform. The answer therefore, only indicates the average resistive force. ◀

▶ ***Example 5.3*** The motion of a particle of mass m is described by y = $ut + \frac{1}{2}gt^2$. Find the force acting on the particle.

Answer We know

$$y = ut + \frac{1}{2}gt^2$$

Now,

$$v = \frac{dy}{dt} = u + gt$$

acceleration, $a = \frac{dv}{dt} = g$

Then the force is given by Eq. (5.5)

F = ma = mg ◀

Thus the given equation describes the motion of a particle under acceleration due to gravity and y is the position coordinate in the direction of g.

Impulse

We sometimes encounter examples where a large force acts for a very short duration producing a finite change in momentum of the body. For example, when a ball hits a wall and bounces back, the force on the ball by the wall acts for a very short time when the two are in contact, yet the force is large enough to reverse the momentum of the ball. Often, in these situations, the force and the time duration are difficult to ascertain separately. However, the product of force and time, which is the change in momentum of the body remains a measurable quantity. This product is called impulse:

Impulse = Force × time duration

= Change in momentum (5.7)

A large force acting for a short time to produce a finite change in momentum is called an *impulsive force*. In the history of science, impulsive forces were put in a conceptually different category from ordinary forces. Newtonian mechanics has no such distinction. Impulsive force is like any other force – except that it is large and acts for a short time.

▶ ***Example 5.4*** A batsman hits back a ball straight in the direction of the bowler without changing its initial speed of 12 m s^{-1}. If the mass of the ball is 0.15 kg, determine the impulse imparted to the ball. (Assume linear motion of the ball)

Answer Change in momentum

= 0.15 × 12–(–0.15×12)

= 3.6 N s,

Impulse = 3.6 N s,

in the direction from the batsman to the bowler.

This is an example where the force on the ball by the batsman and the time of contact of the ball and the bat are difficult to know, but the impulse is readily calculated. ◀

5.6 NEWTON'S THIRD LAW OF MOTION

The second law relates the external force on a body to its acceleration. What is the origin of the external force on the body ? What agency provides the external force ? The simple answer in Newtonian mechanics is that the external force on a body always arises due to some other body. Consider a pair of bodies *A* and *B*. *B* gives rise to an external force on *A*. A natural question is: Does *A* in turn give rise to an external force on *B* ? In some examples, the answer seems clear. If you press a coiled spring, the spring is compressed by the force of your hand. The compressed spring in turn exerts a force on your hand and you can feel it. But what if the bodies are not in contact ? The earth pulls a stone downwards due to gravity. Does the stone exert a force on the earth ? The answer is not obvious since we hardly see the effect of the stone on the earth. The answer according to Newton is: Yes, the stone does exert an equal and opposite force on the earth. We do not notice it since the earth is very massive and the effect of a small force on its motion is negligible.

Thus, according to Newtonian mechanics, force never occurs singly in nature. Force is the mutual interaction between two bodies. Forces

always occur in pairs. Further, the mutual forces between two bodies are always equal and opposite. This idea was expressed by Newton in the form of the **third law of motion**.

To every action, there is always an equal and opposite reaction.

Newton's wording of the third law is so crisp and beautiful that it has become a part of common language. For the same reason perhaps, misconceptions about the third law abound. Let us note some important points about the third law, particularly in regard to the usage of the terms : action and reaction.

1. The terms action and reaction in the third law mean nothing else but 'force'. Using different terms for the same physical concept can sometimes be confusing. A simple and clear way of stating the third law is as follows :

   Forces always occur in pairs. Force on a body A by B is equal and opposite to the force on the body B by A.

2. The terms action and reaction in the third law may give a wrong impression that action comes before reaction i.e action is the cause and reaction the effect. **There is no cause-effect relation implied in the third law. The force on *A* by *B* and the force on *B* by *A* act at the same instant.** By the same reasoning, any one of them may be called action and the other reaction.

3. Action and reaction forces act on different bodies, not on the same body. Consider a pair of bodies *A* and *B*. According to the third law,

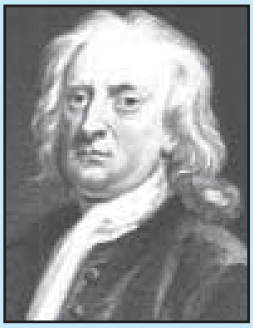
$$\mathbf{F}_{AB} = -\mathbf{F}_{BA} \qquad (5.8)$$

(force on *A* by *B*) = – (force on *B* by *A*)

Thus if we are considering the motion of any one body (*A* or *B*), only one of the two forces is relevant. It is an error to add up the two forces and claim that the net force is zero.

However, if you are considering the system of two bodies as a whole, $\mathbf{F}_{AB}$ and $\mathbf{F}_{BA}$ are internal forces of the system ($A + B$). They add up to give a null force. Internal forces in a body or a system of particles thus cancel away in pairs. This is an important fact that enables the second law to be applicable to a body or a system of particles (See Chapter 7).

Isaac Newton (1642 – 1727)



Isaac Newton was born in Woolsthorpe, England in 1642, the year Galileo died. His extraordinary mathematical ability and mechanical aptitude remained hidden from others in his school life. In 1662, he went to Cambridge for undergraduate studies. A plague epidemic in 1665 forced the university town to close and Newton had to return to his mother's farm. There in two years of solitude, his dormant creativity blossomed in a deluge of fundamental discoveries in mathematics and physics : binomial theorem for negative and fractional exponents, the beginning of calculus, the inverse square law of gravitation, the spectrum of white light, and so on. Returning to Cambridge, he pursued his investigations in optics and devised a reflecting telescope.

In 1684, encouraged by his friend Edmund Halley, Newton embarked on writing what was to be one of the greatest scientific works ever published : The Principia Mathematica. In it, he enunciated the three laws of motion and the universal law of gravitation, which explained all the three Kepler's laws of planetary motion. The book was packed with a host of path-breaking achievements : basic principles of fluid mechanics, mathematics of wave motion, calculation of masses of the earth, the sun and other planets, explanation of the precession of equinoxes, theory of tides, etc. In 1704, Newton brought out another masterpiece Opticks that summarized his work on light and colour.

The scientific revolution triggered by Copernicus and steered vigorously ahead by Kepler and Galileo was brought to a grand completion by Newton. Newtonian mechanics unified terrestrial and celestial phenomena. The same mathematical equation governed the fall of an apple to the ground and the motion of the moon around the earth. The age of reason had dawned.

▶ ***Example 5.5*** Two identical billiard balls strike a rigid wall with the same speed but at different angles, and get reflected without any change in speed, as shown in Fig. 5.6. What is (i) the direction of the force on the wall due to each ball? (ii) the ratio of the magnitudes of impulses imparted to the balls by the wall ?

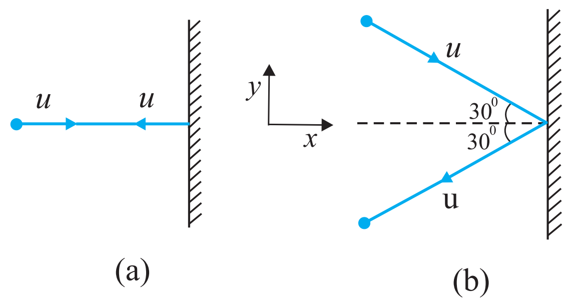


Fig. 5.6

Answer An instinctive answer to (i) might be that the force on the wall in case (a) is normal to the wall, while that in case (b) is inclined at 30° to the normal. This answer is wrong. The force on the wall is normal to the wall in both cases.

How to find the force on the wall? The trick is to consider the force (or impulse) on the ball due to the wall using the second law, and then use the third law to answer (i). Let u be the speed of each ball before and after collision with the wall, and m the mass of each ball. Choose the x and y axes as shown in the figure, and consider the change in momentum of the ball in each case :

Case (a)

$$(p_x)_{\text{initial}} = mu \qquad (p_y)_{\text{initial}} = 0$$

$$(p_x)_{\text{final}} = -mu \qquad (p_y)_{\text{final}} = 0$$

Impulse is the change in momentum vector. Therefore,

x-component of impulse = – 2 $m\,u$
y-component of impulse = 0

Impulse and force are in the same direction. Clearly, from above, the force on the ball due to the wall is normal to the wall, along the negative x-direction. Using Newton's third law of motion, the force on the wall due to the ball is normal to the wall along the positive x-direction. The magnitude of force cannot be ascertained since the small time taken for the collision has not been specified in the problem.

Case (b)

$$(p_x)_{initial} = m\,u\cos 30^\circ,\ (p_y)_{initial} = -\,m\,u\sin 30^\circ$$

$$(p_x)_{final} = -\,m\,u\cos 30^\circ,\ (p_y)_{final} = -\,m\,u\sin 30^\circ$$

Note, while p_x changes sign after collision, p_y does not. Therefore,

x-component of impulse = –2 $m\,u$ cos 30°
y-component of impulse = 0

The direction of impulse (and force) is the same as in (a) and is normal to the wall along the negative x direction. As before, using Newton's third law, the force on the wall due to the ball is normal to the wall along the positive x direction.

The ratio of the magnitudes of the impulses imparted to the balls in (a) and (b) is

$$2\,m\,u/\left(2\,m\,u\cos 30^\circ\right) = \frac{2}{\sqrt{3}} \approx 1.2$$ ◀

5.7 CONSERVATION OF MOMENTUM

The second and the third laws of motion lead to an important consequence: the law of conservation of momentum. Take a familiar example. A bullet is fired from a gun. If the force on the bullet by the gun is **F**, the force on the gun by the bullet is –**F**, according to the third law. The two forces act for a common interval of time Δt. According to the second law, **F** Δt is the change in momentum of the bullet and –**F** Δt is the change in momentum of the gun. Since initially, both are at rest, the change in momentum equals the final momentum for each. Thus if $\mathbf{p}_b$ is the momentum of the bullet after firing and $\mathbf{p}_g$ is the recoil momentum of the gun, $\mathbf{p}_g = -\mathbf{p}_b$ i.e. $\mathbf{p}_b + \mathbf{p}_g = 0$. That is, the total momentum of the (bullet + gun) system is conserved.

Thus in an isolated system (i.e. a system with no external force), mutual forces between pairs of particles in the system can cause momentum change in individual particles, but since the mutual forces for each pair are equal and opposite, the momentum changes cancel in pairs and the total momentum remains unchanged. This fact is known as the **law of conservation of momentum** :

The total momentum of an isolated system of interacting particles is conserved.

An important example of the application of the law of conservation of momentum is the collision of two bodies. Consider two bodies A and B, with initial momenta $\mathbf{p}_A$ and $\mathbf{p}_B$. The bodies collide, get apart, with final momenta $\mathbf{p}'_A$ and $\mathbf{p}'_B$ respectively. By the Second Law

$$\mathbf{F}_{AB}\Delta t = \mathbf{p}'_A - \mathbf{p}_A \quad \text{and}$$

$$\mathbf{F}_{BA}\Delta t = \mathbf{p}'_B - \mathbf{p}_B$$

(where we have taken a common interval of time for both forces i.e. the time for which the two bodies are in contact.)

Since $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$ by the third law,

$$\mathbf{p}'_A - \mathbf{p}_A = -(\mathbf{p}'_B - \mathbf{p}_B)$$

i.e. $$\mathbf{p}'_A + \mathbf{p}'_B = \mathbf{p}_A + \mathbf{p}_B \qquad (5.9)$$

which shows that the total final momentum of the isolated system equals its initial momentum. Notice that this is true whether the collision is elastic or inelastic. In elastic collisions, there is a second condition that the total initial kinetic energy of the system equals the total final kinetic energy (See Chapter 6).

5.8 EQUILIBRIUM OF A PARTICLE

Equilibrium of a particle in mechanics refers to the situation when the net external force on the particle is zero.* According to the first law, this means that, the particle is either at rest or in uniform motion.

If two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, act on a particle, equilibrium requires

$$\mathbf{F}_1 = -\mathbf{F}_2 \qquad (5.10)$$

i.e. the two forces on the particle must be equal and opposite. Equilibrium under three concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ requires that the vector sum of the three forces is zero.

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0 \qquad (5.11)$$

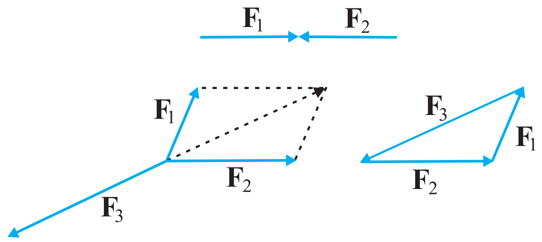


Fig. 5.7 *Equilibrium under concurrent forces.*

In other words, the resultant of any two forces say $\mathbf{F}_1$ and $\mathbf{F}_2$, obtained by the parallelogram law of forces must be equal and opposite to the third force, $\mathbf{F}_3$. As seen in Fig. 5.7, the three forces in equilibrium can be represented by the sides of a triangle with the vector arrows taken in the same sense. The result can be generalised to any number of forces. A particle is in equilibrium under the action of forces $\mathbf{F}_1$, $\mathbf{F}_2$,... $\mathbf{F}_n$ if they can be represented by the sides of a closed n-sided polygon with arrows directed in the same sense.

Equation (5.11) implies that

$$F_{1x} + F_{2x} + F_{3x} = 0$$
$$F_{1y} + F_{2y} + F_{3y} = 0$$
$$F_{1z} + F_{2z} + F_{3z} = 0 \qquad (5.12)$$

where F_{1x}, F_{1y} and F_{1z} are the components of F_1 along x, y and z directions respectively.

▶ ***Example 5.6*** See Fig. 5.8 A mass of 6 kg is suspended by a rope of length 2 m from the ceiling. A force of 50 N in the horizontal direction is applied at the mid-point P of the rope, as shown. What is the angle the rope makes with the vertical in equilibrium ? (Take g = 10 m s^{-2}). Neglect the mass of the rope.

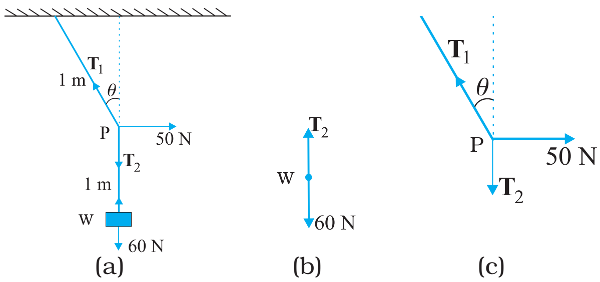


Fig. 5.8

* *Equilibrium of a body requires not only translational equilibrium (zero net external force) but also rotational equilibrium (zero net external torque), as we shall see in Chapter 7.*

Answer Figures 5.8(b) and 5.8(c) are known as free-body diagrams. Figure 5.8(b) is the free-body diagram of W and Fig. 5.8(c) is the free-body diagram of point P.

Consider the equilibrium of the weight W. Clearly, $T_2 = 6 \times 10 = 60$ N.

Consider the equilibrium of the point P under the action of three forces - the tensions T_1 and T_2, and the horizontal force 50 N. The horizontal and vertical components of the resultant force must vanish separately :

$$T_1 \cos\theta = T_2 = 60 \text{ N}$$

$$T_1 \sin\theta = 50 \text{ N}$$

which gives that

$$\tan\boldsymbol{\theta} = \frac{5}{6} \text{ or } \boldsymbol{\theta} = \tan^{-1}\left(\frac{5}{6}\right) = 40^{\circ}$$

Note the answer does not depend on the length of the rope (assumed massless) nor on the point at which the horizontal force is applied. ◀

5.9 COMMON FORCES IN MECHANICS

In mechanics, we encounter several kinds of forces. The gravitational force is, of course, all pervasive. Every object on the earth experiences the force of gravity due to the earth. Gravity also governs the motion of celestial bodies. The gravitational force can act at a distance without the need of any intervening medium.

All the other forces common in mechanics are contact forces.* As the name suggests, a contact force on an object arises due to contact with some other object: solid or fluid. When bodies are in contact (e.g. a book resting on a table, a system of rigid bodies connected by rods, hinges and other types of supports), there are mutual contact forces (for each pair of bodies) satisfying the third law. The component of contact force normal to the surfaces in contact is called normal reaction. The component parallel to the surfaces in contact is called friction. Contact forces arise also when solids are in contact with fluids. For example, for a solid immersed in a fluid, there is an upward bouyant force equal to the weight of the fluid displaced. The viscous force, air resistance, etc are also examples of contact forces (Fig. 5.9).

Two other common forces are tension in a string and the force due to spring. When a spring is compressed or extended by an external force, a restoring force is generated. This force is usually proportional to the compression or elongation (for small displacements). The spring force F is written as $F = -k\,x$ where x is the displacement and k is the force constant. The negative sign denotes that the force is opposite to the displacement from the unstretched state. For an inextensible string, the force constant is very high. The restoring force in a string is called tension. It is customary to use a constant tension T throughout the string. This assumption is true for a string of negligible mass.

In Chapter 1, we learnt that there are four fundamental forces in nature. Of these, the weak and strong forces appear in domains that do not concern us here. Only the gravitational and electrical forces are relevant in the context of mechanics. The different contact forces of mechanics mentioned above fundamentally arise from electrical forces. This may seem surprising

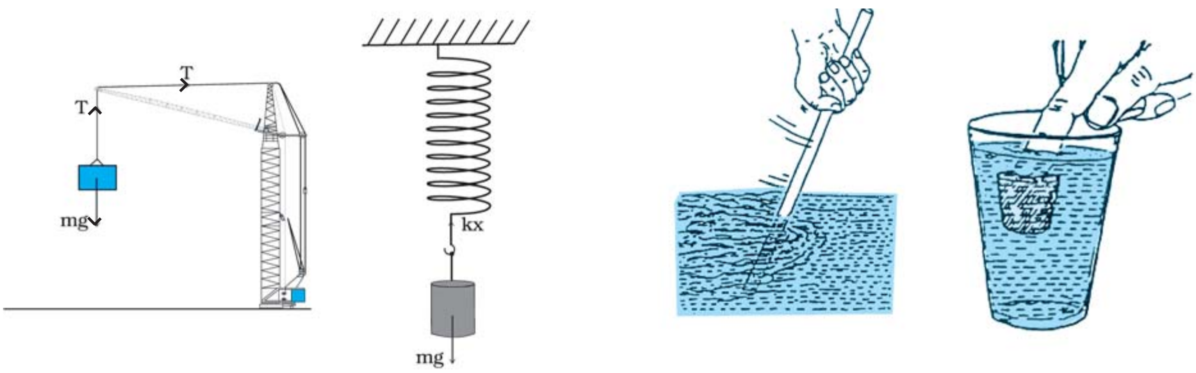


Fig. 5.9 *Some examples of contact forces in mechanics.*

* *We are not considering, for simplicity, charged and magnetic bodies. For these, besides gravity, there are electrical and magnetic non-contact forces.*

since we are talking of uncharged and non-magnetic bodies in mechanics. At the microscopic level, all bodies are made of charged constituents (nuclei and electrons) and the various contact forces arising due to elasticity of bodies, molecular collisions and impacts, etc. can ultimately be traced to the electrical forces between the charged constituents of different bodies. The detailed microscopic origin of these forces is, however, complex and not useful for handling problems in mechanics at the macroscopic scale. This is why they are treated as different types of forces with their characteristic properties determined empirically.

5.9.1 Friction

Let us return to the example of a body of mass m at rest on a horizontal table. The force of gravity (*mg*) is cancelled by the normal reaction force (N) of the table. Now suppose a force *F* is applied horizontally to the body. We know from experience that a small applied force may not be enough to move the body. But if the applied force *F* were the only external force on the body, it must move with acceleration F/m, however small. Clearly, the body remains at rest because some other force comes into play in the horizontal direction and opposes the applied force *F*, resulting in zero net force on the body. This force f_s parallel to the surface of the body in contact with the table is known as frictional force, or simply friction (Fig. 5.10(a)). The subscript stands for static friction to distinguish it from kinetic friction f_k that we consider later (Fig. 5.10(b)). Note that static friction does not exist by itself. When there is no applied force, there is no static friction. It comes into play the moment there is an applied force. As the applied force *F* increases, f_s also increases, remaining equal and opposite to the applied force (up to a certain limit), keeping the body at rest. Hence, it is called **static friction**. Static friction opposes **impending motion**. The term impending motion means motion that would take place (but does not actually take place) under the applied force, if friction were absent.

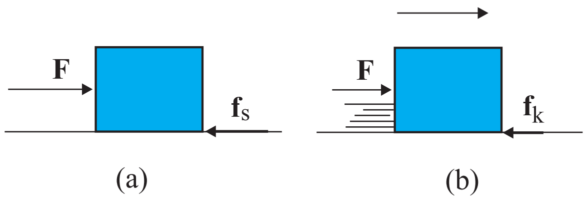


Fig. 5.10 *Static and sliding friction: (a) Impending motion of the body is opposed by static friction. When external force exceeds the maximum limit of static friction, the body begins to move. (b) Once the body is in motion, it is subject to sliding or kinetic friction which opposes relative motion between the two surfaces in contact. Kinetic friction is usually less than the maximum value of static friction.*

We know from experience that as the applied force exceeds a certain limit, the body begins to move. It is found experimentally that the limiting value of static friction $\left(f_s\right)_{\max}$ is independent of the area of contact and varies with the normal force(*N*) approximately as :

$$\left(f_s\right)_{\max} = \mu_s \mathrm{N} \qquad (5.13)$$

where μ_s is a constant of proportionality depending only on the nature of the surfaces in contact. The constant μ_s is called the coefficient of static friction. The law of static friction may thus be written as

$$f_s \le \mu_s \ \mathrm{N} \qquad (5.14)$$

If the applied force *F* exceeds $\left(f_s\right)_{\max}$ the body begins to slide on the surface. It is found experimentally that when relative motion has started, the frictional force decreases from the static maximum value $\left(f_s\right)_{\max}$. Frictional force that opposes relative motion between surfaces in contact is called kinetic or sliding friction and is denoted by f_k. Kinetic friction, like static friction, is found to be independent of the area of contact. Further, it is nearly independent of the velocity. It satisfies a law similar to that for static friction:

$$f_k = \mu_k \mathbf{N} \qquad (5.15)$$

where μ_k, the coefficient of kinetic friction, depends only on the surfaces in contact. As mentioned above, experiments show that μ_k is less than μ_s. When relative motion has begun, the acceleration of the body according to the Second Law is $(F - f_k)/m$. For a body moving with constant velocity, $F = f_k$. If the applied force on the body is removed, its acceleration is $-f_k/m$ and it eventually comes to a stop.

The laws of friction given above do not have the status of fundamental laws like those for gravitational, electric and magnetic forces. They are empirical relations that are only

approximately true. Yet they are very useful in practical calculations in mechanics.

Thus, when two bodies are in contact, each experiences a contact force by the other. Friction, by definition, is the component of the contact force parallel to the surfaces in contact, which opposes impending or actual relative motion between the two surfaces. Note that it is not motion, but **relative motion** that the frictional force opposes. Consider a box lying in the compartment of a train that is accelerating. If the box is stationary relative to the train, it is in fact accelerating along with the train. What forces cause the acceleration of the box? Clearly, the only conceivable force in the horizontal direction is the force of friction. If there were no friction, the floor of the train would slip by and the box would remain at its initial position due to inertia (and hit the back side of the train). This impending relative motion is opposed by the static friction f_s. Static friction provides the same acceleration to the box as that of the train, keeping it stationary relative to the train.

Example 5.7 Determine the maximum acceleration of the train in which a box lying on its floor will remain stationary, given that the co-efficient of static friction between the box and the train's floor is 0.15.

Answer Since the acceleration of the box is due to the static friction,

$$ma = f_s \le \mu_s N = \mu_s\ m g$$

i.e. $a \le \mu_s g$

$$\therefore a_{max} = \mu_s g = 0.15 \text{ x } 10 \text{ m s}^{-2}$$
$$= 1.5 \text{ m s}^{-2}$$

Example 5.8 See Fig. 5.11. A mass of 4 kg rests on a horizontal plane. The plane is gradually inclined until at an angle θ = 15° with the horizontal, the mass just begins to slide. What is the coefficient of static friction between the block and the surface ?

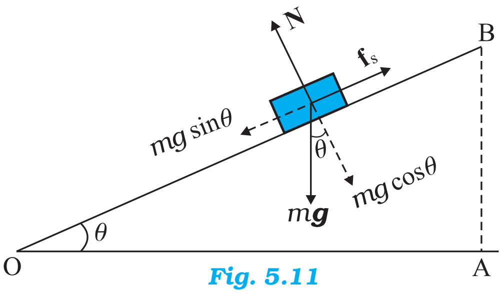


Fig. 5.11

Answer The forces acting on a block of mass *m* at rest on an inclined plane are (i) the weight mg acting vertically downwards (ii) the normal force *N* of the plane on the block, and (iii) the static frictional force f_s opposing the impending motion. In equilibrium, the resultant of these forces must be zero. Resolving the weight *mg* along the two directions shown, we have

$$m g \sin\theta = f_s \ , \quad m g \cos\theta = N$$

As θ increases, the self-adjusting frictional force f_s increases until at $\theta = \theta_{max}$, f_s achieves its maximum value, $(f_s)_{max} = \mu_s N$.

Therefore,

$$\tan\theta_{max} = \mu_s \quad or \quad \theta_{max} = \tan^{-1} \mu_s$$

When θ becomes just a little more than θ_{max}, there is a small net force on the block and it begins to slide. Note that θ_{max} depends only on μ_s and is independent of the mass of the block.

For θ_{max} = 15°,
μ_s = tan 15°
= 0.27

Example 5.9 What is the acceleration of the block and trolley system shown in a Fig. 5.12(a), if the coefficient of kinetic friction between the trolley and the surface is 0.04? What is the tension in the string? (Take g = 10 m s^{-2}). Neglect the mass of the string.

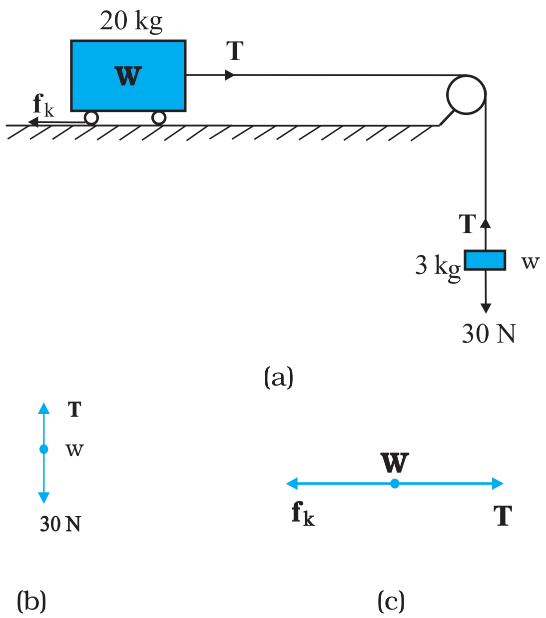


Fig. 5.12

Answer As the string is inextensible, and the pully is smooth, the 3 kg block and the 20 kg trolley both have same magnitude of acceleration. Applying second law to motion of the block (Fig. 5.12(b)),

$30 - T = 3a$

Apply the second law to motion of the trolley (Fig. 5.12(c)),

$T - f_k = 20\ a.$

Now $f_k = \mu_k\ N,$

Here $\mu_k = 0.04,$

$N = 20 \times 10$

$= 200$ N.

Thus the equation for the motion of the trolley is

$T - 0.04 \times 200 = 20\ a$ Or $T - 8 = 20a.$

These equations give $a = \frac{22}{23}$ m s^{-2} = 0.96 m s^{-2} and $T = 27.1$ N. ◄

Rolling friction

A body like a ring or a sphere rolling without slipping over a horizontal plane will suffer no friction, in principle. At every instant, there is just one point of contact between the body and the plane and this point has no motion relative to the plane. In this ideal situation, kinetic or static friction is zero and the body should continue to roll with constant velocity. We know, in practice, this will not happen and some resistance to motion (rolling friction) does occur, i.e. to keep the body rolling, some applied force is needed. For the same weight, rolling friction is much smaller (even by 2 or 3 orders of magnitude) than static or sliding friction. This is the reason why discovery of the wheel has been a major milestone in human history.

Rolling friction again has a complex origin, though somewhat different from that of static and sliding friction. During rolling, the surfaces in contact get momentarily deformed a little, and this results in a finite area (not a point) of the body being in contact with the surface. The net effect is that the component of the contact force parallel to the surface opposes motion.

We often regard friction as something undesirable. In many situations, like in a machine with different moving parts, friction does have a negative role. It opposes relative motion and thereby dissipates power in the form of heat, etc. Lubricants are a way of reducing kinetic friction in a machine. Another way is to use ball bearings between two moving parts of a machine. (Fig. 5.13(a)) Since the rolling friction between ball bearings and the surfaces in contact is very small, power dissipation is reduced. A thin cushion of air maintained between solid surfaces in relative motion is another effective way of reducing friction (Fig. 5.13(a)).

In many practical situations, however, friction is critically needed. Kinetic friction that dissipates power is nevertheless important for quickly stopping relative motion. It is made use of by brakes in machines and automobiles. Similarly, static friction is important in daily life. We are able to walk because of friction. It is impossible for a car to move on a very slippery road. On an ordinary road, the friction between the tyres and the road provides the necessary external force to accelerate the car.

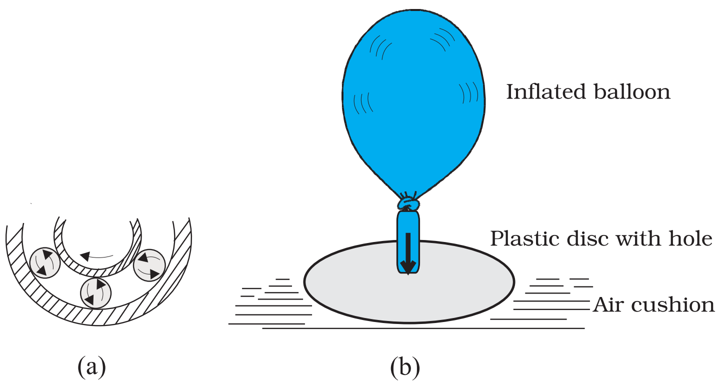


Fig. 5.13 *Some ways of reducing friction. (a) Ball bearings placed between moving parts of a machine. (b) Compressed cushion of air between surfaces in relative motion.*

5.10 CIRCULAR MOTION

We have seen in Chapter 4 that acceleration of a body moving in a circle of radius R with uniform speed v is v^2/R directed towards the centre. According to the second law, the force f providing this acceleration is :

$$f = \frac{mv^2}{R} \qquad (5.16)$$

where m is the mass of the body. This force directed forwards the centre is called the centripetal force. For a stone rotated in a circle by a string, the centripetal force is provided by the tension in the string. The centripetal force for motion of a planet around the sun is the gravitational force on the planet due to the sun. For a car taking a circular turn on a horizontal road, the centripetal force is the force of friction.

The circular motion of a car on a flat and banked road give interesting application of the laws of motion.

Motion of a car on a level road

Three forces act on the car. (Fig. 5.14(a)

(i) The weight of the car, mg

(ii) Normal reaction, N

(iii) Frictional force, f

As there is no acceleration in the vertical direction

$N – mg = 0$

$N = mg \qquad (5.17)$

The centripetal force required for circular motion is along the surface of the road, and is provided by the component of the contact force between road and the car tyres along the surface. This by definition is the frictional force. Note that it is the static friction that provides the centripetal acceleration. Static friction opposes the impending motion of the car moving away from the circle. Using equation (5.14) & (5.16) we get the result

$$f \le \mu_s N = \frac{mv^2}{R}$$

$$v^2 \le \frac{\mu_s RN}{m} = \mu_s Rg \qquad [\because N = mg]$$

which is independent of the mass of the car. This shows that for a given value of μ_s and R, there is a maximum speed of circular motion of the car possible, namely

$$v_{max} = \sqrt{\mu_s Rg} \qquad (5.18)$$

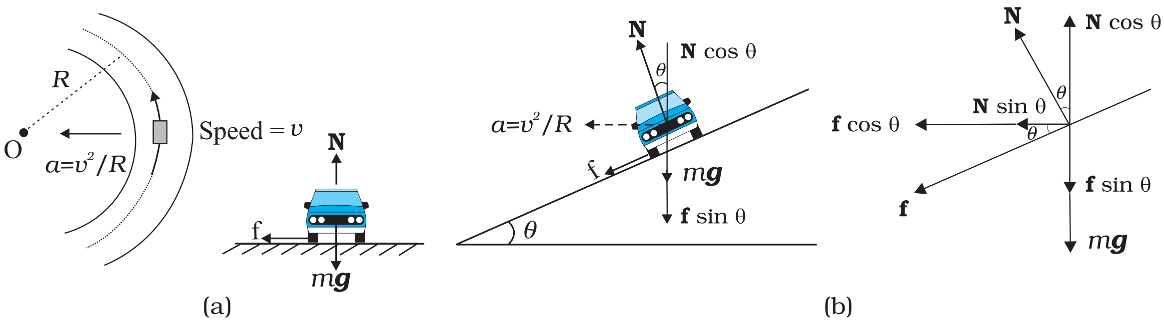


Fig. 5.14 *Circular motion of a car on (a) a level road, (b) a banked road.*

Motion of a car on a banked road

We can reduce the contribution of friction to the circular motion of the car if the road is banked (Fig. 5.14(b)). Since there is no acceleration along the vertical direction, the net force along this direction must be zero. Hence,

$$N \cos \theta = mg + f \sin \theta \qquad (5.19a)$$

The centripetal force is provided by the horizontal components of N and f.

$$N \sin \theta + f \cos \theta = \frac{mv^2}{R} \qquad (5.20a)$$

But $f \le \mu_s N$

Thus to obtain v_{max} we put

$f = \mu_s N$.

Then Eqs. (5.19) & (5.20) become

$$N \cos \theta = mg + \mu_s N \sin \theta \qquad (5.19b)$$

$N \sin\theta + \mu_s N \cos\theta = mv^2/R$ (5.20b)

We obtain

$$N = \frac{mg}{\cos\theta - \mu_s \sin\theta}$$

Substituting value of N in Eq. (5.20b), we get

$$\frac{mg(\sin\theta - \mu_s \cos\theta)}{\cos\theta - \mu_s \sin\theta} = \frac{mv_{max}^2}{R}$$

$$\text{or } v_{max} = \left(Rg\frac{\mu_s + \tan\theta}{1 - \mu_s \tan\theta}\right)^{\frac{1}{2}} \quad (5.21)$$

Comparing this with Eq. (5.18) we see that maximum possible speed of a car on a banked road is greater than that on a flat road.

For $\mu_s = 0$ in Eq. (5.21),

$v_o = (R g \tan\theta)^{1/2}$ (5.22)

At this speed, frictional force is not needed at all to provide the necessary centripetal force. Driving at this speed on a banked road will cause little wear and tear of the tyres. The same equation also tells you that for $v < v_o$, frictional force will be up the slope and that a car can be parked only if $\tan\theta \le \mu_s$.

Example 5.10 A cyclist speeding at 18 km/h on a level road takes a sharp circular turn of radius 3 m without reducing the speed. The co-efficient of static friction between the tyres and the road is 0.1. Will the cyclist slip while taking the turn ?

Answer On an unbanked road, frictional force alone can provide the centripetal force needed to keep the cyclist moving on a circular turn without slipping. If the speed is too large, or if the turn is too sharp (i.e. of too small a radius) or both, the frictional force is not sufficient to provide the necessary centripetal force, and the cyclist slips. The condition for the cyclist not to slip is given by Eq. (5.18) :

$$v^2 \le \mu_s R g$$

Now, $R = 3$ m, $g = 9.8$ m s^{-2}, $\mu_s = 0.1$. That is, $\mu_s R g = 2.94$ m^2 s^{-2}. $v = 18$ km/h $= 5$ m s^{-1}; i.e., $v^2 = 25$ m^2 s^{-2}. The condition is not obeyed. The cyclist will slip while taking the circular turn. ◀

Example 5.11 A circular racetrack of radius 300 m is banked at an angle of 15°. If the coefficient of friction between the wheels of a race-car and the road is 0.2, what is the (a) optimum speed of the race-car to avoid wear and tear on its tyres, and (b) maximum permissible speed to avoid slipping ?

Answer On a banked road, the horizontal component of the normal force and the frictional force contribute to provide centripetal force to keep the car moving on a circular turn without slipping. At the optimum speed, the normal reaction's component is enough to provide the needed centripetal force, and the frictional force is not needed. The optimum speed v_o is given by Eq. (5.22):

$v_o = (R g \tan\theta)^{1/2}$

Here $R = 300$ m, $\theta = 15°$, $g = 9.8$ m s^{-2}; we have

$v_o = 28.1$ m s^{-1}.

The maximum permissible speed v_{max} is given by Eq. (5.21):

$$v_{max} = \left(Rg\frac{\mu_s + \tan\theta}{1 - \mu_s \tan\theta}\right)^{1/2} = 38.1 \text{ m s}^{-1}$$ ◀

5.11 SOLVING PROBLEMS IN MECHANICS

The three laws of motion that you have learnt in this chapter are the foundation of mechanics. You should now be able to handle a large variety of problems in mechanics. A typical problem in mechanics usually does not merely involve a single body under the action of given forces. More often, we will need to consider an assembly of different bodies exerting forces on each other. Besides, each body in the assembly experiences the force of gravity. When trying to solve a problem of this type, it is useful to remember the fact that we can choose any part of the assembly and apply the laws of motion to that part provided we include all forces on the chosen part due to the remaining parts of the assembly. We may call the chosen part of the assembly as the system and the remaining part of the assembly (plus any other agencies of forces) as the environment. We have followed the same

method in solved examples. To handle a typical problem in mechanics systematically, one should use the following steps :

(i) Draw a diagram showing schematically the various parts of the assembly of bodies, the links, supports, etc.

(ii) Choose a convenient part of the assembly as one system.

(iii) Draw a separate diagram which shows this system and all the forces on the system by the remaining part of the assembly. Include also the forces on the system by other agencies. **Do not include the forces on the environment by the system**. A diagram of this type is known as 'a free-body diagram'. (Note this does not imply that the system under consideration is without a net force).

(iv) In a free-body diagram, include information about forces (their magnitudes and directions) that are either given or you are sure of (e.g., the direction of tension in a string along its length). The rest should be treated as unknowns to be determined using laws of motion.

(v) If necessary, follow the same procedure for another choice of the system. In doing so, employ Newton's third law. That is, if in the free-body diagram of A, the force on A due to B is shown as **F**, then in the free-body diagram of B, the force on B due to A should be shown as –**F**.

The following example illustrates the above procedure :

▶ ***Example 5.12*** See (Fig. 5.15) A wooden block of mass 2 kg rests on a soft horizontal floor. When an iron cylinder of mass 25 kg is placed on top of the block, the floor yields steadily and the block and the cylinder together go down with an acceleration of 0.1 m s^{-2}. What is the action of the block on the floor (a) before and (b) after the floor yields ? Take g = 10 m s^{-2}. Identify the action-reaction pairs in the problem.

Answer

(a) The block is at rest on the floor. Its free-body diagram shows two forces on the block, the force of gravitational attraction by the earth equal to 2 × 10 = 20 N; and the normal force R of the floor on the block. By the First Law, the net force on the block must be zero i.e., R = 20 N. Using third law the action of the block (i.e. the force exerted on the floor by the block) is equal to 20 N and directed vertically downwards.

(b) The system (block + cylinder) accelerates downwards with 0.1 m s^{-2}. The free-body diagram of the system shows two forces on the system : the force of gravity due to the earth (270 N); and the normal force R' by the floor. Note, the free-body diagram of the system does not show the internal forces between the block and the cylinder. Applying the second law to the system,

$$270 - R' = 27 \times 0.1\text{N}$$
$$\text{ie. } R' = 267.3 \text{ N}$$

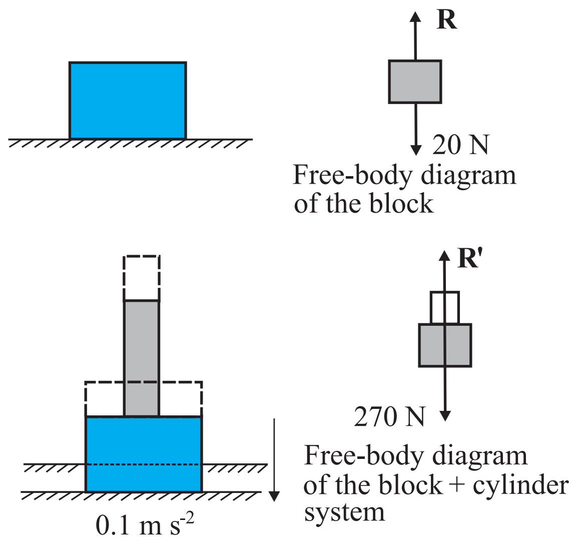


Fig. 5.15

By the third law, the action of the system on the floor is equal to 267.3 N vertically downward.

Action-reaction pairs

For (a): (i) the force of gravity (20 N) on the block by the earth (say, action); the force of gravity on the earth by the block (reaction) equal to 20 N directed upwards (not shown in the figure).
(ii) the force on the floor by the block (action); the force on the block by the floor (reaction).

For (b): (i) the force of gravity (270 N) on the system by the earth (say, action); the force of gravity on the earth by the system (reaction), equal to 270 N,

directed upwards (not shown in the figure).
(ii) the force on the floor by the system (action); the force on the system by the floor (reaction). In addition, for (b), the force on the block by the cylinder and the force on the cylinder by the block also constitute an action-reaction pair.

The important thing to remember is that an action-reaction pair consists of mutual forces which are always equal and opposite between two bodies. Two forces on the same body which happen to be equal and opposite can never constitute an action-reaction pair. The force of gravity on the mass in (a) or (b) and the normal force on the mass by the floor are not action-reaction pairs. These forces happen to be equal and opposite for (a) since the mass is at rest. They are not so for case (b), as seen already. The weight of the system is 270 N, while the normal force R' is 267.3 N. ◀

The practice of drawing free-body diagrams is of great help in solving problems in mechanics. It allows you to clearly define your system and consider all forces on the system due to objects that are not part of the system itself. A number of exercises in this and subsequent chapters will help you cultivate this practice.

SUMMARY

1. Aristotle's view that a force is necessary to keep a body in uniform motion is wrong. A force is necessary in practice to counter the opposing force of friction.
2. Galileo extrapolated simple observations on motion of bodies on inclined planes, and arrived at the law of inertia. Newton's first law of motion is the same law rephrased thus: *"Everybody continues to be in its state of rest or of uniform motion in a straight line, unless compelled by some external force to act otherwise"*. In simple terms, the First Law is **"If external force on a body is zero, its acceleration is zero"**.
3. Momentum (**p**) of a body is the product of its mass *(m)* and velocity ***(v)*** :

   $$\mathbf{p} = m\mathbf{v}$$
4. Newton's second law of motion :
   The rate of change of momentum of a body is proportional to the applied force and takes place in the direction in which the force acts. Thus

   $$\mathbf{F} = k\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = k\,m\,\mathbf{a}$$

   where **F** is the net external force on the body and **a** its acceleration. We set the constant of proportionality $k = 1$ in S.I. Then

   $$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\mathbf{a}$$

   The SI unit of force is newton : 1 N = 1 kg m s^{-2}.
   (a) The second law is consistent with the First Law (**F** = 0 implies **a** = 0)
   (b) It is a vector equation
   (c) It is applicable to a particle, and also to a body or a system of particles, provided **F** is the total external force on the system and **a** is the acceleration of the system as a whole.
   (d) **F** at a point at a certain instant determines **a** at the same point at that instant. That is the Second Law is a local law; **a** at an instant does not depend on the history of motion.
5. Impulse is the product of force and time which equals change in momentum.
   The notion of impulse is useful when a large force acts for a short time to produce a measurable change in momentum. Since the time of action of the force is very short, one can assume that there is no appreciable change in the position of the body during the action of the impulsive force.
6. Newton's third law of motion:
   To every action, there is always an equal and opposite reaction

In simple terms, the law can be stated thus :

Forces in nature always occur between pairs of bodies. Force on a body A by body B is equal and opposite to the force on the body B by A.

Action and reaction forces are simultaneous forces. There is no cause-effect relation between action and reaction. Any of the two mutual forces can be called action and the other reaction. Action and reaction act on different bodies and so they cannot be cancelled out. The internal action and reaction forces between different parts of a body do, however, sum to zero.

7. *Law of Conservation of Momentum*

The total momentum of an isolated system of particles is conserved. The law follows from the second and third law of motion.

8. *Friction*

Frictional force opposes (impending or actual) relative motion between two surfaces in contact. It is the component of the contact force along the common tangent to the surface in contact. Static friction f_s opposes impending relative motion; kinetic friction f_k opposes actual relative motion. They are independent of the area of contact and satisfy the following approximate laws :

$$f_s \le (f_s)_{max} = \mu_s R$$

$$f_k = \mu_k R$$

μ_s (co-efficient of static friction) and μ_k (co-efficient of kinetic friction) are constants characteristic of the pair of surfaces in contact. It is found experimentally that μ_k is less than μ_s .

Quantity	Symbol	Units	Dimensions	Remarks
Momentum	p	kg m s^{-1} or N s	[MLT^{-1}]	Vector
Force	F	N	[MLT^{-2}]	F = ma Second Law
Impulse		kg m s^{-1} or N s	[M LT^{-1}]	Impulse = force x time = change in momentum
Static friction	f_s	N	[MLT^{-2}]	$f_s \le \mu_s$ N
Kinetic friction	f_k	N	[MLT^{-2}]	$f_k = \mu_k$ N

POINTS TO PONDER

1. Force is not always in the direction of motion. Depending on the situation, **F** may be along **v**, opposite to **v**, normal to **v** or may make some other angle with **v**. In every case, it is parallel to acceleration.
2. If **v** = 0 at an instant, i.e. if a body is momentarily at rest, it does not mean that force or acceleration are necessarily zero at that instant. For example, when a ball thrown upward reaches its maximum height, **v** = 0 but the force continues to be its weight *mg* and the acceleration is not zero but *g*.
3. Force on a body at a given time is determined by the situation at the location of the body at that time. Force is not 'carried' by the body from its earlier history of motion. The moment after a stone is released out of an accelerated train, there is no horizontal force (or acceleration) on the stone, if the effects of the surrounding air are neglected. The stone then has only the vertical force of gravity.
4. In the second law of motion **F** = *m* **a**, **F** stands for the net force due to all material agencies external to the body. **a** is the effect of the force. *m***a** should

not be regarded as yet another force, besides **F**.

5. The centripetal force should not be regarded as yet another kind of force. It is simply a name given to the force that provides inward radial acceleration to a body in circular motion. We should always look for some material force like tension, gravitational force, electrical force, friction, etc as the centripetal force in any circular motion.
6. Static friction is a self-adjusting force up to its limit $\mu_s N$ ($f_s \le \mu_s N$). Do not put $f_s = \mu_s N$ without being sure that the maximum value of static friction is coming into play.
7. The familiar equation $mg = R$ for a body on a table is true only if the body is in equilibrium. The two forces mg and R can be different (e.g. a body in an accelerated lift). The equality of mg and R has no connection with the third law.
8. The terms 'action' and 'reaction' in the third Law of Motion simply stand for simultaneous mutual forces between a pair of bodies. Unlike their meaning in ordinary language, action does not precede or cause reaction. Action and reaction act on different bodies.
9. The different terms like 'friction', 'normal reaction' 'tension', 'air resistance', 'viscous drag', 'thrust', 'buoyancy' 'weight' 'centripetal force' all stand for 'force' in different contexts. For clarity, every force and its equivalent terms encountered in mechanics should be reduced to the phrase 'force on *A* by *B*'.
10. For applying the second law of motion, there is no conceptual distinction between inanimate and animate objects. An animate object such as a human also requires an external force to accelerate. For example, without the external force of friction, we cannot walk on the ground.
11. The objective concept of force in physics should not be confused with the subjective concept of the 'feeling of force'. On a merry-go-around, all parts of our body are subject to an inward force, but we have a feeling of being pushed outward – the direction of impending motion.

EXERCISES

(For simplicity in numerical calculations, take $g = 10 \text{ m s}^{-2}$)

5.1 Give the magnitude and direction of the net force acting on
(a) a drop of rain falling down with a constant speed,
(b) a cork of mass 10 g floating on water,
(c) a kite skillfully held stationary in the sky,
(d) a car moving with a constant velocity of 30 km/h on a rough road,
(e) a high-speed electron in space far from all material objects, and free of electric and magnetic fields.

5.2 A pebble of mass 0.05 kg is thrown vertically upwards. Give the direction and magnitude of the net force on the pebble,
(a) during its upward motion,
(b) during its downward motion,
(c) at the highest point where it is momentarily at rest. Do your answers change if the pebble was thrown at an angle of 45° with the horizontal direction?

Ignore air resistance.

5.3 Give the magnitude and direction of the net force acting on a stone of mass 0.1 kg,
(a) just after it is dropped from the window of a stationary train,
(b) just after it is dropped from the window of a train running at a constant velocity of 36 km/h,
(c) just after it is dropped from the window of a train accelerating with 1 m s^{-2},
(d) lying on the floor of a train which is accelerating with 1 m s^{-2}, the stone being at rest relative to the train.

Neglect air resistance throughout.

5.4 One end of a string of length l is connected to a particle of mass m and the other to a small peg on a smooth horizontal table. If the particle moves in a circle with speed v the net force on the particle (directed towards the centre) is :

(i) T, (ii) $T - \frac{mv^2}{l}$, (iii) $T + \frac{mv^2}{l}$, (iv) 0

T is the tension in the string. [Choose the correct alternative].

5.5 A constant retarding force of 50 N is applied to a body of mass 20 kg moving initially with a speed of 15 m s^{-1}. How long does the body take to stop ?

5.6 A constant force acting on a body of mass 3.0 kg changes its speed from 2.0 m s^{-1} to 3.5 m s^{-1} in 25 s. The direction of the motion of the body remains unchanged. What is the magnitude and direction of the force ?

5.7 A body of mass 5 kg is acted upon by two perpendicular forces 8 N and 6 N. Give the magnitude and direction of the acceleration of the body.

5.8 The driver of a three-wheeler moving with a speed of 36 km/h sees a child standing in the middle of the road and brings his vehicle to rest in 4.0 s just in time to save the child. What is the average retarding force on the vehicle ? The mass of the three-wheeler is 400 kg and the mass of the driver is 65 kg.

5.9 A rocket with a lift-off mass 20,000 kg is blasted upwards with an initial acceleration of 5.0 m s^{-2}. Calculate the initial thrust (force) of the blast.

5.10 A body of mass 0.40 kg moving initially with a constant speed of 10 m s^{-1} to the north is subject to a constant force of 8.0 N directed towards the south for 30 s. Take the instant the force is applied to be $t = 0$, the position of the body at that time to be $x = 0$, and predict its position at t = –5 s, 25 s, 100 s.

5.11 A truck starts from rest and accelerates uniformly at 2.0 m s^{-2}. At $t = 10$ s, a stone is dropped by a person standing on the top of the truck (6 m high from the ground). What are the (a) velocity, and (b) acceleration of the stone at t = 11s ? (Neglect air resistance.)

5.12 A bob of mass 0.1 kg hung from the ceiling of a room by a string 2 m long is set into oscillation. The speed of the bob at its mean position is 1 m s^{-1}. What is the trajectory of the bob if the string is cut when the bob is (a) at one of its extreme positions, (b) at its mean position.

5.13 A man of mass 70 kg stands on a weighing scale in a lift which is moving
(a) upwards with a uniform speed of 10 m s^{-1},
(b) downwards with a uniform acceleration of 5 m s^{-2},
(c) upwards with a uniform acceleration of 5 m s^{-2}.
What would be the readings on the scale in each case?
(d) What would be the reading if the lift mechanism failed and it hurtled down freely under gravity ?

5.14 Figure 5.16 shows the position-time graph of a particle of mass 4 kg. What is the (a) force on the particle for $t < 0$, $t > 4$ s, $0 < t < 4$ s? (b) impulse at $t = 0$ and t = 4 s ? (Consider one-dimensional motion only).

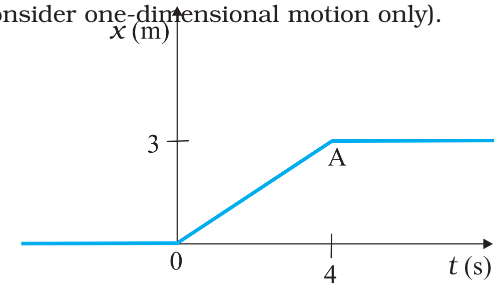


Fig. 5.16

5.15 Two bodies of masses 10 kg and 20 kg respectively kept on a smooth, horizontal surface are tied to the ends of a light string. a horizontal force F = 600 N is applied to (i) A, (ii) B along the direction of string. What is the tension in the

string in each case?

5.16 Two masses 8 kg and 12 kg are connected at the two ends of a light inextensible string that goes over a frictionless pulley. Find the acceleration of the masses, and the tension in the string when the masses are released.

5.17 A nucleus is at rest in the laboratory frame of reference. Show that if it disintegrates into two smaller nuclei the products must move in opposite directions.

5.18 Two billiard balls each of mass 0.05 kg moving in opposite directions with speed 6 m s^{-1} collide and rebound with the same speed. What is the impulse imparted to each ball due to the other ?

5.19 A shell of mass 0.020 kg is fired by a gun of mass 100 kg. If the muzzle speed of the shell is 80 m s^{-1}, what is the recoil speed of the gun ?

5.20 A batsman deflects a ball by an angle of 45° without changing its initial speed which is equal to 54 km/h. What is the impulse imparted to the ball ? (Mass of the ball is 0.15 kg.)

5.21 A stone of mass 0.25 kg tied to the end of a string is whirled round in a circle of radius 1.5 m with a speed of 40 rev./min in a horizontal plane. What is the tension in the string ? What is the maximum speed with which the stone can be whirled around if the string can withstand a maximum tension of 200 N ?

5.22 If, in Exercise 5.21, the speed of the stone is increased beyond the maximum permissible value, and the string breaks suddenly, which of the following correctly describes the trajectory of the stone after the string breaks :
(a) the stone moves radially outwards,
(b) the stone flies off tangentially from the instant the string breaks,
(c) the stone flies off at an angle with the tangent whose magnitude depends on the speed of the particle ?

5.23 Explain why
(a) a horse cannot pull a cart and run in empty space,
(b) passengers are thrown forward from their seats when a speeding bus stops suddenly,
(c) it is easier to pull a lawn mower than to push it,
(d) a cricketer moves his hands backwards while holding a catch.

Additional Exercises

5.24 Figure 5.17 shows the position-time graph of a body of mass 0.04 kg. Suggest a suitable physical context for this motion. What is the time between two consecutive impulses received by the body ? What is the magnitude of each impulse ?

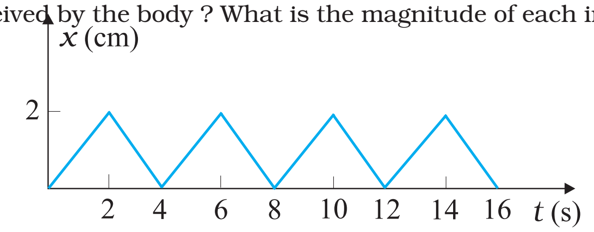


Fig. 5.17

5.25 Figure 5.18 shows a man standing stationary with respect to a horizontal conveyor belt that is accelerating with 1 m s^{-2}. What is the net force on the man? If the coefficient of static friction between the man's shoes and the belt is 0.2, up to what acceleration of the belt can the man continue to be stationary relative to the belt ? (Mass of the man = 65 kg.)

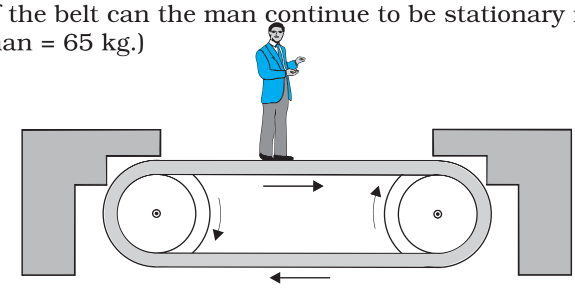


Fig. 5.18

5.26 A stone of mass m tied to the end of a string revolves in a vertical circle of radius R. The net forces at the lowest and highest points of the circle directed vertically downwards are : [Choose the correct alternative]

	Lowest Point	Highest Point
(a)	$mg - T_1$	$mg + T_2$
(b)	$mg + T_1$	$mg - T_2$
(c)	$mg + T_1 - (m\,v_1^2)\,/\,R$	$mg - T_2 + (m\,v_1^2)\,/\,R$
(d)	$mg - T_1 - (m\,v_1^2)\,/\,R$	$mg + T_2 + (m\,v_1^2)\,/\,R$

T_1 and v_1 denote the tension and speed at the lowest point. T_2 and v_2 denote corresponding values at the highest point.

5.27 A helicopter of mass 1000 kg rises with a vertical acceleration of 15 m s^{-2}. The crew and the passengers weigh 300 kg. Give the magnitude and direction of the
(a) force on the floor by the crew and passengers,
(b) action of the rotor of the helicopter on the surrounding air,
(c) force on the helicopter due to the surrounding air.

5.28 A stream of water flowing horizontally with a speed of 15 m s^{-1} gushes out of a tube of cross-sectional area 10^{-2} m^2, and hits a vertical wall nearby. What is the force exerted on the wall by the impact of water, assuming it does not rebound ?

5.29 Ten one-rupee coins are put on top of each other on a table. Each coin has a mass m. Give the magnitude and direction of
(a) the force on the 7^{th} coin (counted from the bottom) due to all the coins on its top,
(b) the force on the 7^{th} coin by the eighth coin,
(c) the reaction of the 6^{th} coin on the 7^{th} coin.

5.30 An aircraft executes a horizontal loop at a speed of 720 km/h with its wings banked at 15°. What is the radius of the loop ?

5.31 A train runs along an unbanked circular track of radius 30 m at a speed of 54 km/h. The mass of the train is 10^6 kg. What provides the centripetal force required for this purpose — The engine or the rails ? What is the angle of banking required to prevent wearing out of the rail ?

5.32 A block of mass 25 kg is raised by a 50 kg man in two different ways as shown in Fig. 5.19. What is the action on the floor by the man in the two cases ? If the floor yields to a normal force of 700 N, which mode should the man adopt to lift the block without the floor yielding ?

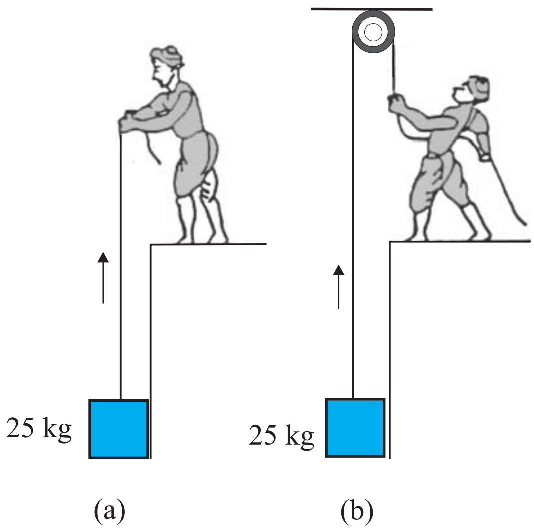


Fig. 5.19

5.33 A monkey of mass 40 kg climbs on a rope (Fig. 5.20) which can stand a maximum tension of 600 N. In which of the following cases will the rope break: the monkey
(a) climbs up with an acceleration of 6 m s^{-2}
(b) climbs down with an acceleration of 4 m s^{-2}
(c) climbs up with a uniform speed of 5 m s^{-1}
(d) falls down the rope nearly freely under gravity?
(Ignore the mass of the rope).

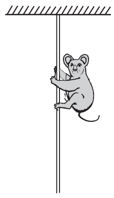


Fig. 5.20

5.34 Two bodies *A* and *B* of masses 5 kg and 10 kg in contact with each other rest on a table against a rigid wall (Fig. 5.21). The coefficient of friction between the bodies and the table is 0.15. A force of 200 N is applied horizontally to *A*. What are (a) the reaction of the partition (b) the action-reaction forces between *A* and *B*? What happens when the wall is removed? Does the answer to (b) change, when the bodies are in motion? Ignore the difference between μ_s and μ_k.

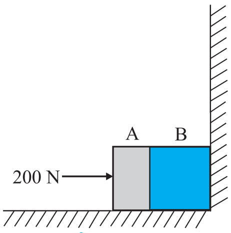


Fig. 5.21

5.35 A block of mass 15 kg is placed on a long trolley. The coefficient of static friction between the block and the trolley is 0.18. The trolley accelerates from rest with 0.5 m s^{-2} for 20 s and then moves with uniform velocity. Discuss the motion of the block as viewed by (a) a stationary observer on the ground, (b) an observer moving with the trolley.

5.36 The rear side of a truck is open and a box of 40 kg mass is placed 5 m away from the open end as shown in Fig. 5.22. The coefficient of friction between the box and the surface below it is 0.15. On a straight road, the truck starts from rest and accelerates with 2 m s^{-2}. At what distance from the starting point does the box fall off the truck? (Ignore the size of the box).

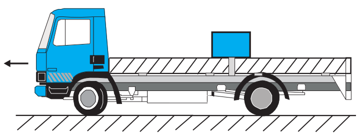


Fig. 5.22

5.37 A disc revolves with a speed of $33\frac{1}{3}$ rev/min, and has a radius of 15 cm. Two coins are placed at 4 cm and 14 cm away from the centre of the record. If the co-efficient of friction between the coins and the record is 0.15, which of the coins will revolve with the record ?

5.38 You may have seen in a circus a motorcyclist driving in vertical loops inside a 'death-well' (a hollow spherical chamber with holes, so the spectators can watch from outside). Explain clearly why the motorcyclist does not drop down when he is at the uppermost point, with no support from below. What is the minimum speed required at the uppermost position to perform a vertical loop if the radius of the chamber is 25 m ?

5.39 A 70 kg man stands in contact against the inner wall of a hollow cylindrical drum of radius 3 m rotating about its vertical axis with 200 rev/min. The coefficient of friction between the wall and his clothing is 0.15. What is the minimum rotational speed of the cylinder to enable the man to remain stuck to the wall (without falling) when the floor is suddenly removed ?

5.40 A thin circular loop of radius *R* rotates about its vertical diameter with an angular frequency ω Show that a small bead on the wire loop remains at its lowermost point for $\omega \le \sqrt{g/R}$. What is the angle made by the radius vector joining the centre to the bead with the vertical downward direction for $\omega = \sqrt{2g/R}$? Neglect friction.

CHAPTER SIX

WORK, ENERGY AND POWER

6.1 INTRODUCTION

The terms 'work', 'energy' and 'power' are frequently used in everyday language. A farmer ploughing the field, a construction worker carrying bricks, a student studying for a competitive examination, an artist painting a beautiful landscape, all are said to be working. In physics, however, the word 'Work' covers a definite and precise meaning. Somebody who has the capacity to work for 14-16 hours a day is said to have a large stamina or energy. We admire a long distance runner for her stamina or energy. Energy is thus our capacity to do work. In Physics too, the term 'energy' is related to work in this sense, but as said above the term 'work' itself is defined much more precisely. The word 'power' is used in everyday life with different shades of meaning. In karate or boxing we talk of 'powerful' punches. These are delivered at a great speed. This shade of meaning is close to the meaning of the word 'power' used in physics. We shall find that there is at best a loose correlation between the physical definitions and the physiological pictures these terms generate in our minds. The aim of this chapter is to develop an understanding of these three physical quantities. Before we proceed to this task, we need to develop a mathematical prerequisite, namely the scalar product of two vectors.

6.1.1 The Scalar Product

We have learnt about vectors and their use in Chapter 4. Physical quantities like displacement, velocity, acceleration, force etc. are vectors. We have also learnt how vectors are added or subtracted. We now need to know how vectors are multiplied. There are two ways of multiplying vectors which we shall come across : one way known as the scalar product gives a scalar from two vectors and the other known as the vector product produces a new vector from two vectors. We shall look at the vector product in Chapter 7. Here we take up the scalar product of two vectors. The scalar product or dot product of any two vectors **A** and **B**, denoted as **A·B** (read

A dot **B**) is defined as

$$\mathbf{A{\cdot}B} = A\,B\cos\theta \qquad (6.1a)$$

where θ is the angle between the two vectors as shown in Fig. 6.1a. Since A, B and cos θ are scalars, the dot product of **A** and **B** is a scalar quantity. Each vector, **A** and **B**, has a direction but their scalar product does not have a direction.

From Eq. (6.1a), we have

$$\mathbf{A{\cdot}B} = A\,(B\cos\theta)$$
$$= B\,(A\cos\theta)$$

Geometrically, $B\cos\theta$ is the projection of **B** onto **A** in Fig.6.1 (b) and $A\cos\theta$ is the projection of **A** onto **B** in Fig. 6.1 (c). So, **A·B** is the product of the magnitude of **A** and the component of **B** along **A**. Alternatively, it is the product of the magnitude of **B** and the component of **A** along **B**.

Equation (6.1a) shows that the scalar product follows the commutative law :

$$\mathbf{A{\cdot}B} = \mathbf{B{\cdot}A}$$

Scalar product obeys the **distributive law**:

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A{\cdot}B} + \mathbf{A{\cdot}C}$$

Further, $\mathbf{A}\cdot(\lambda\,\mathbf{B}) = \lambda\,(\mathbf{A{\cdot}B})$

where λ is a real number.

The proofs of the above equations are left to you as an exercise.

For unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ we have

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$$
$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0$$

Given two vectors

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$
$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

their scalar product is

$$\mathbf{A{\cdot}B} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\cdot\left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right)$$
$$= A_xB_x + A_yB_y + A_zB_z \qquad (6.1b)$$

From the definition of scalar product and, (Eq. 6.1b) we have :

(i) $\mathbf{A{\cdot}A} = A_xA_x + A_yA_y + A_zA_z$

Or, $A^2 = A_x^2 + A_y^2 + A_z^2 \qquad (6.1c)$

since $\mathbf{A{\cdot}A} = |\mathbf{A}|\,|\mathbf{A}|\cos 0 = A^2$.

(ii) $\mathbf{A{\cdot}B} = 0$, if **A** and **B** are perpendicular.

> ▶ ***Example 6.1*** Find the angle between force $\mathbf{F} = (3\,\hat{i} + 4\,\hat{j}\ \ 5\,\hat{k})$ unit and displacement $\mathbf{d} = (5\,\hat{i} + 4\,\hat{j} + 3\,\hat{k})$ unit. Also find the projection of **F** on **d**.

Answer $\mathbf{F{\cdot}d} = F_xd_x + F_yd_y + F_zd_z$

$= 3\,(5) + 4\,(4) + (-3)\,(3)$

$= 16$ unit

Hence $\mathbf{F{\cdot}d} = F\,d\,\cos\theta = 16$ unit

Now $\mathbf{F{\cdot}F} = F^2 = F_x^2 + F_y^2 + F_z^2$

$= 9 + 16 + 25$

$= 50$ unit

and $\mathbf{d{\cdot}d} = d^2 = d_x^2 + d_y^2 + d_z^2$

$= 25 + 16 + 9$

$= 50$ unit

$$\therefore\ \cos\theta = \frac{16}{\sqrt{50}\sqrt{50}} = \frac{16}{50} = 0.32,$$

$\theta = \cos^{-1} 0.32$ ◀

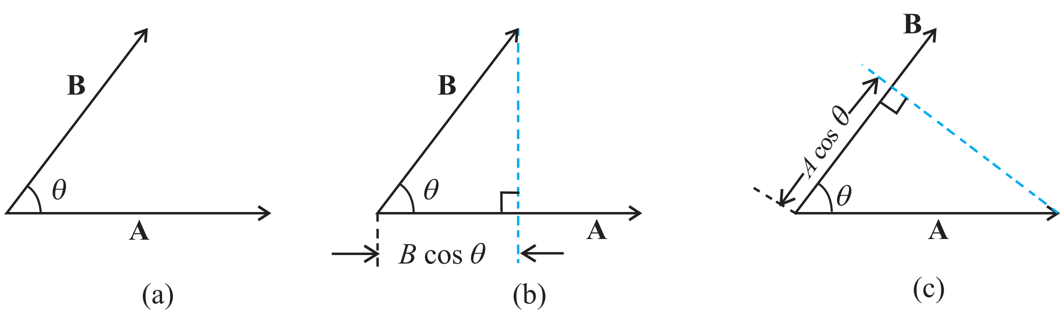


Fig. 6.1 (a) *The scalar product of two vectors* **A** *and* **B** *is a scalar :* $\mathbf{A{\cdot}B} = A\,B\cos\theta$. (b) $B\cos\theta$ *is the projection of* **B** *onto* **A**. (c) $A\cos\theta$ *is the projection of* **A** *onto* **B**.

6.2 NOTIONS OF WORK AND KINETIC ENERGY: THE WORK-ENERGY THEOREM

The following relation for rectilinear motion under constant acceleration a has been encountered in Chapter 3,

$$v^2 - u^2 = 2\,as$$

where u and v are the initial and final speeds and s the distance traversed. Multiplying both sides by $m/2$, we have

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mas = Fs \qquad (6.2a)$$

where the last step follows from Newton's Second Law. We can generalise Eq. (6.1) to three dimensions by employing vectors

$$v^2 - u^2 = 2\,\mathbf{a.d}$$

Once again multiplying both sides by $m/2$, we obtain

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = m\,\mathbf{a.d} = \mathbf{F.d} \qquad (6.2b)$$

The above equation provides a motivation for the definitions of work and kinetic energy. The left side of the equation is the difference in the quantity 'half the mass times the square of the speed' from its initial value to its final value. We call each of these quantities the 'kinetic energy', denoted by K. The right side is a product of the displacement and the component of the force along the displacement. This quantity is called 'work' and is denoted by W. Eq. (6.2) is then

$$K_f - K_i = W \qquad (6.3)$$

where K_i and K_f are respectively the initial and final kinetic energies of the object. Work refers to the force and the displacement over which it acts. **Work is done by a force on the body over a certain displacement**.

Equation (6.2) is also a special case of the work-energy (WE) theorem : **The change in kinetic energy of a particle is equal to the work done on it by the net force**. We shall generalise the above derivation to a varying force in a later section.

▶ ***Example 6.2*** It is well known that a raindrop falls under the influence of the downward gravitational force and the opposing resistive force. The latter is known to be proportional to the speed of the drop but is otherwise undetermined. Consider a drop of mass 1.00 g falling from a height 1.00 km. It hits the ground with a speed of 50.0 m s^{-1}. (a) What is the work done by the gravitational force ? What is the work done by the unknown resistive force?

Answer (a) The change in kinetic energy of the drop is

$$\Delta K = \frac{1}{2}m\,v^2 - 0$$

$$= \frac{1}{2} \times 10^{-3} \times 50 \times 50$$

$$= 1.25 \text{ J}$$

where we have assumed that the drop is initially at rest.

Assuming that g is a constant with a value 10 m/s^2, the work done by the gravitational force is,

$$W_g = mgh$$

$$= 10^{-3} \times 10 \times 10^3$$

$$= 10.0 \text{ J}$$

(b) From the work-energy theorem

$$\Delta K = W_g + W_r$$

where W_r is the work done by the resistive force on the raindrop. Thus

$$W_r = \Delta K - W_g$$

$$= 1.25 - 10$$

$$= -8.75 \text{ J}$$

is negative. ◀

6.3 WORK

As seen earlier, work is related to force and the displacement over which it acts. Consider a constant force $\mathbf{F}$ acting on an object of mass m. The object undergoes a displacement $\mathbf{d}$ in the positive x-direction as shown in Fig. 6.2.

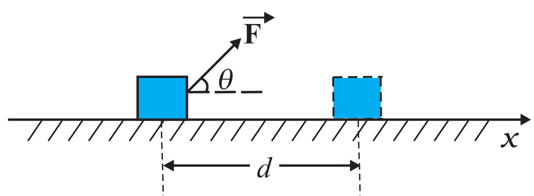


Fig. 6.2 *An object undergoes a displacement* **d** *under the influence of the force* **F**.

The work done by the force is defined to be the product of component of the force in the direction of the displacement and the magnitude of this displacement. Thus

$$W = (F \cos \theta) d = \mathbf{F.d} \quad (6.4)$$

We see that if there is no displacement, there is no work done even if the force is large. Thus, when you push hard against a rigid brick wall, the force you exert on the wall does no work. Yet your muscles are alternatively contracting and relaxing and internal energy is being used up and you do get tired. Thus, the meaning of work in physics is different from its usage in everyday language.

No work is done if :

(i) the displacement is zero as seen in the example above. A weightlifter holding a 150 kg mass steadily on his shoulder for 30 s does no work on the load during this time.

(ii) the force is zero. A block moving on a smooth horizontal table is not acted upon by a horizontal force (since there is no friction), but may undergo a large displacement.

(iii) the force and displacement are mutually perpendicular. This is so since, for $\theta = \pi/2$ rad (= 90°), cos ($\pi/2$) = 0. For the block moving on a smooth horizontal table, the gravitational force mg does no work since it acts at right angles to the displacement. If we assume that the moon's orbits around the earth is perfectly circular then the earth's gravitational force does no work. The moon's instantaneous displacement is tangential while the earth's force is radially inwards and $\theta = \pi/2$.

Work can be both positive and negative. If θ is between 0° and 90°, cos θ in Eq. (6.4) is positive. If θ is between 90° and 180°, cos θ is negative. In many examples the frictional force opposes displacement and θ = 180°. Then the work done by friction is negative (cos 180° = –1).

From Eq. (6.4) it is clear that work and energy have the same dimensions, [ML^2T^{-2}]. The SI unit of these is joule (J), named after the famous British physicist James Prescott Joule (1811-1869). Since work and energy are so widely used as physical concepts, alternative units abound and some of these are listed in Table 6.1.

Table 6.1 Alternative Units of Work/Energy in J

erg	10^{-7} J
electron volt (eV)	1.6×10^{-19} J
calorie (cal)	4.186 J
kilowatt hour (kWh)	3.6×10^{6} J

▶ ***Example 6.3*** A cyclist comes to a skidding stop in 10 m. During this process, the force on the cycle due to the road is 200 N and is directly opposed to the motion. (a) How much work does the road do on the cycle ? (b) How much work does the cycle do on the road ?

Answer Work done on the cycle by the road is the work done by the stopping (frictional) force on the cycle due to the road.

(a) The stopping force and the displacement make an angle of 180° (π rad) with each other. Thus, work done by the road,

$$W_r = Fd \cos\theta$$
$$= 200 \times 10 \times \cos \pi$$
$$= -2000 \text{ J}$$

It is this negative work that brings the cycle to a halt in accordance with WE theorem.

(b) From Newton's Third Law an equal and opposite force acts on the road due to the cycle. Its magnitude is 200 N. However, the road undergoes no displacement. Thus, work done by cycle on the road is zero. ◀

The lesson of this example is that though the force on a body A exerted by the body B is always equal and opposite to that on B by A (Newton's Third Law); the work done on A by B is not necessarily equal and opposite to the work done on B by A.

6.4 KINETIC ENERGY

As noted earlier, if an object of mass m has velocity $\mathbf{v}$, its kinetic energy K is

$$K = \frac{1}{2} m\, \mathbf{v.v} = \frac{1}{2} mv^2 \quad (6.5)$$

Kinetic energy is a scalar quantity. The kinetic energy of an object is a measure of the work an

Table 6.2 **Typical kinetic energies (*K*)**

Object	Mass (kg)	Speed (m s^{-1})	*K* (J)
Car	2000	25	6.3×10^{5}
Running athlete	70	10	3.5×10^{3}
Bullet	5×10^{-2}	200	10^{3}
Stone dropped from 10 m	1	14	10^{2}
Rain drop at terminal speed	3.5×10^{-5}	9	1.4×10^{-3}
Air molecule	$\simeq 10^{-26}$	500	$\simeq 10^{-21}$

object can do by the virtue of its motion. This notion has been intuitively known for a long time. The kinetic energy of a fast flowing stream has been used to grind corn. Sailing ships employ the kinetic energy of the wind. Table 6.2 lists the kinetic energies for various objects.

Example 6.4 In a ballistics demonstration a police officer fires a bullet of mass 50.0 g with speed 200 m s^{-1} (see Table 6.2) on soft plywood of thickness 2.00 cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet ?

Answer The initial kinetic energy of the bullet is $mv^2/2$ = 1000 J. It has a final kinetic energy of 0.1×1000 = 100 J. If v_f is the emergent speed of the bullet,

$$\frac{1}{2}mv_f^2 = 100 \text{ J}$$

$$v_f = \sqrt{\frac{2\times100 \text{ J}}{0.05 \text{ kg}}}$$

$$= 63.2 \text{ m s}^{-1}$$

The speed is reduced by approximately 68% (not 90%).

6.5 WORK DONE BY A VARIABLE FORCE

A constant force is rare. It is the variable force, which is more commonly encountered. Fig. 6.2 is a plot of a varying force in one dimension.

If the displacement Δx is small, we can take the force $F(x)$ as approximately constant and the work done is then

$$\Delta W = F(x)\,\Delta x$$

This is illustrated in Fig. 6.3(a). Adding successive rectangular areas in Fig. 6.3(a) we get the total work done as

$$W \cong \sum_{x_i}^{x_f} F(x)\Delta x \qquad (6.6)$$

where the summation is from the initial position x_i to the final position x_f.

If the displacements are allowed to approach zero, then the number of terms in the sum increases without limit, but the sum approaches a definite value equal to the area under the curve in Fig. 6.3(b). Then the work done is

$$W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F(x)\Delta x$$

$$= \int_{x_i}^{x_f} F(x)\, dx \qquad (6.7)$$

where '*lim*' stands for the limit of the sum when Δx tends to zero. Thus, for a varying force the work done can be expressed as a definite integral of force over displacement (see also Appendix 3.1).

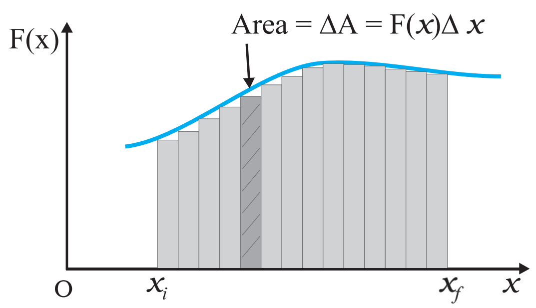


Fig. 6.3(a)

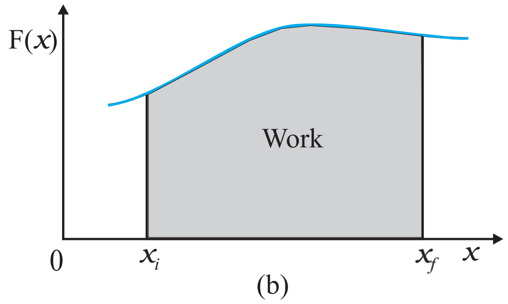


Fig. 6.3 *(a) The shaded rectangle represents the work done by the varying force F(x), over the small displacement Δx, $\Delta W = F(x)\Delta x$. (b) adding the areas of all the rectangles we find that for $\Delta x \to 0$, the area under the curve is exactly equal to the work done by F(x).*

> **Example 6.5** A woman pushes a trunk on a railway platform which has a rough surface. She applies a force of 100 N over a distance of 10 m. Thereafter, she gets progressively tired and her applied force reduces linearly with distance to 50 N. The total distance through which the trunk has been moved is 20 m. Plot the force applied by the woman and the frictional force, which is 50 N. Calculate the work done by the two forces over 20 m.

Answer

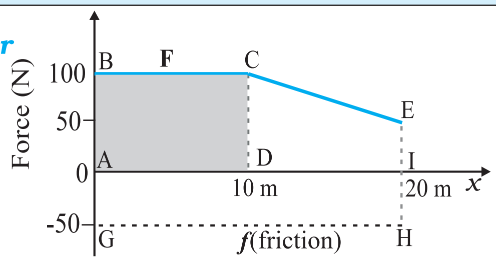


Fig. 6.4 *Plot of the force F applied by the woman and the opposing frictional force f.*

The plot of the applied force is shown in Fig. 6.4. At x = 20 m, F = 50 N ($\neq$ 0). We are given that the frictional force f is $|\mathbf{f}| = 50$ N. It opposes motion and acts in a direction opposite to **F**. It is therefore, shown on the negative side of the force axis.

The work done by the woman is

$W_F \to$ area of the rectangle ABCD + area of the trapezium CEID

$$W_F = 100 \times 10 + \frac{1}{2}(100 + 50) \times 10$$
$$= 1000 + 750$$
$$= 1750 \text{ J}$$

The work done by the frictional force is

$W_F \to$ area of the rectangle AGHI

$$W_f = (-50) \times 20$$
$$= -1000 \text{ J}$$

The area on the negative side of the force axis has a negative sign. ◀

6.6 THE WORK-ENERGY THEOREM FOR A VARIABLE FORCE

We are now familiar with the concepts of work and kinetic energy to prove the work-energy theorem for a variable force. We confine ourselves to one dimension. The time rate of change of kinetic energy is

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v^2\right)$$
$$= m \frac{dv}{dt} v$$
$$= F v \text{ (from Newton's Second Law)}$$
$$= F \frac{dx}{dt}$$

Thus

$$dK = F dx$$

Integrating from the initial position (x_i) to final position (x_f), we have

$$\int_{K_i}^{K_f} dK = \int_{x_i}^{x_f} F dx$$

where, K_i and K_f are the initial and final kinetic energies corresponding to x_i and x_f.

or $$K_f - K_i = \int_{x_i}^{x_f} F dx \qquad (6.8a)$$

From Eq. (6.7), it follows that

$$K_f - K_i = W \qquad (6.8b)$$

Thus, the WE theorem is proved for a variable force.

While the WE theorem is useful in a variety of problems, it does not, in general, incorporate the complete dynamical information of Newton's Second Law. It is an integral form of Newton's second law. Newton's second law is a relation between acceleration and force at any instant of time. Work-energy theorem involves an integral over an interval of time. In this sense, the temporal (time) information contained in the statement of Newton's second law is 'integrated over' and is

not available explicitly. Another observation is that Newton's second law for two or three dimensions is in vector form whereas the work-energy theorem is in scalar form. In the scalar form, information with respect to directions contained in Newton's second law is not present.

> **Example 6.6** A block of mass m = 1 kg, moving on a horizontal surface with speed v_i = 2 ms^{-1} enters a rough patch ranging from x = 0.10 m to x = 2.01 m. The retarding force F_r on the block in this range is inversely proportional to x over this range,
>
> $$F_r = \frac{-k}{x} \quad \text{for } 0.1 < x < 2.01 \text{ m}$$
>
> = 0 for x < 0.1m and x > 2.01 m
> where k = 0.5 J. What is the final kinetic energy and speed v_f of the block as it crosses this patch ?

Answer From Eq. (6.8a)

$$K_f = K_i + \int_{0.1}^{2.01} \frac{(-k)}{x} \mathrm{d}x$$

$$= \frac{1}{2} m v_i^2 - k \ln(x)\Big|_{0.1}^{2.01}$$

$$= \frac{1}{2} m v_i^2 - k \ln(2.01/0.1)$$

$$= 2 - 0.5 \ln(20.1)$$

$$= 2 - 1.5 = 0.5 \text{ J}$$

$$v_f = \sqrt{2K_f/m} = 1 \text{ m s}^{-1}$$

Here, note that ln is a symbol for the natural logarithm to the base e and not the logarithm to the base 10 [ln X = $\log_e$ X = 2.303 $\log_{10}$ X]. ◀

6.7 THE CONCEPT OF POTENTIAL ENERGY

The word potential suggests possibility or capacity for action. The term potential energy brings to one's mind 'stored' energy. A stretched bow-string possesses potential energy. When it is released, the arrow flies off at a great speed. The earth's crust is not uniform, but has discontinuities and dislocations that are called fault lines. These fault lines in the earth's crust are like 'compressed springs'. They possess a large amount of potential energy. An earthquake results when these fault lines readjust. Thus, potential energy is the 'stored energy' by virtue of the position or configuration of a body. The body left to itself releases this stored energy in the form of kinetic energy. Let us make our notion of potential energy more concrete.

The gravitational force on a ball of mass m is mg . g may be treated as a constant near the earth surface. By 'near' we imply that the height h of the ball above the earth's surface is very small compared to the earth's radius R_E ($h \ll R_E$) so that we can ignore the variation of g near the earth's surface*. In what follows we have taken the upward direction to be positive. Let us raise the ball up to a height h. The work done by the external agency against the gravitational force is mgh. This work gets stored as potential energy. Gravitational potential energy of an object, as a function of the height h, is denoted by $V(h)$ and it is the negative of work done by the gravitational force in raising the object to that height.

$$V(h) = mgh$$

If h is taken as a variable, it is easily seen that the gravitational force F equals the negative of the derivative of $V(h)$ with respect to h. Thus,

$$F = -\frac{\mathrm{d}}{\mathrm{d}h} V(h) = -m\,g$$

The negative sign indicates that the gravitational force is downward. When released, the ball comes down with an increasing speed. Just before it hits the ground, its speed is given by the kinematic relation,

$$v^2 = 2gh$$

This equation can be written as

$$\frac{1}{2} m\,v^2 = m\,g\,h$$

which shows that the gravitational potential energy of the object at height h, when the object is released, manifests itself as kinetic energy of the object on reaching the ground.

Physically, the notion of potential energy is applicable only to the class of forces where work done against the force gets 'stored up' as energy. When external constraints are removed, it manifests itself as kinetic energy. Mathematically, (for simplicity, in one dimension) the potential

* *The variation of g with height is discussed in Chapter 8 on Gravitation.*

energy $V(x)$ is defined if the force $F(x)$ can be written as

$$F(x) = -\frac{dV}{dx}$$

This implies that

$$\int_{x_i}^{x_f} F(x)dx = -\int_{V_i}^{V_f} dV = V_i - V_f$$

The work done by a conservative force such as gravity depends on the initial and final positions only. In the previous chapter we have worked on examples dealing with inclined planes. If an object of mass m is released from rest, from the top of a smooth (frictionless) inclined plane of height h, its speed at the bottom is $\sqrt{2gh}$ irrespective of the angle of inclination. Thus, at the bottom of the inclined plane it acquires a kinetic energy, mgh. If the work done or the kinetic energy did depend on other factors such as the velocity or the particular path taken by the object, the force would be called non-conservative.

The dimensions of potential energy are $[ML^2T^{-2}]$ and the unit is joule (J), the same as kinetic energy or work. To reiterate, the change in potential energy, for a conservative force, ΔV is equal to the negative of the work done by the force

$$\Delta V = -F(x)\ \Delta x \qquad (6.9)$$

In the example of the falling ball considered in this section we saw how potential energy was converted to kinetic energy. This hints at an important principle of conservation in mechanics, which we now proceed to examine.

6.8 THE CONSERVATION OF MECHANICAL ENERGY

For simplicity we demonstrate this important principle for one-dimensional motion. Suppose that a body undergoes displacement Δx under the action of a conservative force F. Then from the WE theorem we have,

$$\Delta K = F(x)\ \Delta x$$

If the force is conservative, the potential energy function $V(x)$ can be defined such that

$$-\Delta V = F(x)\ \Delta x$$

The above equations imply that

$$\Delta K + \Delta V = 0$$

$$\Delta(K + V) = 0 \qquad (6.10)$$

which means that $K + V$, the sum of the kinetic and potential energies of the body is a constant. Over the whole path, x_i to x_f, this means that

$$K_i + V(x_i) = K_f + V(x_f) \qquad (6.11)$$

The quantity $K + V(x)$, is called the total mechanical energy of the system. Individually the kinetic energy K and the potential energy $V(x)$ may vary from point to point, but the sum is a constant. The aptness of the term 'conservative force' is now clear.

Let us consider some of the definitions of a conservative force.

- A force $F(x)$ is conservative if it can be derived from a scalar quantity $V(x)$ by the relation given by Eq. (6.9). The three-dimensional generalisation requires the use of a vector derivative, which is outside the scope of this book.
- The work done by the conservative force depends only on the end points. This can be seen from the relation,
  $W = K_f - K_i = V(x_i) - V(x_f)$
  which depends on the end points.
- A third definition states that the work done by this force in a closed path is zero. This is once again apparent from Eq. (6.11) since $x_i = x_f$.

Thus, the principle of conservation of total mechanical energy can be stated as

The total mechanical energy of a system is conserved if the forces, doing work on it, are conservative.

The above discussion can be made more concrete by considering the example of the gravitational force once again and that of the spring force in the next section. Fig. 6.5 depicts a ball of mass m being dropped from a cliff of height H.

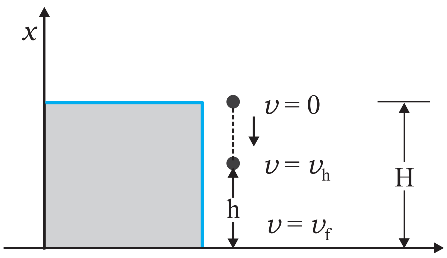


Fig. 6.5 *The conversion of potential energy to kinetic energy for a ball of mass m dropped from a height H.*

The total mechanical energies E_0, E_h, and E_H of the ball at the indicated heights zero (ground level), h and H, are

$$E_H = mgH \quad (6.11\ a)$$

$$E_h = mgh + \frac{1}{2}mv_h^2 \quad (6.11\ b)$$

$$E_0 = (1/2)\ mv^2 \quad (6.11\ c)$$

The constant force is a special case of a spatially dependent force $F(x)$. Hence, the mechanical energy is conserved. Thus

$$E_H = E_0$$

or, $$mgH = \frac{1}{2}mv_f^2$$

$$v_f = \sqrt{2gH}$$

a result that was obtained in section 3.7 for a freely falling body.

Further,

$$E_H = E_h$$

which implies,

$$v_h^2 = 2g(H - h) \quad (6.11\ d)$$

and is a familiar result from kinematics.

At the height H, the energy is purely potential. It is partially converted to kinetic at height h and is fully kinetic at ground level. This illustrates the conservation of mechanical energy.

Example 6.7 A bob of mass m is suspended by a light string of length L. It is imparted a horizontal velocity v_o at the lowest point A such that it completes a semi-circular trajectory in the vertical plane with the string becoming slack only on reaching the topmost point, C. This is shown in Fig. 6.6. Obtain an expression for (i) v_o; (ii) the speeds at points B and C; (iii) the ratio of the kinetic energies (K_B/K_C) at B and C. Comment on the nature of the trajectory of the bob after it reaches the point C.

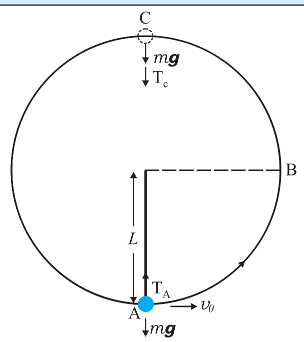


Fig. 6.6

Answer (i) There are two external forces on the bob : gravity and the tension (T) in the string. The latter does no work since the displacement of the bob is always normal to the string. The potential energy of the bob is thus associated with the gravitational force only. The total mechanical energy E of the system is conserved. We take the potential energy of the system to be zero at the lowest point A. Thus, at A :

$$E = \frac{1}{2}mv_0^2 \quad (6.12)$$

$$T_A - mg = \frac{mv_0^2}{L} \quad \text{[Newton's Second Law]}$$

where T_A is the tension in the string at A. At the highest point C, the string slackens, as the tension in the string (T_C) becomes zero.

Thus, at C

$$E = \frac{1}{2}mv_c^2 + 2mgL \quad (6.13)$$

$$mg = \frac{mv_c^2}{L} \quad \text{[Newton's Second Law]} \quad (6.14)$$

where v_C is the speed at C. From Eqs. (6.13) and (6.14)

$$E = \frac{5}{2}mgL$$

Equating this to the energy at A

$$\frac{5}{2}mgL = \frac{m}{2}v_0^2$$

or, $$v_0 = \sqrt{5gL}$$

(ii) It is clear from Eq. (6.14)

$$v_C = \sqrt{gL}$$

At B, the energy is

$$E = \frac{1}{2}mv_B^2 + mgL$$

Equating this to the energy at A and employing the result from (i), namely $v_0^2 = 5gL$,

$$\frac{1}{2}mv_B^2 + mgL = \frac{1}{2}mv_0^2$$

$$= \frac{5}{2}m\,g\,L$$

$$\therefore v_B = \sqrt{3gL}$$

(iii) The ratio of the kinetic energies at B and C is :

$$\frac{K_B}{K_C} = \frac{\frac{1}{2}mv_B^2}{\frac{1}{2}mv_C^2} = \frac{3}{1}$$

At point C, the string becomes slack and the velocity of the bob is horizontal and to the left. If the connecting string is cut at this instant, the bob will execute a projectile motion with horizontal projection akin to a rock kicked horizontally from the edge of a cliff. Otherwise the bob will continue on its circular path and complete the revolution. ◀

6.9 THE POTENTIAL ENERGY OF A SPRING

The spring force is an example of a variable force which is conservative. Fig. 6.7 shows a block attached to a spring and resting on a smooth horizontal surface. The other end of the spring is attached to a rigid wall. The spring is light and may be treated as massless. In an ideal spring, the spring force F_s is proportional to x where x is the displacement of the block from the equilibrium position. The displacement could be either positive [Fig. 6.7(b)] or negative [Fig. 6.7(c)]. This force law for the spring is called Hooke's law and is mathematically stated as

$$F_s = -kx$$

The constant k is called the spring constant. Its unit is N m^{-1}. The spring is said to be stiff if k is large and soft if k is small.

Suppose that we pull the block outwards as in Fig. 6.7(b). If the extension is x_m, the work done by the spring force is

$$W_s = \int_0^{x_m} F_s \, dx \quad = -\int_0^{x_m} kx \, dx$$

$$= -\frac{k\,x_m^2}{2} \qquad (6.15)$$

This expression may also be obtained by considering the area of the triangle as in Fig. 6.7(d). Note that the work done by the external pulling force F is positive since it overcomes the spring force.

$$W = +\frac{k\,x_m^2}{2} \qquad (6.16)$$

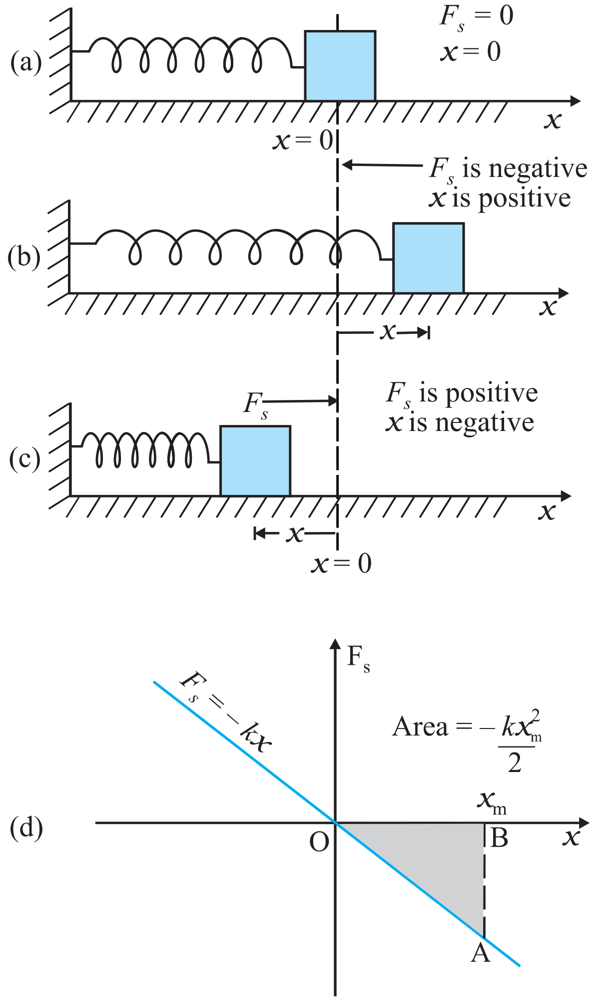


Fig. 6.7 *Illustration of the spring force with a block attached to the free end of the spring. (a) The spring force F_s is zero when the displacement x from the equilibrium position is zero. (b) For the stretched spring $x > 0$ and $F_s < 0$ (c) For the compressed spring $x < 0$ and $F_s > 0$.(d) The plot of F_s versus x. The area of the shaded triangle represents the work done by the spring force. Due to the opposing signs of F_s and x, this work done is negative, $W_s = -kx_m^2 / 2$.*

The same is true when the spring is compressed with a displacement x_c (< 0). The spring force does work $W_s = -kx_c^2/2$ while the

external force F does work $+ kx_c^2/2$. If the block is moved from an initial displacement x_i to a final displacement x_f, the work done by the spring force W_s is

$$W_s = -\int_{x_i}^{x_f} k\,x\,\mathrm{d}x = \frac{k\,x_i^2}{2} - \frac{k\,x_f^2}{2} \qquad (6.17)$$

Thus the work done by the spring force depends only on the end points. Specifically, if the block is pulled from x_i and allowed to return to x_i;

$$W_s = -\int_{x_i}^{x_i} k\,x\,\mathrm{d}x = \frac{k\,x_i^2}{2} - \frac{k\,x_i^2}{2}$$

$$= 0 \qquad (6.18)$$

The work done by the spring force in a cyclic process is zero. We have explicitly demonstrated that the spring force (i) is position dependent only as first stated by Hooke, ($F_s = -kx$); (ii) does work which only depends on the initial and final positions, e.g. Eq. (6.17). Thus, the spring force is a **conservative force**.

We define the potential energy $V(x)$ of the spring to be zero when block and spring system is in the equilibrium position. For an extension (or compression) x the above analysis suggests that

$$V(x) = \frac{kx^2}{2} \qquad (6.19)$$

You may easily verify that $-\mathrm{d}V/\mathrm{d}x = -k\,x$, the spring force. If the block of mass m in Fig. 6.7 is extended to x_m and released from rest, then its total mechanical energy at any arbitrary point x, where x lies between $-x_m$ and $+x_m$, will be given by

$$\frac{1}{2}k\,x_m^2 = \frac{1}{2}k\,x^2 + \frac{1}{2}m\,v^2$$

where we have invoked the conservation of mechanical energy. This suggests that the speed and the kinetic energy will be maximum at the equilibrium position, $x = 0$, i.e.,

$$\frac{1}{2}m\,v_m^2 = \frac{1}{2}k\,x_m^2$$

where v_m is the maximum speed.

or $\quad v_m = \sqrt{\frac{k}{m}}\,x_m$

Note that k/m has the dimensions of [T^{-2}] and our equation is dimensionally correct. The kinetic energy gets converted to potential energy and vice versa, however, the total mechanical energy remains constant. This is graphically depicted in Fig. 6.8.

▶ ***Example 6.8*** To simulate car accidents, auto manufacturers study the collisions of moving cars with mounted springs of different spring constants. Consider a typical simulation with a car of mass 1000 kg moving with a speed 18.0 km/h on a smooth road and colliding with a horizontally mounted spring of spring constant 6.25×10^3 N m^{-1}. What is the maximum compression of the spring ?

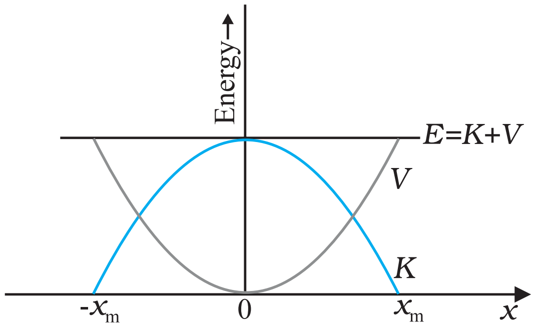


Fig. 6.8 *Parabolic plots of the potential energy V and kinetic energy K of a block attached to a spring obeying Hooke's law. The two plots are complementary, one decreasing as the other increases. The total mechanical energy E = K + V remains constant.*

Answer At maximum compression the kinetic energy of the car is converted entirely into the potential energy of the spring.

The kinetic energy of the moving car is

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2} \times 10^3 \times 5 \times 5$$

$$K = 1.25 \times 10^4 \text{ J}$$

where we have converted 18 km h^{-1} to 5 m s^{-1} **[It is useful to remember that 36 km h^{-1} = 10 m s^{-1}]**. At maximum compression x_m, the potential energy V of the spring is equal to the kinetic energy K of the moving car from the principle of conservation of mechanical energy.

$$V = \frac{1}{2}k\,x_m^2$$

$= 1.25 \times 10^4$ J

We obtain

$$x_m = 2.00 \text{ m}$$

We note that we have idealised the situation. The spring is considered to be massless. The surface has been considered to possess negligible friction. ◀

We conclude this section by making a few remarks on conservative forces.

(i) Information on time is absent from the above discussions. In the example considered above, we can calculate the compression, but not the time over which the compression occurs. A solution of Newton's Second Law for this system is required for temporal information.

(ii) Not all forces are conservative. Friction, for example, is a non-conservative force. The principle of conservation of energy will have to be modified in this case. This is illustrated in Example 6.9.

(iii) The zero of the potential energy is arbitrary. It is set according to convenience. For the spring force we took $V(x) = 0$, at $x = 0$, i.e. the unstretched spring had zero potential energy. For the constant gravitational force mg, we took $V = 0$ on the earth's surface. In a later chapter we shall see that for the force due to the universal law of gravitation, the zero is best defined at an infinite distance from the gravitational source. However, once the zero of the potential energy is fixed in a given discussion, it must be consistently adhered to throughout the discussion. You cannot change horses in midstream !

▶ **Example 6.9** Consider Example 6.7 taking the coefficient of friction, μ, to be 0.5 and calculate the maximum compression of the spring.

Answer In presence of friction, both the spring force and the frictional force act so as to oppose the compression of the spring as shown in Fig. 6.9.

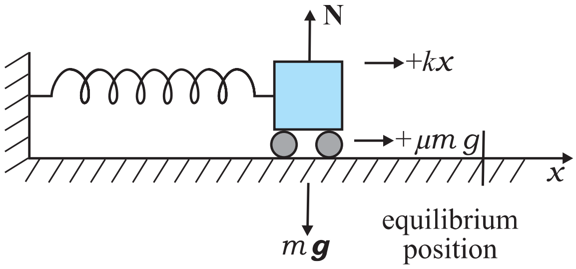


Fig. 6.9 *The forces acting on the car.*

We invoke the work-energy theorem, rather than the conservation of mechanical energy.

The change in kinetic energy is

$$\Delta K = K_f - K_i = 0 - \frac{1}{2} m v^2$$

The work done by the net force is

$$W = -\frac{1}{2} k x_m^2 - \mu m g x_m$$

Equating we have

$$\frac{1}{2} m v^2 = \frac{1}{2} k x_m^2 + \mu m g x_m$$

Now $\mu mg = 0.5 \times 10^3 \times 10 = 5 \times 10^3$ N (taking $g = 10.0$ m s^{-2}). After rearranging the above equation we obtain the following quadratic equation in the unknown x_m.

$$k x_m^2 + 2\mu m g x_m - m v^2 = 0$$

$$x_m = \frac{-\mu m g + \left[\mu^2 m^2 g^2 + m k v^2\right]^{1/2}}{k}$$

where we take the positive square root since x_m is positive. Putting in numerical values we obtain

$$x_m = 1.35 \text{ m}$$

which, as expected, is less than the result in Example 6.8.

If the two forces on the body consist of a conservative force F_c and a non-conservative force F_{nc}, the conservation of mechanical energy formula will have to be modified. By the WE theorem

$$(F_c + F_{nc})\, \Delta x = \Delta K$$

But $\quad F_c\, \Delta x = -\Delta V$

Hence, $\quad \Delta(K + V) = F_{nc}\, \Delta x$

$$\Delta E = F_{nc}\, \Delta x$$

where E is the total mechanical energy. Over the path this assumes the form

$$E_f - E_i = W_{nc}$$

Where W_{nc} is the total work done by the non-conservative forces over the path. Note that

unlike the conservative force, W_{nc} depends on the particular path i to f. ◀

6.10 VARIOUS FORMS OF ENERGY : THE LAW OF CONSERVATION OF ENERGY

In the previous section we have discussed mechanical energy. We have seen that it can be classified into two distinct categories : one based on motion, namely kinetic energy; the other on configuration (position), namely potential energy. Energy comes in many a forms which transform into one another in ways which may not often be clear to us.

6.10.1 Heat

We have seen that the frictional force is excluded from the category of conservative forces. However, work is associated with the force of friction. A block of mass m sliding on a rough horizontal surface with speed v_0 comes to a halt over a distance x_0. The work done by the force of kinetic friction f over x_0 is $-f\,x_0$. By the work-energy theorem $m v_o^2/2 = f\,x_0$. If we confine our scope to mechanics, we would say that the kinetic energy of the block is 'lost' due to the frictional force. On examination of the block and the table we would detect a slight increase in their temperatures. The work done by friction is not 'lost', but is transferred as heat energy. This raises the internal energy of the block and the table. In winter, in order to feel warm, we generate heat by vigorously rubbing our palms together. We shall see later that the internal energy is associated with the ceaseless, often random, motion of molecules. A quantitative idea of the transfer of heat energy is obtained by noting that 1 kg of water releases about 42000 J of energy when it cools by 10 °C.

6.10.2 Chemical Energy

One of the greatest technical achievements of humankind occurred when we discovered how to ignite and control fire. We learnt to rub two flint stones together (mechanical energy), got them to heat up and to ignite a heap of dry leaves (chemical energy), which then provided sustained warmth. A matchstick ignites into a bright flame when struck against a specially prepared chemical surface. The lighted matchstick, when applied to a firecracker, results in a spectacular display of sound and light.

Chemical energy arises from the fact that the molecules participating in the chemical reaction have different binding energies. A stable chemical compound has less energy than the separated parts. A chemical reaction is basically a rearrangement of atoms. If the total energy of the reactants is more than the products of the reaction, heat is released and the reaction is said to be an **exothermic** reaction. If the reverse is true, heat is absorbed and the reaction is **endothermic**. Coal consists of carbon and a kilogram of it when burnt releases 3×10^7 J of energy.

Chemical energy is associated with the forces that give rise to the stability of substances. These forces bind atoms into molecules, molecules into polymeric chains, etc. The chemical energy arising from the combustion of coal, cooking gas, wood and petroleum is indispensable to our daily existence.

6.10.3 Electrical Energy

The flow of electrical current causes bulbs to glow, fans to rotate and bells to ring. There are laws governing the attraction and repulsion of charges and currents, which we shall learn later. Energy is associated with an electric current. An urban Indian household consumes about 200 J of energy per second on an average.

6.10.4 The Equivalence of Mass and Energy

Till the end of the nineteenth century, physicists believed that in every physical and chemical process, the mass of an isolated system is conserved. Matter might change its phase, e.g. glacial ice could melt into a gushing stream, but matter is neither created nor destroyed; Albert Einstein (1879-1955) however, showed that mass and energy are equivalent and are related by the relation

$$E = m\,c^2 \tag{6.20}$$

where c, the speed of light in vacuum is approximately 3×10^8 m s^{-1}. Thus, a staggering amount of energy is associated with a mere kilogram of matter

$$E = 1\times\,(3 \times 10^8)^2\ \text{J} = 9 \times 10^{16}\ \text{J}.$$

This is equivalent to the annual electrical output of a large (3000 MW) power generating station.

6.10.5 Nuclear Energy

The most destructive weapons made by man, the fission and fusion bombs are manifestations of

Table 6.3 Approximate energy associated with various phenomena

Description	Energy (J)
Big Bang	10^{68}
Radio energy emitted by the galaxy during its lifetime	10^{55}
Rotational energy of the Milky Way	10^{52}
Energy released in a supernova explosion	10^{44}
Ocean's hydrogen in fusion	10^{34}
Rotational energy of the earth	10^{29}
Annual solar energy incident on the earth	5×10^{24}
Annual wind energy dissipated near earth's surface	10^{22}
Annual global energy usage by human	3×10^{20}
Annual energy dissipated by the tides	10^{20}
Energy release of 15-megaton fusion bomb	10^{17}
Annual electrical output of large generating plant	10^{16}
Thunderstorm	10^{15}
Energy released in burning 1000 kg of coal	3×10^{10}
Kinetic energy of a large jet aircraft	10^{9}
Energy released in burning 1 litre of gasoline	3×10^{7}
Daily food intake of a human adult	10^{7}
Work done by a human heart per beat	0.5
Turning this page	10^{-3}
Flea hop	10^{-7}
Discharge of a single neuron	10^{-10}
Typical energy of a proton in a nucleus	10^{-13}
Typical energy of an electron in an atom	10^{-18}
Energy to break one bond in DNA	10^{-20}

the above equivalence of mass and energy [Eq. (6.20)]. On the other hand the explanation of the life-nourishing energy output of the sun is also based on the above equation. In this case effectively four light hydrogen nuclei fuse to form a helium nucleus whose mass is less than the sum of the masses of the reactants. This mass difference, called the mass defect Δm is the source of energy $(\Delta m)c^2$. In fission, a heavy nucleus like uranium $^{235}_{92}U$, is split by a neutron into lighter nuclei. Once again the final mass is less than the initial mass and the mass difference translates into energy, which can be tapped to provide electrical energy as in nuclear power plants (controlled nuclear fission) or can be employed in making nuclear weapons (uncontrolled nuclear fission). Strictly, the energy ΔE released in a chemical reaction can also be related to the mass defect $\Delta m = \Delta E/c^2$. However, for a chemical reaction, this mass defect is much smaller than for a nuclear reaction. Table 6.3 lists the total energies for a variety of events and phenomena.

▶ ***Example 6.10*** Examine Tables 6.1-6.3 and express (a) The energy required to break one bond in DNA in eV; (b) The kinetic energy of an air molecule (10^{-21} J) in eV; (c) The daily intake of a human adult in kilocalories.

Answer (a) Energy required to break one bond of DNA is

$$\frac{10^{-20}\,\text{J}}{1.6\times10^{-19}\,\text{J/eV}} \simeq 0.06 \text{ eV}$$

where the symbol '~' stands for approximate. Note 0.1 eV = 100 meV (100 millielectron volt).

(b) The kinetic energy of an air molecule is

$$\frac{10^{-21}\,\text{J}}{1.6\times10^{-19}\,\text{J/eV}} \simeq 0.0062 \text{ eV}$$

This is the same as 6.2 meV.

(c) The average human consumption in a day is

$$\frac{10^{7}\,\text{J}}{4.2\times10^{3}\ \text{J/kcal}} \simeq 2400 \text{ kcal}$$

We point out a common misconception created by newspapers and magazines. They mention food values in calories and urge us to restrict diet intake to below 2400 calories. What they should be saying is kilocalories (kcal) and not calories. A person consuming 2400 calories a day will soon starve to death! 1 food calorie is 1 kcal. ◀

6.10.6 The Principle of Conservation of Energy

We have seen that the total mechanical energy of the system is conserved if the forces doing work on it are conservative. If some of the forces involved are non-conservative, part of the mechanical energy may get transformed into other forms such as heat, light and sound. However, the total energy of an isolated system does not change, as long as one accounts for all forms of energy. Energy may be transformed from one form to another but the total energy of an isolated system remains constant. Energy can neither be created, nor destroyed.

Since the universe as a whole may be viewed as an isolated system, the total energy of the universe is constant. If one part of the universe loses energy, another part must gain an equal amount of energy.

The principle of conservation of energy cannot be proved. However, no violation of this principle has been observed. The concept of conservation and transformation of energy into various forms links together various branches of physics, chemistry and life sciences. It provides a unifying, enduring element in our scientific pursuits. From engineering point of view all electronic, communication and mechanical devices rely on some forms of energy transformation.

6.11 POWER

Often it is interesting to know not only the work done on an object, but also the rate at which this work is done. We say a person is physically fit if he not only climbs four floors of a building but climbs them fast. **Power** is defined as the time rate at which work is done or energy is transferred.

The average power of a force is defined as the ratio of the work, W, to the total time t taken

$$P_{av} = \frac{W}{t}$$

The instantaneous power is defined as the limiting value of the average power as time interval approaches zero,

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{6.21}$$

The work dW done by a force F for a displacement $d\mathbf{r}$ is $dW = \mathbf{F}.d\mathbf{r}$. The instantaneous power can also be expressed as

$$P = \mathbf{F}.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

$$= \mathbf{F}.\mathbf{v} \tag{6.22}$$

where $\mathbf{v}$ is the instantaneous velocity when the force is $\mathbf{F}$.

Power, like work and energy, is a scalar quantity. Its dimensions are $[ML^2T^{-3}]$. In the SI, its unit is called a watt (W). The watt is 1 J s^{-1}. The unit of power is named after James Watt, one of the innovators of the steam engine in the eighteenth century.

There is another unit of power, namely the horse-power (hp)

$$1 \text{ hp} = 746 \text{ W}$$

This unit is still used to describe the output of automobiles, motorbikes, etc.

We encounter the unit watt when we buy electrical goods such as bulbs, heaters and refrigerators. A 100 watt bulb which is on for 10 hours uses 1 kilowatt hour (kWh) of energy.

$$\begin{aligned} & 100 \text{ (watt)} \times 10 \text{ (hour)} \\ & = 1000 \text{ watt hour} \\ & = 1 \text{ kilowatt hour (kWh)} \\ & = 10^3 \text{ (W)} \times 3600 \text{ (s)} \\ & = 3.6 \times 10^6 \text{ J} \end{aligned}$$

Our electricity bills carry the energy consumption in units of kWh. Note that kWh is a unit of energy and not of power.

▶ ***Example 6.11*** An elevator can carry a maximum load of 1800 kg (elevator + passengers) is moving up with a constant speed of 2 m s^{-1}. The frictional force opposing the motion is 4000 N. Determine the minimum power delivered by the motor to the elevator in watts as well as in horse power.

Answer The downward force on the elevator is

$$F = m g + F_f = (1800 \times 10) + 4000 = 22000 \text{ N}$$

The motor must supply enough power to balance this force. Hence,

$$P = \mathbf{F} . \mathbf{v} = 22000 \times 2 = 44000 \text{ W} = 59 \text{ hp}$$ ◀

6.12 COLLISIONS

In physics we study motion (change in position). At the same time, we try to discover physical quantities, which do not change in a physical process. The laws of momentum and energy conservation are typical examples. In this section we shall apply these laws to a commonly encountered phenomena, namely collisions. Several games such as billiards, marbles or carrom involve collisions.We shall study the collision of two masses in an idealised form.

Consider two masses m_1 and m_2. The particle m_1 is moving with speed v_{1i}, the subscript '*i*' implying initial. We can cosider m_2 to be at rest. No loss of generality is involved in making such a selection. In this situation the mass m_1 collides with the stationary mass m_2 and this is depicted in Fig. 6.10.

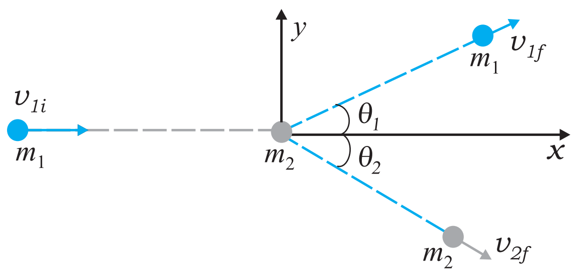


Fig. 6.10 *Collision of mass* m_1*, with a stationary mass* m_2*.*

The masses m_1 and m_2 fly-off in different directions. We shall see that there are relationships, which connect the masses, the velocities and the angles.

6.12.1 Elastic and Inelastic Collisions

In all collisions the total linear momentum is conserved; the initial momentum of the system is equal to the final momentum of the system. One can argue this as follows. When two objects collide, the mutual impulsive forces acting over the collision time Δt cause a change in their respective momenta :

$$\Delta \mathbf{p}_1 = \mathbf{F}_{12} \ \Delta t$$
$$\Delta \mathbf{p}_2 = \mathbf{F}_{21} \ \Delta t$$

where $\mathbf{F}_{12}$ is the force exerted on the first particle by the second particle. $\mathbf{F}_{21}$ is likewise the force exerted on the second particle by the first particle. Now from Newton's Third Law, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. This implies

$$\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = \mathbf{0}$$

The above conclusion is true even though the forces vary in a complex fashion during the collision time Δt. Since the third law is true at every instant, the total impulse on the first object is equal and opposite to that on the second.

On the other hand, the total kinetic energy of the system is not necessarily conserved. The impact and deformation during collision may generate heat and sound. Part of the initial kinetic energy is transformed into other forms of energy. A useful way to visualise the deformation during collision is in terms of a 'compressed spring'. If the 'spring' connecting the two masses regains its original shape without loss in energy, then the initial kinetic energy is equal to the final kinetic energy but the kinetic energy during the collision time Δt is not constant. Such a collision is called an **elastic collision**. On the other hand the deformation may not be relieved and the two bodies could move together after the collision. A collision in which the two particles move together after the collision is called a **completely inelastic collision**. The intermediate case where the deformation is partly relieved and some of the initial kinetic energy is lost is more common and is appropriately called an **inelastic collision**.

6.12.2 Collisions in One Dimension

Consider first a **completely inelastic collision** in one dimension. Then, in Fig. 6.10,

$$\theta_1 = \theta_2 = 0$$

$$m_1 v_{1i} = (m_1 + m_2) v_f \quad \text{(momentum conservation)}$$

$$v_f = \frac{m_1}{m_1 + m_2} v_{1i} \qquad (6.23)$$

The loss in kinetic energy on collision is

$$\Delta K = \frac{1}{2} m_1 v_{1i}^2 - \frac{1}{2}(m_1 + m_2) v_f^2$$

$$= \frac{1}{2} m_1 v_{1i}^2 - \frac{1}{2} \frac{m_1^2}{m_1 + m_2} v_{1i}^2 \quad \text{[using Eq. (6.23)]}$$

$$= \frac{1}{2} m_1 v_{1i}^2 \left[1 - \frac{m_1}{m_1 + m_2} \right]$$

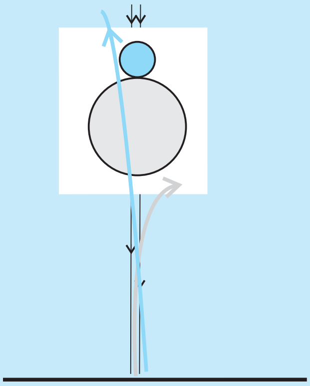
An experiment on head-on collision

In performing an experiment on collision on a horizontal surface, we face three difficulties. One, there will be friction and bodies will not travel with uniform velocities. Two, if two bodies of different sizes collide on a table, it would be difficult to arrange them for a head-on collision unless their centres of mass are at the same height above the surface. Three, it will be fairly difficult to measure velocities of the two bodies just before and just after collision.

By performing this experiment in a vertical direction, all the three difficulties vanish. Take two balls, one of which is heavier (basketball/football/volleyball) and the other lighter (tennis ball/rubber ball/table tennis ball). First take only the heavier ball and drop it vertically from some height, say 1 m. Note to which it rises. This gives the velocities near the floor or ground, just before and just after the bounce (by using $v^2 = 2gh$). Hence you will get the coefficient of restitution.

Now take the big ball and a small ball and hold them in your hands one over the other, with the heavier ball below the lighter one, as shown here. Drop them together, taking care that they remain together while falling, and see what happens. You will find that the heavier ball rises less than when it was dropped alone, while the lighter one shoots up to about 3 m. With practice, you will be able to hold the ball properly so that the lighter ball rises vertically up and does not fly sideways. This is head-on collision.

You can try to find the best combination of balls which gives you the best effect. You can measure the masses on a standard balance. We leave it to you to think how you can determine the initial and final velocities of the balls.



$$= \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_{1i}^2$$

which is a positive quantity as expected.

Consider next an elastic collision. Using the above nomenclature with $\theta_1 = \theta_2 = 0$, the momentum and kinetic energy conservation equations are

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \tag{6.24}$$

$$m_1 v_{1i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2 \tag{6.25}$$

From Eqs. (6.24) and (6.25) it follows that,

$$m_1 v_{1i}(v_{2f} - v_{1i}) = m_1 v_{1f}(v_{2f} - v_{1f})$$

or, $$v_{2f}(v_{1i} - v_{1f}) = v_{1i}^2 - v_{1f}^2$$

$$= (v_{1i} - v_{1f})(v_{1i} + v_{1f})$$

Hence, $$\therefore v_{2f} = v_{1i} + v_{1f} \tag{6.26}$$

Substituting this in Eq. (6.24), we obtain

$$v_{1f} = \frac{(m_1 - m_2)}{m_1 + m_2} v_{1i} \tag{6.27}$$

and $$v_{2f} = \frac{2 m_1 v_{1i}}{m_1 + m_2} \tag{6.28}$$

Thus, the 'unknowns' $\{v_{1f}, v_{2f}\}$ are obtained in terms of the 'knowns' $\{m_1, m_2, v_{1i}\}$. Special cases of our analysis are interesting.

Case I : If the two masses are equal

$$v_{1f} = 0$$
$$v_{2f} = v_{1i}$$

The first mass comes to rest and pushes off the second mass with its initial speed on collision.

Case II : If one mass dominates, e.g. $m_2 >> m_1$

$$v_{1f} \simeq -v_{1i} \qquad v_{2f} \simeq 0$$

The heavier mass is undisturbed while the lighter mass reverses its velocity.

▶ ***Example 6.12*** **Slowing down of neutrons:** In a nuclear reactor a neutron of high speed (typically 10^7 m s^{-1}) must be slowed

to 10^3 m s^{-1} so that it can have a high probability of interacting with isotope $^{235}_{92}U$ and causing it to fission. Show that a neutron can lose most of its kinetic energy in an elastic collision with a light nuclei like deuterium or carbon which has a mass of only a few times the neutron mass. The material making up the light nuclei, usually heavy water (D_2O) or graphite, is called a moderator.

Answer The initial kinetic energy of the neutron is

$$K_{1i} = \frac{1}{2} m_1 v_{1i}^2$$

while its final kinetic energy from Eq. (6.27)

$$K_{1f} = \frac{1}{2} m_1 v_{1f}^2 = \frac{1}{2} m_1 \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 v_{1i}^2$$

The fractional kinetic energy lost is

$$f_1 = \frac{K_{1f}}{K_{1i}} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2$$

while the fractional kinetic energy gained by the moderating nuclei K_{2f}/K_{1i} is

$f_2 = 1 - f_1$ (elastic collision)

$$= \frac{4 m_1 m_2}{(m_1 + m_2)^2}$$

One can also verify this result by substituting from Eq. (6.28).

For deuterium $m_2 = 2m_1$ and we obtain $f_1 = 1/9$ while $f_2 = 8/9$. Almost 90% of the neutron's energy is transferred to deuterium. For carbon $f_1 = 71.6\%$ and $f_2 = 28.4\%$. In practice, however, this number is smaller since head-on collisions are rare. ◀

If the initial velocities and final velocities of both the bodies are along the same straight line, then it is called a one-dimensional collision, or **head-on collision.** In the case of small spherical bodies, this is possible if the direction of travel of body 1 passes through the centre of body 2 which is at rest. In general, the collision is two-dimensional, where the initial velocities and the final velocities lie in a plane.

6.12.3 Collisions in Two Dimensions

Fig. 6.10 also depicts the collision of a moving mass m_1 with the stationary mass m_2. Linear momentum is conserved in such a collision. Since momentum is a vector this implies three equations for the three directions {x, y, z}. Consider the plane determined by the final velocity directions of m_1 and m_2 and choose it to be the x-y plane. The conservation of the z-component of the linear momentum implies that the entire collision is in the x-y plane. The x- and y-component equations are

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2 \qquad (6.29)$$

$$0 = m_1 v_{1f} \sin\theta_1 - m_2 v_{2f} \sin\theta_2 \qquad (6.30)$$

One knows {m_1, m_2, v_{1i}} in most situations. There are thus four unknowns {v_{1f}, v_{2f}, θ_1 *and* θ_2}, and only two equations. If $\theta_1 = \theta_2 = 0$, we regain Eq. (6.24) for one dimensional collision.

If, further the collision is elastic,

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \qquad (6.31)$$

We obtain an additional equation. That still leaves us one equation short. At least one of the four unknowns, say θ_1, must be made known for the problem to be solvable. For example, θ_1 can be determined by moving a detector in an angular fashion from the x to the y axis. Given {m_1, m_2, v_{1i}, θ_1} we can determine {v_{1f}, v_{2f}, θ_2} from Eqs. (6.29)-(6.31).

▶ ***Example 6.13*** Consider the collision depicted in Fig. 6.10 to be between two billiard balls with equal masses $m_1 = m_2$. The first ball is called the cue while the second ball is called the target. The billiard player wants to 'sink' the target ball in a corner pocket, which is at an angle $\theta_2 = 37°$. Assume that the collision is elastic and that friction and rotational motion are not important. Obtain θ_1.

Answer From momentum conservation, since the masses are equal

$$\mathbf{v}_{1i} = \mathbf{v}_{1f} + \mathbf{v}_{2f}$$

or

$$v_{1i}^2 = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f})$$

$$= v_{1f}^2 + v_{2f}^2 + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

$$= \left\{ v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}\cos\left(\theta_1 + 37°\right) \right\} \quad (6.32)$$

Since the collision is elastic and $m_1 = m_2$ it follows from conservation of kinetic energy that

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 \quad (6.33)$$

Comparing Eqs. (6.32) and (6.33), we get

$$\cos(\theta_1 + 37°) = 0$$

or $\quad \theta_1 + 37° = 90°$

Thus, $\quad \theta_1 = 53°$

This proves the following result : when two equal masses undergo a glancing elastic collision with one of them at rest, after the collision, they will move at right angles to each other. ◀

The matter simplifies greatly if we consider spherical masses with smooth surfaces, and assume that collision takes place only when the bodies touch each other. This is what happens in the games of marbles, carrom and billiards.

In our everyday world, collisions take place only when two bodies touch each other. But consider a comet coming from far distances to the sun, or alpha particle coming towards a nucleus and going away in some direction. Here we have to deal with forces involving action at a distance. Such an event is called scattering. The velocities and directions in which the two particles go away depend on their initial velocities as well as the type of interaction between them, their masses, shapes and sizes.

SUMMARY

1. The *work-energy theorem* states that the change in kinetic energy of a body is the work done by the net force on the body.
   $$K_f - K_i = W_{net}$$
2. A force is *conservative* if (i) work done by it on an object is path independent and depends only on the end points $\{x_i, x_j\}$, or (ii) the work done by the force is zero for an arbitrary closed path taken by the object such that it returns to its initial position.
3. For a conservative force in one dimension, we may define a *potential energy* function $V(x)$ such that
   $$F(x) = -\frac{dV(x)}{dx}$$
   or
   $$V_i - V_f = \int_{x_i}^{x_f} F(x)\,dx$$
4. The principle of conservation of mechanical energy states that the total mechanical energy of a body remains constant if the only forces that act on the body are conservative.
5. The *gravitational potential energy* of a particle of mass m at a height x about the earth's surface is
   $$V(x) = m\,g\,x$$
   where the variation of g with height is ignored.
6. The elastic potential energy of a spring of force constant k and extension x is
   $$V(x) = \frac{1}{2}k\,x^2$$
7. The scalar or dot product of two vectors **A** and **B** is written as **A.B** and is a scalar quantity given by : **A·B** = $AB\cos\theta$, where θ is the angle between **A** and **B**. It can be positive, negative or zero depending upon the value of θ. The scalar product of two vectors can be interpreted as the product of magnitude of one vector and component of the other vector along the first vector. For unit vectors :
   $$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1 \text{ and } \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0$$
   Scalar products obey the commutative and the distributive laws.

Physical Quantity	Symbol	Dimensions	Units	Remarks
Work	W	$[ML^2T^{-2}]$	J	W = **F.d**
Kinetic energy	K	$[ML^2T^{-2}]$	J	$K = \frac{1}{2}mv^2$
Potential energy	$V(x)$	$[ML^2T^{-2}]$	J	$F(x) = -\frac{dV(x)}{dx}$
Mechanical energy	E	$[ML^2T^{-2}]$	J	$E = K + V$
Spring constant	k	$[MT^{-2}]$	N m^{-1}	$F = -kx$ $V(x) = \frac{1}{2}kx^2$
Power	P	$[ML^2T^{-3}]$	W	P = **F.v** $P = \frac{dW}{dt}$

POINTS TO PONDER

1. The phrase 'calculate the work done' is incomplete. We should refer (or imply clearly by context) to the work done by a specific force or a group of forces on a given body over a certain displacement.
2. Work done is a scalar quantity. It can be positive or negative unlike mass and kinetic energy which are positive scalar quantities. The work done by the friction or viscous force on a moving body is negative.
3. For two bodies, the sum of the mutual forces exerted between them is zero from Newton's Third Law,

   $$\mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

   But the sum of the work done by the two forces need not *always cancel, i.e.*

   $$W_{12} + W_{21} \neq 0$$

   However, it *may sometimes be true.*
4. The work done by a force can be calculated sometimes even if the exact nature of the force is not known. This is clear from Example 6.1 where the WE theorem is used in such a situation.
5. The WE theorem is not independent of Newton's Second Law. The WE theorem may be viewed as a scalar form of the Second Law. The principle of conservation of mechanical energy may be viewed as a consequence of the WE theorem for conservative forces.
6. The WE theorem holds in all inertial frames. It can also be extended to non-inertial frames provided we include the pseudoforces in the calculation of the net force acting on the body under consideration.
7. The potential energy of a body subjected to a conservative force is always undetermined upto a constant. For example, the point where the potential energy is zero is a matter of choice. For the gravitational potential energy mgh, the zero of the potential energy is chosen to be the ground. For the spring potential energy $kx^2/2$, the zero of the potential energy is the equilibrium position of the oscillating mass.
8. Every force encountered in mechanics does not have an associated potential energy. For example, work done by friction over a closed path is not zero and no potential energy can be associated with friction.
9. During a collision : (a) the total linear momentum is conserved at each instant of the collision ; (b) the kinetic energy conservation (even if the collision is elastic) applies after the collision is over and does not hold at every instant of the collision. In fact the two colliding objects are deformed and may be momentarily at rest with respect to each other.

EXERCISES

6.1 The sign of work done by a force on a body is important to understand. State carefully if the following quantities are positive or negative:

(a) work done by a man in lifting a bucket out of a well by means of a rope tied to the bucket.

(b) work done by gravitational force in the above case,

(c) work done by friction on a body sliding down an inclined plane,

(d) work done by an applied force on a body moving on a rough horizontal plane with uniform velocity,

(e) work done by the resistive force of air on a vibrating pendulum in bringing it to rest.

6.2 A body of mass 2 kg initially at rest moves under the action of an applied horizontal force of 7 N on a table with coefficient of kinetic friction = 0.1. Compute the

(a) work done by the applied force in 10 s,

(b) work done by friction in 10 s,

(c) work done by the net force on the body in 10 s,

(d) change in kinetic energy of the body in 10 s,

and interpret your results.

6.3 Given in Fig. 6.11 are examples of some potential energy functions in one dimension. The total energy of the particle is indicated by a cross on the ordinate axis. In each case, specify the regions, if any, in which the particle cannot be found for the given energy. Also, indicate the minimum total energy the particle must have in each case. Think of simple physical contexts for which these potential energy shapes are relevant.

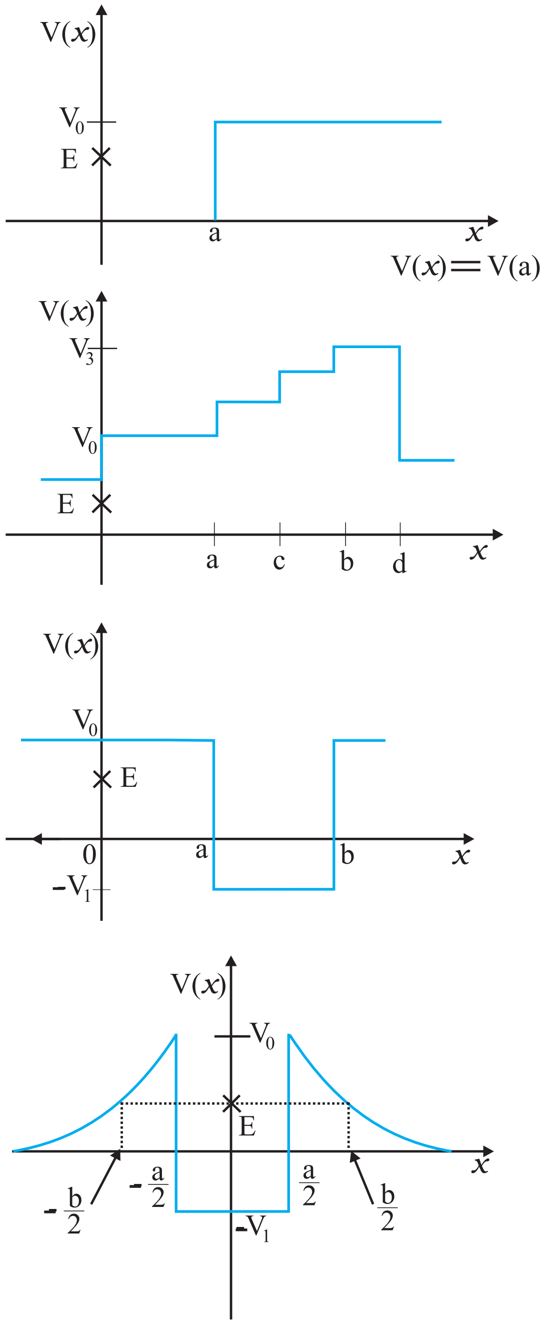


Fig. 6.11

6.4 The potential energy function for a particle executing linear simple harmonic motion is given by $V(x) = kx^2/2$, where k is the force constant of the oscillator. For $k = 0.5$ N m^{-1}, the graph of $V(x)$ versus x is shown in Fig. 6.12. Show that a particle of total energy 1 J moving under this potential must 'turn back' when it reaches $x = \pm 2$ m.

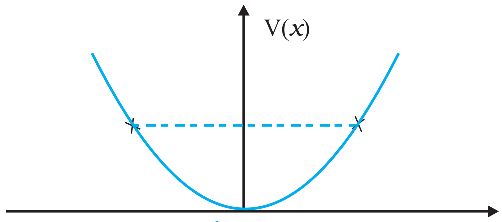


Fig. 6.12

6.5 Answer the following :

(a) The casing of a rocket in flight burns up due to friction. At whose expense is the heat energy required for burning obtained? The rocket or the atmosphere?

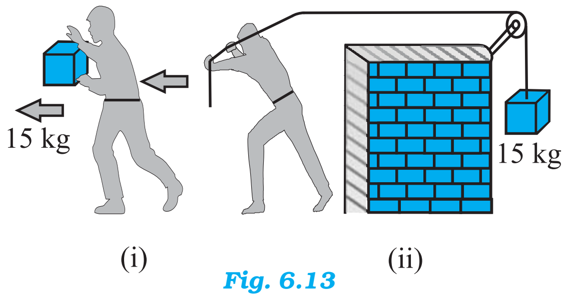


Fig. 6.13

(b) Comets move around the sun in highly elliptical orbits. The gravitational force on the comet due to the sun is not normal to the comet's velocity in general. Yet the work done by the gravitational force over every complete orbit of the comet is zero. Why ?

(c) An artificial satellite orbiting the earth in very thin atmosphere loses its energy gradually due to dissipation against atmospheric resistance, however small. Why then does its speed increase progressively as it comes closer and closer to the earth ?

(d) In Fig. 6.13(i) the man walks 2 m carrying a mass of 15 kg on his hands. In Fig. 6.13(ii), he walks the same distance pulling the rope behind him. The rope goes over a pulley, and a mass of 15 kg hangs at its other end. In which case is the work done greater ?

6.6 Underline the correct alternative :

(a) When a conservative force does positive work on a body, the potential energy of the body increases/decreases/remains unaltered.

(b) Work done by a body against friction always results in a loss of its kinetic/potential energy.

(c) The rate of change of total momentum of a many-particle system is proportional to the external force/sum of the internal forces on the system.

(d) In an inelastic collision of two bodies, the quantities which do not change after the collision are the total kinetic energy/total linear momentum/total energy of the system of two bodies.

6.7 State if each of the following statements is true or false. Give reasons for your answer.

(a) In an elastic collision of two bodies, the momentum and energy of each body is conserved.

(b) Total energy of a system is always conserved, no matter what internal and external forces on the body are present.

(c) Work done in the motion of a body over a closed loop is zero for every force in nature.

(d) In an inelastic collision, the final kinetic energy is always less than the initial kinetic energy of the system.

6.8 Answer carefully, with reasons :

(a) In an elastic collision of two billiard balls, is the total kinetic energy conserved during the short time of collision of the balls (i.e. when they are in contact) ?

(b) Is the total linear momentum conserved during the short time of an elastic collision of two balls ?

(c) What are the answers to (a) and (b) for an inelastic collision ?

(d) If the potential energy of two billiard balls depends only on the separation distance between their centres, is the collision elastic or inelastic ? (Note, we are talking here of potential energy corresponding to the force during collision, not gravitational potential energy).

6.9 A body is initially at rest. It undergoes one-dimensional motion with constant acceleration. The power delivered to it at time t is proportional to

(i) $t^{1/2}$ (ii) t (iii) $t^{3/2}$ (iv) t^2

6.10 A body is moving unidirectionally under the influence of a source of constant power. Its displacement in time t is proportional to

(i) $t^{1/2}$ (ii) t (iii) $t^{3/2}$ (iv) t^2

6.11 A body constrained to move along the z-axis of a coordinate system is subject to a constant force **F** given by

$$\mathbf{F} = -\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}\ \text{N}$$

where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors along the x-, y- and z-axis of the system respectively. What is the work done by this force in moving the body a distance of 4 m along the z-axis ?

6.12 An electron and a proton are detected in a cosmic ray experiment, the first with kinetic energy 10 keV, and the second with 100 keV. Which is faster, the electron or the proton ? Obtain the ratio of their speeds. (electron mass = 9.11×10^{-31} kg, proton mass = 1.67×10^{-27} kg, 1 eV = 1.60×10^{-19} J).

6.13 A rain drop of radius 2 mm falls from a height of 500 m above the ground. It falls with decreasing acceleration (due to viscous resistance of the air) until at half its original height, it attains its maximum (terminal) speed, and moves with uniform speed thereafter. What is the work done by the gravitational force on the drop in the first and second half of its journey ? What is the work done by the resistive force in the entire journey if its speed on reaching the ground is 10 m s^{-1} ?

6.14 A molecule in a gas container hits a horizontal wall with speed 200 m s^{-1} and angle 30° with the normal, and rebounds with the same speed. Is momentum conserved in the collision ? Is the collision elastic or inelastic ?

6.15 A pump on the ground floor of a building can pump up water to fill a tank of volume 30 m^3 in 15 min. If the tank is 40 m above the ground, and the efficiency of the pump is 30%, how much electric power is consumed by the pump ?

6.16 Two identical ball bearings in contact with each other and resting on a frictionless table are hit head-on by another ball bearing of the same mass moving initially with a speed V. If the collision is elastic, which of the following (Fig. 6.14) is a possible result after collision ?

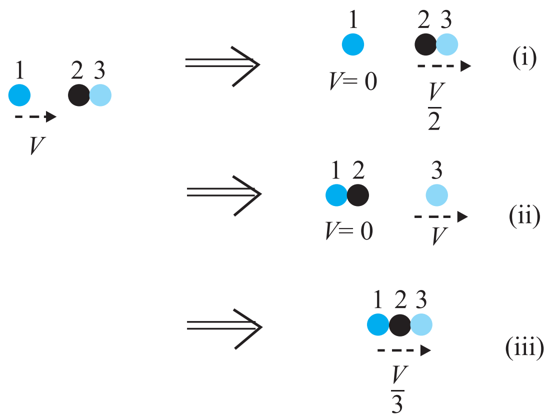


Fig. 6.14

6.17 The bob A of a pendulum released from 30° to the vertical hits another bob B of the same mass at rest on a table as shown in Fig. 6.15. How high does the bob A rise after the collision ? Neglect the size of the bobs and assume the collision to be elastic.

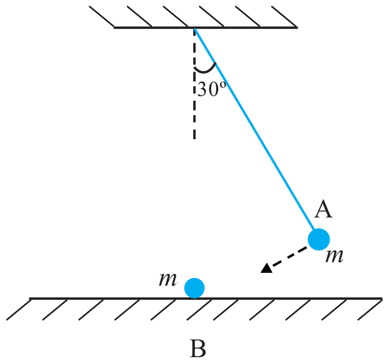


Fig. 6.15

6.18 The bob of a pendulum is released from a horizontal position. If the length of the pendulum is 1.5 m, what is the speed with which the bob arrives at the lowermost point, given that it dissipated 5% of its initial energy against air resistance ?

6.19 A trolley of mass 300 kg carrying a sandbag of 25 kg is moving uniformly with a speed of 27 km/h on a frictionless track. After a while, sand starts leaking out of a hole on the floor of the trolley at the rate of 0.05 kg s^{-1}. What is the speed of the trolley after the entire sand bag is empty ?

6.20 A body of mass 0.5 kg travels in a straight line with velocity $v = a\, x^{3/2}$ where a = 5 $m^{-1/2}$ s^{-1}. What is the work done by the net force during its displacement from x = 0 to x = 2 m ?

6.21 The blades of a windmill sweep out a circle of area A. (a) If the wind flows at a velocity v perpendicular to the circle, what is the mass of the air passing through it in time t ? (b) What is the kinetic energy of the air ? (c) Assume that the windmill converts 25% of the wind's energy into electrical energy, and that A = 30 m^2, v = 36 km/h and the density of air is 1.2 kg m^{-3}. What is the electrical power produced ?

6.22 A person trying to lose weight (dieter) lifts a 10 kg mass, one thousand times, to a height of 0.5 m each time. Assume that the potential energy lost each time she lowers the mass is dissipated. (a) How much work does she do against the gravitational force ? (b) Fat supplies 3.8×10^7J of energy per kilogram which is converted to mechanical energy with a 20% efficiency rate. How much fat will the dieter use up?

6.23 A family uses 8 kW of power. (a) Direct solar energy is incident on the horizontal surface at an average rate of 200 W per square meter. If 20% of this energy can be converted to useful electrical energy, how large an area is needed to supply 8 kW? (b) Compare this area to that of the roof of a typical house.

Additional Exercises

6.24 A bullet of mass 0.012 kg and horizontal speed 70 m s^{-1} strikes a block of wood of mass 0.4 kg and instantly comes to rest with respect to the block. The block is suspended from the ceiling by means of thin wires. Calculate the height to which the block rises. Also, estimate the amount of heat produced in the block.

6.25 Two inclined frictionless tracks, one gradual and the other steep meet at A from where two stones are allowed to slide down from rest, one on each track (Fig. 6.16). Will the stones reach the bottom at the same time? Will they reach there with the same speed? Explain. Given $\theta_1 = 30^0$, $\theta_2 = 60^0$, and h = 10 m, what are the speeds and times taken by the two stones ?

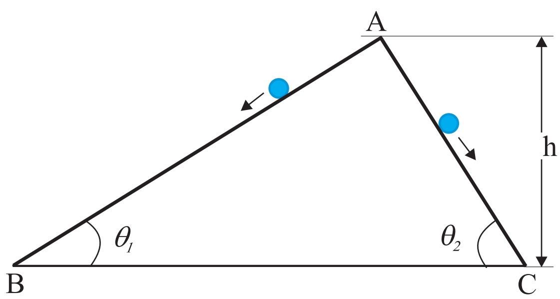


Fig. 6.16

6.26 A 1 kg block situated on a rough incline is connected to a spring of spring constant 100 N m^{-1} as shown in Fig. 6.17. The block is released from rest with the spring in the unstretched position. The block moves 10 cm down the incline before coming to rest. Find the coefficient of friction between the block and the incline. Assume that the spring has a negligible mass and the pulley is frictionless.

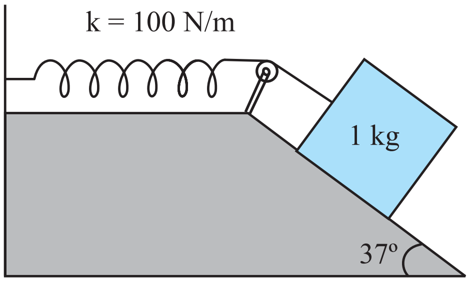


Fig. 6.17

6.27 A bolt of mass 0.3 kg falls from the ceiling of an elevator moving down with an uniform speed of 7 m s^{-1}. It hits the floor of the elevator (length of the elevator = 3 m) and does not rebound. What is the heat produced by the impact ? Would your answer be different if the elevator were stationary ?

6.28 A trolley of mass 200 kg moves with a uniform speed of 36 km/h on a frictionless track. A child of mass 20 kg runs on the trolley from one end to the other (10 m away) with a speed of 4 m s^{-1} relative to the trolley in a direction opposite to the its motion, and jumps out of the trolley. What is the final speed of the trolley ? How much has the trolley moved from the time the child begins to run ?

6.29 Which of the following potential energy curves in Fig. 6.18 cannot possibly describe the elastic collision of two billiard balls ? Here r is the distance between centres of the balls.

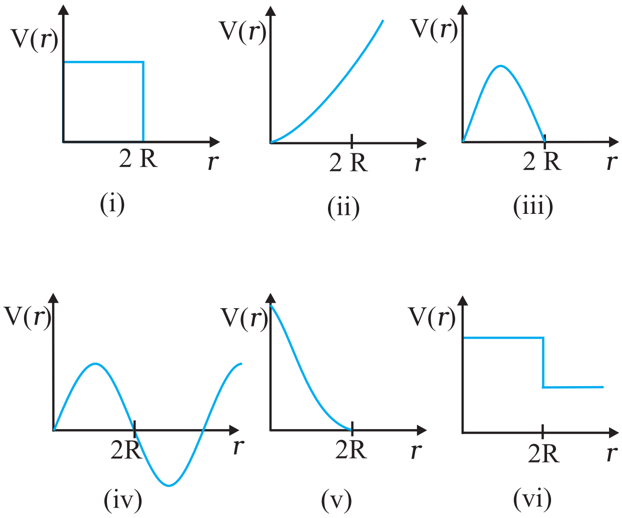


Fig. 6.18

6.30 Consider the decay of a free neutron at rest : $n \to p + e^-$

Show that the two-body decay of this type must necessarily give an electron of fixed energy and, therefore, cannot account for the observed continuous energy distribution in the β-decay of a neutron or a nucleus (Fig. 6.19).

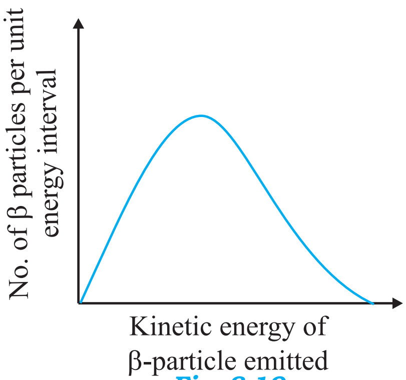


Fig. 6.19

[Note: The simple result of this exercise was one among the several arguments advanced by W. Pauli to predict the existence of a third particle in the decay products of β-decay. This particle is known as neutrino. We now know that it is a particle of intrinsic spin ½ (like e^-, p or n), but is neutral, and either massless or having an extremely small mass (compared to the mass of electron) and which interacts very weakly with matter. The correct decay process of neutron is : $n \rightarrow p + e^- + \nu$]

APPENDIX 6.1 : POWER CONSUMPTION IN WALKING

The table below lists the approximate power expended by an adult human of mass 60 kg.

Table 6.4 **Approximate power consumption**

Activity	Power (W)
Sleeping	75
Slow Walking	200
Bicycling	500
Heart beat	1.2

Mechanical work must not be confused with the everyday usage of the term work. A woman standing with a very heavy load on her head may get very tired. But no mechanical work is involved. That is not to say that mechanical work cannot be estimated in ordinary human activity.

Consider a person walking with constant speed v_0. The mechanical work he does may be estimated simply with the help of the work-energy theorem. Assume :

(a) The major work done in walking is due to the acceleration and deceleration of the legs with each stride (See Fig. 6.20).
(b) Neglect air resistance.
(c) Neglect the small work done in lifting the legs against gravity.
(d) Neglect the swinging of hands etc. as is common in walking.

As we can see in Fig. 6.20, in each stride the leg is brought from rest to a speed, approximately equal to the speed of walking, and then brought to rest again.

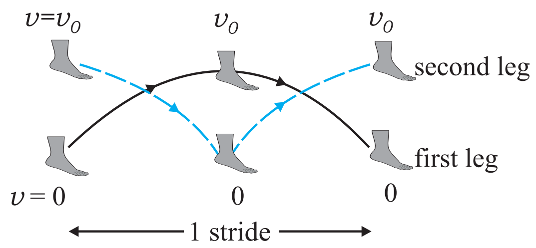


Fig. 6.20 *An illustration of a single stride in walking. While the first leg is maximally off the round, the second leg is on the ground and vice-versa*

The work done by one leg in each stride is $m_l\, v_0^2$ by the work-energy theorem. Here m_l is the mass of the leg.

Note $m_l\, v_0^2/2$ energy is expended by one set of leg muscles to bring the foot from rest to speed v_0 while an additional $m_l\, v_0^2/2$ is expended by a complementary set of leg muscles to bring the foot to rest from speed v_0. Hence work done by both legs in one stride is (study Fig. 6.20 carefully)

$$W_s = 2m_l\, v_0^2 \tag{6.34}$$

Assuming m_l = 10 kg and slow running of a nine-minute mile which translates to 3 m s^{-1} in SI units, we obtain

$$W_s = 180 \text{ J / stride}$$

If we take a stride to be 2 m long, the person covers 1.5 strides per second at his speed of 3 m s^{-1}. Thus the power expended

$$P = 180 \frac{\text{J}}{\text{stride}} \times 1.5 \frac{\text{stride}}{\text{second}}$$
$$= 270 \text{ W}$$

We must bear in mind that this is a lower estimate since several avenues of power loss (e.g. swinging of hands, air resistance etc.) have been ignored. The interesting point is that we did not worry about the forces involved. The forces, mainly friction and those exerted on the leg by the muscles of the rest of the body, are hard to estimate. Static friction does no work and we bypassed the impossible task of estimating the work done by the muscles by taking recourse to the work-energy theorem. We can also see the advantage of a wheel. The wheel permits smooth locomotion without the continual starting and stopping in mammalian locomotion.

CHAPTER SEVEN

SYSTEMS OF PARTICLES AND ROTATIONAL MOTION

7.1 INTRODUCTION

In the earlier chapters we primarily considered the motion of a single particle (A particle is represented as a point mass. It has practically no size). We applied the results of our study even to the motion of bodies of finite size, assuming that motion of such bodies can be described in terms of the motion of a particle.

Any real body which we encounter in daily life has a finite size. In dealing with the motion of extended bodies (bodies of finite size) often the idealised model of a particle is inadequate. In this chapter we shall try to go beyond this inadequacy. We shall attempt to build an understanding of the motion of extended bodies. An extended body, in the first place, is a system of particles. We shall begin with the consideration of motion of the system as a whole. The centre of mass of a system of particles will be a key concept here. We shall discuss the motion of the centre of mass of a system of particles and usefulness of this concept in understanding the motion of extended bodies.

A large class of problems with extended bodies can be solved by considering them to be rigid bodies. **Ideally a rigid body is a body with a perfectly definite and unchanging shape. The distances between different pairs of such a body do not change.** It is evident from this definition of a rigid body that no real body is truly rigid, since real bodies deform under the influence of forces. But in many situations the deformations are negligible. Thus, in a number of situations involving bodies such as wheels, tops, steel beams, molecules and planets on the other hand, we can ignore that they warp, bend or vibrate and treat them as rigid.

7.1.1 What kind of motion can a rigid body have?

Let us try to explore this question by taking some examples of the motion of rigid bodies. Let us begin with a rectangular block sliding down an inclined plane without any sidewise

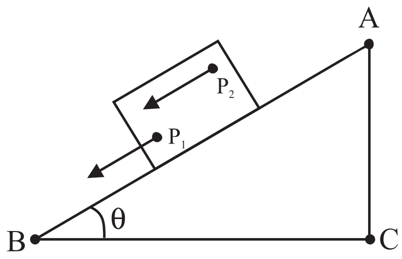


Fig 7.1 *Translational (sliding) motion of a block down an inclined plane. (Any point like P_1 or P_2 of the block moves with the same velocity at any instant of time.)*

movement. The block is a rigid body. Its motion down the plane is such that all the particles of the body are moving together, i.e. they have the same velocity at any instant of time. The rigid body here is in pure translational motion (Fig. 7.1).

In pure translational motion at any instant of time every particle of the body has the same velocity.

Consider now the rolling motion of a solid metallic or wooden cylinder down the same inclined plane (Fig. 7.2). The rigid body in this problem, namely the cylinder, shifts from the top to the bottom of the inclined plane, and thus, has translational motion. But as Fig. 7.2 shows, all its particles are not moving with the same velocity at any instant. The body therefore, is not in pure translation. Its motion is translation plus 'something else.'

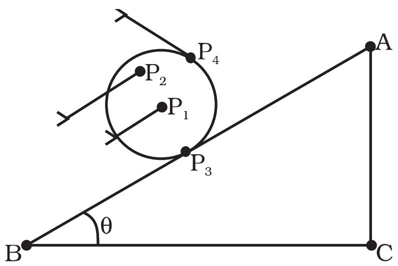


Fig. 7.2 *Rolling motion of a cylinder It is not pure translational motion. Points P_1, P_2, P_3 and P_4 have different velocities (shown by arrows) at any instant of time. In fact, the velocity of the point of contact P_3 is zero at any instant, if the cylinder rolls without slipping.*

In order to understand what this 'something else' is, let us take a rigid body so constrained that it cannot have translational motion. The most common way to constrain a rigid body so that it does not have translational motion is to fix it along a straight line. The only possible motion of such a rigid body is **rotation**. The line along which the body is fixed is termed as its **axis of rotation**. If you look around, you will come across many examples of rotation about an axis, a ceiling fan, a potter's wheel, a giant wheel in a fair, a merry-go-round and so on (Fig 7.3(a) and (b)).

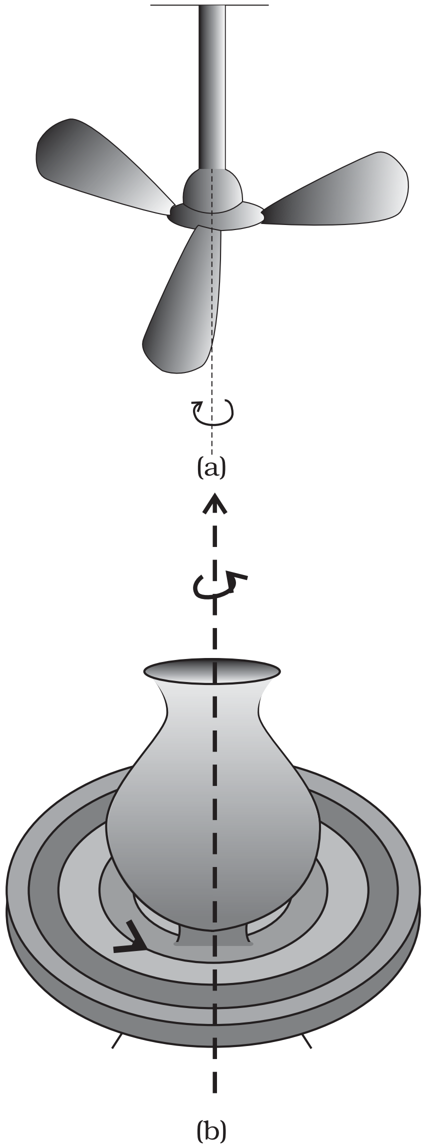


Fig. 7.3 *Rotation about a fixed axis*
(a) A ceiling fan
(b) A potter's wheel.

Let us try to understand what rotation is, what characterises rotation. You may notice that **in rotation of a rigid body about a fixed**

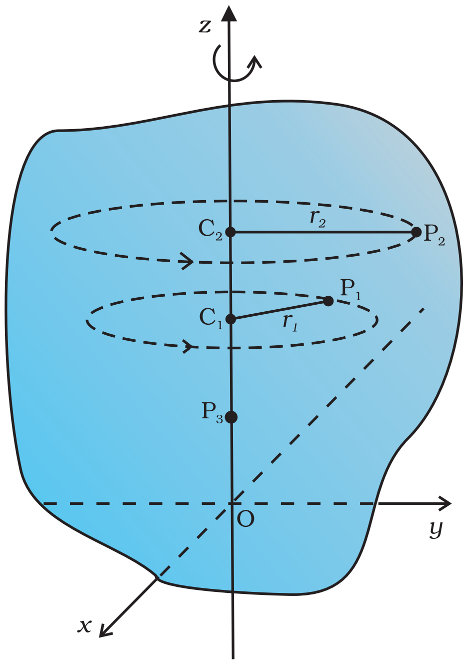


Fig. 7.4 *A rigid body rotation about the z-axis (Each point of the body such as P_1 or P_2 describes a circle with its centre (C_1 or C_2) on the axis. The radius of the circle ($r_1$or r_2) is the perpendicular distance of the point (P_1 or P_2) from the axis. A point on the axis like P_3 remains stationary).*

axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis. Fig. 7.4 shows the rotational motion of a rigid body about a fixed axis (the *z*-axis of the frame of reference). Let P_1 be a particle of the rigid body, arbitrarily chosen and at a distance r_1 from fixed axis. The particle P_1 describes a circle of radius r_1 with its centre C_1 on the fixed axis. The circle lies in a plane perpendicular to the axis. The figure also shows another particle P_2 of the rigid body, P_2 is at a distance r_2 from the fixed axis. The particle P_2 moves in a circle of radius r_2 and with centre C_2 on the axis. This circle, too, lies in a plane perpendicular to the axis. Note that the circles described by P_1 and P_2 may lie in different planes; both these planes, however, are perpendicular to the fixed axis. For any particle on the axis like P_3, $r = 0$. Any such particle remains stationary while the body rotates. This is expected since the axis is fixed.

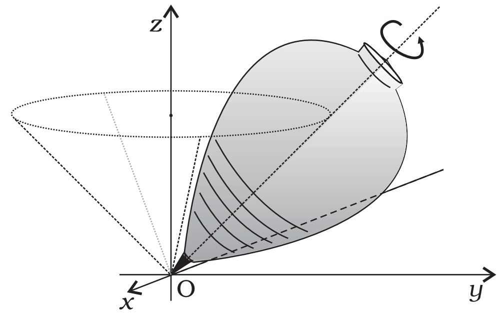


Fig. 7.5 (a) *A spinning top (The point of contact of the top with the ground, its tip O, is fixed.)*

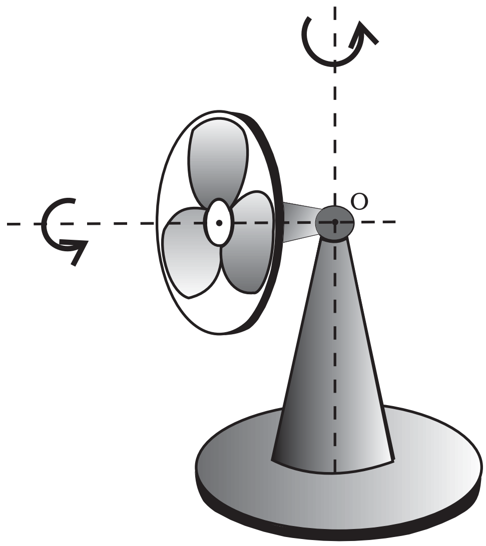


Fig. 7.5 (b) *An oscillating table fan. The pivot of the fan, point O, is fixed.*

In some examples of rotation, however, the axis may not be fixed. A prominent example of this kind of rotation is a top spinning in place [Fig. 7.5(a)]. (We assume that the top does not slip from place to place and so does not have translational motion.) We know from experience that the axis of such a spinning top moves around the vertical through its point of contact with the ground, sweeping out a cone as shown in Fig. 7.5(a). (This movement of the axis of the top around the vertical is termed **precession**.) Note, the **point of contact of the top with ground is fixed**. The axis of rotation of the top at any instant passes through the point of contact. Another simple example of this kind of rotation is the oscillating table fan or a pedestal fan. You may have observed that the axis of

rotation of such a fan has an oscillating (sidewise) movement in a horizontal plane about the vertical through the point at which the axis is pivoted (point O in Fig. 7.5(b)).

While the fan rotates and its axis moves sidewise, this point is fixed. Thus, in more general cases of rotation, such as the rotation of a top or a pedestal fan, **one point and not one line,** of the rigid body is fixed. In this case the axis is not fixed, though it always passes through the fixed point. In our study, however, we mostly deal with the simpler and special case of rotation in which one line (i.e. the axis) is fixed. Thus, for us rotation will be about a fixed axis only unless stated otherwise.

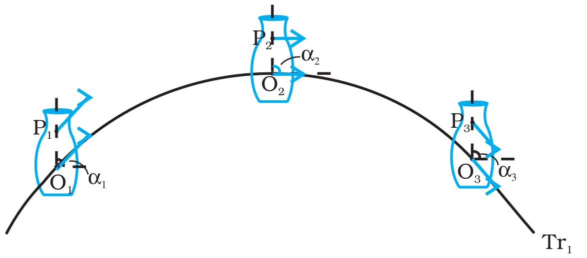


Fig. 7.6(a) *Motion of a rigid body which is pure translation.*

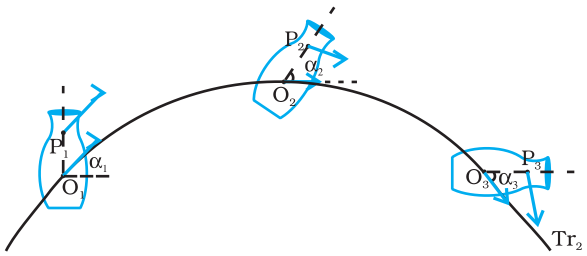


Fig. 7.6(b) *Motion of a rigid body which is a combination of translation and rotation.*

Fig 7.6 (a) and 7.6 (b) illustrate different motions of the same body. Note P is an arbitrary point of the body; O is the centre of mass of the body, which is defined in the next section. Suffice to say here that the trajectories of O are the translational trajectories Tr_1 and Tr_2 of the body. The positions O and P at three different instants of time are shown by O_1, O_2, and O_3, and P_1, P_2 and P_3 respectively in Fig. 7.6 (a) and (b) both. As seen from Fig. 7.6(a), at any instant the velocities of any particles like O and P of the body are the same in pure translation. Notice, in this case the orientation of OP, i.e. the angle OP makes with a fixed direction, say the horizontal, remains the same, i.e. $\alpha_1 = \alpha_2 = \alpha_3$. Fig. 7.6 (b) illustrates a case of combination of translation and rotation. In this case, at any instants the velocities of O and P differ. Also, α_1, α_2 and α_3 may all be different.

The rolling motion of a cylinder down an inclined plane is a combination of rotation about a fixed axis and translation. Thus, the 'something else' in the case of rolling motion which we referred to earlier is rotational motion. You will find Fig. 7.6(a) and (b) instructive from this point of view. Both these figures show motion of the same body along identical translational trajectory. In one case, Fig. 7.6(a), the motion is a pure translation; in the other case [Fig. 7.6(b)] it is a combination of translation and rotation. (You may try to reproduce the two types of motion shown using a rigid object like a heavy book.)

We now recapitulate the most important observations of the present section: **The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation. The motion of a rigid body which is pivoted or fixed in some way is rotation.** The rotation may be about an axis that is fixed (e.g. a ceiling fan) or moving (e.g. an oscillating table fan). We shall, in the present chapter, consider rotational motion about a fixed axis only.

7.2 CENTRE OF MASS

We shall first see what the centre of mass of a system of particles is and then discuss its significance. For simplicity we shall start with a two particle system. We shall take the line joining the two particles to be the x- axis.

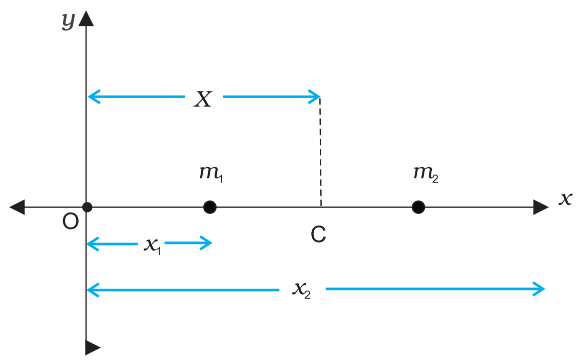


Fig. 7.7

Let the distances of the two particles be x_1 and x_2 respectively from some origin O. Let m_1 and m_2 be respectively the masses of the two

particles. The centre of mass of the system is that point C which is at a distance X from O, where X is given by

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad (7.1)$$

In Eq. (7.1), X can be regarded as the mass-weighted mean of x_1 and x_2. If the two particles have the same mass $m_1 = m_2 = m$ then

$$X = \frac{m x_1 + m x_2}{2m} = \frac{x_1 + x_2}{2}$$

Thus, for two particles of equal mass the centre of mass lies exactly midway between them.

If we have n particles of masses m_1, m_2, ...m_n respectively, along a straight line taken as the x- axis, then by definition the position of the centre of the mass of the system of particles is given by

$$X = \frac{m_1 x_1 + m_2 x_2 + ... + m_n x_n}{m_1 + m_2 + ... + m_n} = \frac{\sum m_i x_i}{\sum m_i} \quad (7.2)$$

where x_1, x_2,...x_n are the distances of the particles from the origin; X is also measured from the same origin. The symbol $\sum$ (the Greek letter sigma) denotes summation, in this case over n particles. The sum

$$\sum m_i = M$$

is the total mass of the system.

Suppose that we have three particles, not lying in a straight line. We may define x and y-axes in the plane in which the particles lie and represent the positions of the three particles by coordinates (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively. Let the masses of the three particles be m_1, m_2 and m_3 respectively. The centre of mass C of the system of the three particles is defined and located by the coordinates (X, Y) given by

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \quad (7.3a)$$

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \quad (7.3b)$$

For the particles of equal mass $m = m_1 = m_2 = m_3$,

$$X = \frac{m(x_1 + x_2 + x_3)}{3m} = \frac{x_1 + x_2 + x_3}{3}$$

$$Y = \frac{m(y_1 + y_2 + y_3)}{3m} = \frac{y_1 + y_2 + y_3}{3}$$

Thus, for three particles of equal mass, the centre of mass coincides with the centroid of the triangle formed by the particles.

Results of Eqs. (7.3a) and (7.3b) are generalised easily to a system of n particles, not necessarily lying in a plane, but distributed in space. The centre of mass of such a system is at (X, Y, Z), where

$$X = \frac{\sum m_i x_i}{M} \quad (7.4a)$$

$$Y = \frac{\sum m_i y_i}{M} \quad (7.4b)$$

and $$Z = \frac{\sum m_i z_i}{M} \quad (7.4c)$$

Here $M = \sum m_i$ is the total mass of the system. The index i runs from 1 to n; m_i is the mass of the i^{th} particle and the position of the i^{th} particle is given by (x_i, y_i, z_i).

Eqs. (7.4a), (7.4b) and (7.4c) can be combined into one equation using the notation of position vectors. Let $\mathbf{r}_i$ be the position vector of the i^{th} particle and $\mathbf{R}$ be the position vector of the centre of mass:

$$\mathbf{r}_i = x_i \hat{\mathbf{i}} + y_i \hat{\mathbf{j}} + z_i \hat{\mathbf{k}}$$

and $$\mathbf{R} = X \hat{\mathbf{i}} + Y \hat{\mathbf{j}} + Z \hat{\mathbf{k}}$$

Then $$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M} \quad (7.4d)$$

The sum on the right hand side is a vector sum.

Note the economy of expressions we achieve by use of vectors. If the origin of the frame of reference (the coordinate system) is chosen to be the centre of mass then $\sum m_i \mathbf{r}_i = 0$ for the given system of particles.

A rigid body, such as a metre stick or a flywheel, is a system of closely packed particles; Eqs. (7.4a), (7.4b), (7.4c) and (7.4d) are therefore, applicable to a rigid body. The number of particles (atoms or molecules) in such a body is so large that it is impossible to carry out the summations over individual particles in these equations. Since the spacing of the particles is

small, we can treat the body as a continuous distribution of mass. We subdivide the body into n small elements of mass; $\Delta m_1, \Delta m_2 \ldots \Delta m_n$; the i^{th} element Δm_i is taken to be located about the point (x_i, y_i, z_i). The coordinates of the centre of mass are then approximately given by

$$X = \frac{\sum(\Delta m_i)x_i}{\sum \Delta m_i}, Y = \frac{\sum(\Delta m_i)y_i}{\sum \Delta m_i}, Z = \frac{\sum(\Delta m_i)z_i}{\sum \Delta m_i}$$

As we make n bigger and bigger and each Δm_i smaller and smaller, these expressions become exact. In that case, we denote the sums over i by integrals. Thus,

$$\sum \Delta m_i \to \int dm = M,$$

$$\sum (\Delta m_i) x_i \to \int x\, dm,$$

$$\sum (\Delta m_i) y_i \to \int y\, dm,$$

and $$\sum (\Delta m_i) z_i \to \int z\, dm$$

Here M is the total mass of the body. The coordinates of the centre of mass now are

$$X = \frac{1}{M}\int x\, dm, \quad Y = \frac{1}{M}\int y\, dm \text{ and } Z = \frac{1}{M}\int z\, dm \quad (7.5a)$$

The vector expression equivalent to these three scalar expressions is

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\, dm \quad (7.5b)$$

If we choose, the centre of mass as the origin of our coordinate system,

$$\mathbf{R}\,(x, y, z) = \mathbf{0}$$

i.e., $\int \mathbf{r}\, dm = \mathbf{0}$

or $$\int x\, dm = \int y\, dm = \int z\, dm = 0 \quad (7.6)$$

Often we have to calculate the centre of mass of homogeneous bodies of regular shapes like rings, discs, spheres, rods etc. (By a homogeneous body we mean a body with uniformly distributed mass.) By using symmetry consideration, we can easily show that the centres of mass of these bodies lie at their geometric centres.

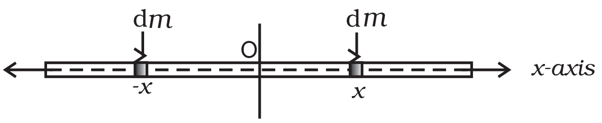


Fig. 7.8 *Determining the CM of a thin rod.*

Let us consider a thin rod, whose width and breath (in case the cross section of the rod is rectangular) or radius (in case the cross section of the rod is cylindrical) is much smaller than its length. Taking the origin to be at the geometric centre of the rod and x-axis to be along the length of the rod, we can say that on account of reflection symmetry, for every element dm of the rod at x, there is an element of the same mass dm located at $-x$ (Fig. 7.8).

The net contribution of every such pair to the integral and hence the integral $\int x\, dm$ itself is zero. From Eq. (7.6), the point for which the integral itself is zero, is the centre of mass. Thus, the centre of mass of a homogenous thin rod coincides with its geometric centre. This can be understood on the basis of reflection symmetry.

The same symmetry argument will apply to homogeneous rings, discs, spheres, or even thick rods of circular or rectangular cross section. For all such bodies you will realise that for every element dm at a point (x,y,z) one can always take an element of the same mass at the point $(-x,-y,-z)$. (In other words, the origin is a point of reflection symmetry for these bodies.) As a result, the integrals in Eq. (7.5 a) all are zero. This means that for all the above bodies, their centre of mass coincides with their geometric centre.

▶ ***Example 7.1*** Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

Answer

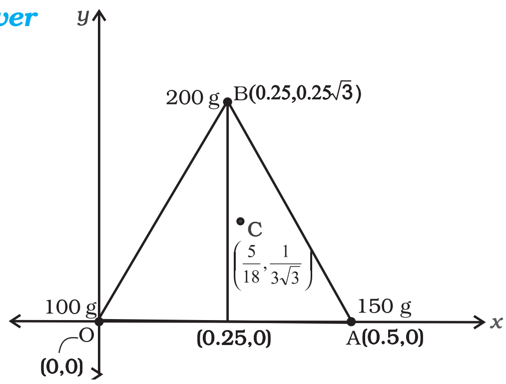


Fig. 7.9

With the X and Y axes chosen as shown in Fig. 7.9, the coordinates of points O, A and B forming the equilateral triangle are respectively (0,0), (0.5,0), (0.25,0.25$\sqrt{3}$). Let the masses 100 g, 150g and 200g be located at O, A and B be respectively. Then,

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{[100(0) + 150(0.5) + 200(0.25)] \text{ g m}}{(100 + 150 + 200) \text{ g}}$$

$$= \frac{75 + 50}{450} \text{m} = \frac{125}{450} \text{m} = \frac{5}{18} \text{m}$$

$$Y = \frac{[100(0) + 150(0) + 200(0.25\sqrt{3})] \text{ g m}}{450 \text{ g}}$$

$$= \frac{50\sqrt{3}}{450} \text{m} = \frac{\sqrt{3}}{9} \text{m} = \frac{1}{3\sqrt{3}} \text{m}$$

The centre of mass C is shown in the figure. Note that it is not the geometric centre of the triangle OAB. Why? ◀

▶ **Example 7.2** Find the centre of mass of a triangular lamina.

Answer The lamina (ΔLMN) may be subdivided into narrow strips each parallel to the base (MN) as shown in Fig. 7.10

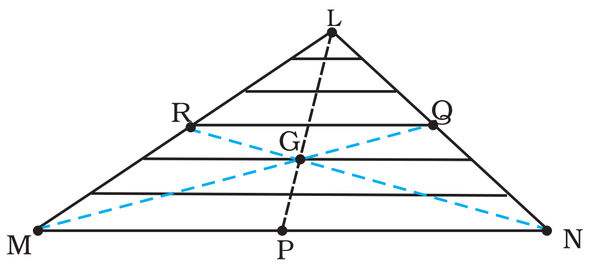


Fig. 7.10

By symmetry each strip has its centre of mass at its midpoint. If we join the midpoint of all the strips we get the median LP. The centre of mass of the triangle as a whole therefore, has to lie on the median LP. Similarly, we can argue that it lies on the median MQ and NR. This means the centre of mass lies on the point of concurrence of the medians, i.e. on the centroid G of the triangle. ◀

▶ **Example 7.3** Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown. The mass of the lamina is 3 kg.

Answer Choosing the X and Y axes as shown in Fig. 7.11 we have the coordinates of the vertices of the L-shaped lamina as given in the figure. We can think of the L-shape to consist of 3 squares each of length 1m. The mass of each square is 1kg, since the lamina is uniform. The centres of mass C_1, C_2 and C_3 of the squares are, by symmetry, their geometric centres and have coordinates (1/2,1/2), (3/2,1/2), (1/2,3/2) respectively. We take the masses of the squares to be concentrated at these points. The centre of mass of the whole L shape (X, Y) is the centre of mass of these mass points.

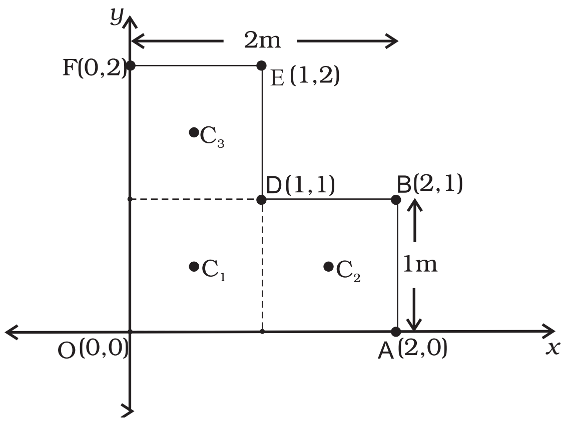


Fig. 7.11

Hence

$$X = \frac{[1(1/2) + 1(3/2) + 1(1/2)] \text{kg m}}{(1 + 1 + 1) \text{kg}} = \frac{5}{6} \text{m}$$

$$Y = \frac{[[1(1/2) + 1(1/2) + 1(3/2)]] \text{kg m}}{(1 + 1 + 1) \text{kg}} = \frac{5}{6} \text{m}$$

The centre of mass of the L-shape lies on the line OD. We could have guessed this without calculations. Can you tell why? Suppose, the three squares that make up the L shaped lamina

of Fig. 7.11 had different masses. How will you then determine the centre of mass of the lamina? ◀

7.3 MOTION OF CENTRE OF MASS

Equipped with the definition of the centre of mass, we are now in a position to discuss its physical importance for a system of particles. We may rewrite Eq.(7.4d) as

$$M\mathbf{R} = \sum m_i \mathbf{r}_i = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n \qquad (7.7)$$

Differentiating the two sides of the equation with respect to time we get

$$M\frac{d\mathbf{R}}{dt} = m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \ldots + m_n\frac{d\mathbf{r}_n}{dt}$$

or

$$M\mathbf{V} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n \qquad (7.8)$$

where $\mathbf{v}_1 (= d\mathbf{r}_1/dt)$ is the velocity of the first particle $\mathbf{v}_2 (= d\mathbf{r}_2/dt)$ is the velocity of the second particle etc. and $\mathbf{V} = d\mathbf{R}/dt$ is the velocity of the centre of mass. Note that we assumed the masses m_1, m_2, ... etc. do not change in time. We have therefore, treated them as constants in differentiating the equations with respect to time.

Differentiating Eq.(7.8) with respect to time, we obtain

$$M\frac{d\mathbf{V}}{dt} = m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} + \ldots + m_n\frac{d\mathbf{v}_n}{dt}$$

or

$$M\mathbf{A} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots + m_n\mathbf{a}_n \qquad (7.9)$$

where $\mathbf{a}_1 (= d\mathbf{v}_1/dt)$ is the acceleration of the first particle, $\mathbf{a}_2 (= d\mathbf{v}_2/dt)$ is the acceleration of the second particle etc. and $\mathbf{A} (= d\mathbf{V}/dt)$ is the acceleration of the centre of mass of the system of particles.

Now, from Newton's second law, the force acting on the first particle is given by $\mathbf{F}_1 = m_1\mathbf{a}_1$. The force acting on the second particle is given by $\mathbf{F}_2 = m_2\mathbf{a}_2$ and so on. Eq. (7.9) may be written as

$$M\mathbf{A} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n \qquad (7.10)$$

Thus, the total mass of a system of particles times the acceleration of its centre of mass is the vector sum of all the forces acting on the system of particles.

Note when we talk of the force $\mathbf{F}_1$ on the first particle, it is not a single force, but the vector sum of all the forces on the first particle; likewise for the second particle etc. Among these forces on each particle there will be **external** forces exerted by bodies outside the system and also **internal** forces exerted by the particles on one another. We know from Newton's third law that these internal forces occur in equal and opposite pairs and in the sum of forces of Eq. (7.10), their contribution is zero. Only the external forces contribute to the equation. We can then rewrite Eq. (7.10) as

$$M\mathbf{A} = \mathbf{F}_{ext} \qquad (7.11)$$

where $\mathbf{F}_{ext}$ represents the sum of all external forces acting on the particles of the system.

Eq. (7.11) states that **the centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.**

Notice, to determine the motion of the centre of mass no knowledge of internal forces of the system of particles is required; for this purpose we need to know only the external forces.

To obtain Eq. (7.11) we did not need to specify the nature of the system of particles. The system may be a collection of particles in which there may be all kinds of internal motions, or it may be a rigid body which has either pure translational motion or a combination of translational and rotational motion. Whatever is the system and the motion of its individual particles, the centre of mass moves according to Eq. (7.11).

Instead of treating extended bodies as single particles as we have done in earlier chapters, we can now treat them as systems of particles. We can obtain the translational component of their motion, i.e. the motion centre of mass of the system, by taking the mass of the whole system to be concentrated at the centre of mass and all the external forces on the system to be acting at the centre of mass.

This is the procedure that we followed earlier in analysing forces on bodies and solving

problems without explicitly outlining and justifying the procedure. We now realise that in earlier studies we assumed, without saying so, that rotational motion and/or internal motion of the particles were either absent or negligible. We no longer need to do this. We have not only found the justification of the procedure we followed earlier; but we also have found how to describe and separate the translational motion of (1) a rigid body which may be rotating as well, or (2) a system of particles with all kinds of internal motion.

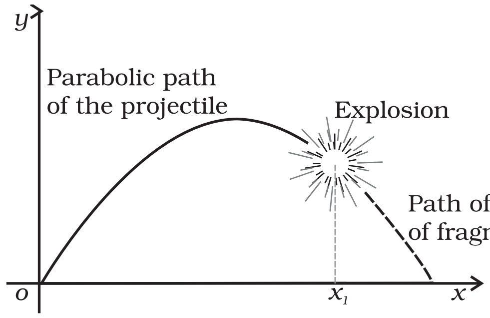


Fig. 7.12 *The centre of mass of the fragments of the projectile continues along the same parabolic path which it would have followed if there were no explosion.*

Figure 7.12 is a good illustration of Eq. (7.11). A projectile, following the usual parabolic trajectory, explodes into fragments midway in air. The forces leading to the explosion are internal forces. They contribute nothing to the motion of the centre of mass. The total external force, namely, the force of gravity acting on the body, is the same before and after the explosion. The centre of mass under the influence of the external force continues, therefore, along the same parabolic trajectory as it would have followed if there were no explosion.

7.4 LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

Let us recall that the linear momentum of a particle is defined as

$$\mathbf{p} = m\,\mathbf{v} \qquad (7.12)$$

Let us also recall that Newton's second law written in symbolic form for a single particle is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad (7.13)$$

where **F** is the force on the particle. Let us consider a system of n particles with masses $m_1, m_2, \ldots m_n$ respectively and velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n$ respectively. The particles may be interacting and have external forces acting on them. The linear momentum of the first particle is $m_1\mathbf{v}_1$, of the second particle is $m_2\mathbf{v}_2$ and so on.

For the system of n particles, the linear momentum of the system is defined to be the vector sum of all individual particles of the system,

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n$$

$$= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n \qquad (7.14)$$

Comparing this with Eq. (7.8)

$$\mathbf{P} = M\,\mathbf{V} \qquad (7.15)$$

Thus, **the total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass**. Differentiating Eq. (7.15) with respect to time,

$$\frac{d\mathbf{P}}{dt} = M\frac{d\mathbf{V}}{dt} = M\mathbf{A} \qquad (7.16)$$

Comparing Eq.(7.16) and Eq. (7.11),

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{ext} \qquad (7.17)$$

This is the statement of **Newton's second law extended to a system of particles.**

Suppose now, that the sum of external forces acting on a system of particles is zero. Then from Eq.(7.17)

$$\frac{d\mathbf{P}}{dt} = 0 \quad \textbf{or} \quad \mathbf{P} = \text{Constant} \qquad (7.18a)$$

Thus, when the total external force acting on a system of particles is zero, the total linear momentum of the system is constant. This is the law of conservation of the total linear momentum of a system of particles. Because of Eq. (7.15), this also means that when the total external force on the system is zero the velocity of the centre of mass remains constant. (We assume throughout the discussion on systems of particles in this chapter that the total mass of the system remains constant.)

Note that on account of the internal forces, i.e. the forces exerted by the particles on one another, the individual particles may have

complicated trajectories. Yet, if the total external force acting on the system is zero, the centre of mass moves with *a* constant velocity, i.e., moves uniformly in a straight line like a free particle.

The vector Eq. (7.18a) is equivalent to three scalar equations,

$$P_x = c_1,\ P_y = c_2 \text{ and } P_z = c_3 \qquad (7.18\text{ b})$$

Here P_x, P_y and P_z are the components of the total linear momentum vector **P** along the *x, y* and *z* axes respectively; c_1, c_2 and c_3 are constants.

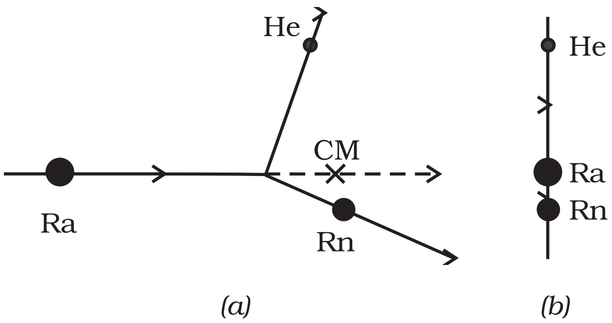


Fig. 7.13 (a) *A heavy nucleus (Ra) splits into a lighter nucleus (Rn) and an alpha particle (He). The CM of the system is in uniform motion.*
(b) *The same spliting of the heavy nucleus (Ra) with the centre of mass at rest. The two product particles fly back to back.*

As an example, let us consider the radioactive decay of a moving unstable particle, like the nucleus of radium. A radium nucleus disintegrates into a nucleus of radon and an alpha particle. The forces leading to the decay are internal to the system and the external forces on the system are negligible. So the total linear momentum of the system is the same before and after decay. The two particles produced in the decay, the radon nucleus and the alpha particle, move in different directions in such a way that their centre of mass moves along the same path along which the original decaying radium nucleus was moving [Fig. 7.13(a)].

If we observe the decay from the frame of reference in which the centre of mass is at rest, the motion of the particles involved in the decay looks particularly simple; the product particles

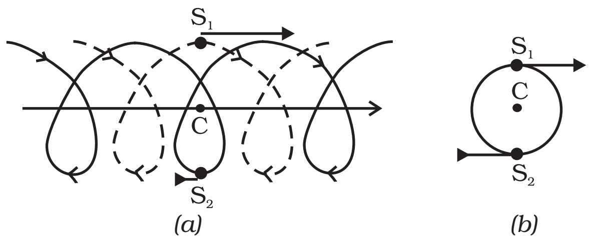


Fig. 7.14 (a) *Trajectories of two stars, S_1 (dotted line) and S_2 (solid line) forming a binary system with their centre of mass C in uniform motion.*
(b) *The same binary system, with the centre of mass C at rest.*

move back to back with their centre of mass remaining at rest as shown in Fig.7.13 (b).

In many problems on the system of particles as in the above radioactive decay problem, it is convenient to work in the centre of mass frame rather than in the laboratory frame of reference.

In astronomy, binary (double) stars is a common occurrence. If there are no external forces, the centre of mass of a double star moves like a free particle, as shown in Fig.7.14 (a). The trajectories of the two stars of equal mass are also shown in the figure; they look complicated. If we go to the centre of mass frame, then we find that there the two stars are moving in a circle, about the centre of mass, which is at rest. Note that the position of the stars have to be diametrically opposite to each other [Fig. 7.14(b)]. Thus in our frame of reference, the trajectories of the stars are a combination of (i) uniform motion in a straight line of the centre of mass and (ii) circular orbits of the stars about the centre of mass.

As can be seen from the two examples, **separating** the motion of different parts of a system into motion **of the centre of mass and motion about the centre of mass** is a very useful technique that helps in understanding the motion of the system.

7.5 VECTOR PRODUCT OF TWO VECTORS

We are already familiar with vectors and their use in physics. In chapter 6 (Work, Energy, Power) we defined the scalar product of two vectors. An important physical quantity, work, is defined as a scalar product of two vector quantities, force and displacement.

We shall now define another product of two vectors. This product is a vector. Two important quantities in the study of rotational motion, namely, moment of a force and angular momentum, are defined as vector products.

Definition of Vector Product

A vector product of two vectors **a** and **b** is a vector **c** such that

(i) magnitude of **c** = $c = ab\sin\theta$ where a and b are magnitudes of **a** and **b** and θ is the angle between the two vectors.

(ii) **c** is perpendicular to the plane containing **a** and **b**.

(iii) if we take a right handed screw with its head lying in the plane of **a** and **b** and the screw perpendicular to this plane, and if we turn the head in the direction from **a** to **b**, then the tip of the screw advances in the direction of **c**. This right handed screw rule is illustrated in Fig. 7.15a.

Alternately, if one curls up the fingers of right hand around a line perpendicular to the plane of the vectors **a** and **b** and if the fingers are curled up in the direction from **a** to **b**, then the stretched thumb points in the direction of **c**, as shown in Fig. 7.15b.

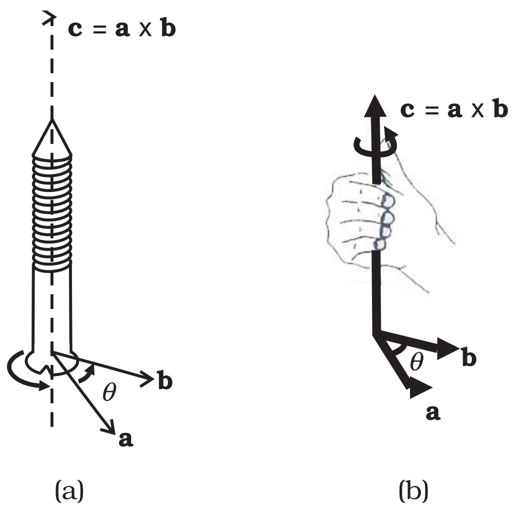


Fig. 7.15 (a) Rule of the right handed screw for defining the direction of the vector product of two vectors.
(b) Rule of the right hand for defining the direction of the vector product.

A simpler version of the right hand rule is the following : Open up your right hand palm and curl the fingers pointing from **a** to **b**. Your stretched thumb points in the direction of **c**.

It should be remembered that there are two angles between any two vectors **a** and **b** . In Fig. 7.15 (a) or (b) they correspond to θ (as shown) and ($360^0 - \theta$). While applying either of the above rules, the rotation should be taken through the smaller angle (<180^0) between **a** and **b**. It is θ here.

Because of the cross used to denote the vector product, it is also referred to as cross product.

- Note that scalar product of two vectors is commutative as said earlier, **a**.**b** = **b.a**

The vector product, however, is not commutative, i.e. **a** × **b** ≠ **b** × **a**

The magnitude of both **a** × **b** and **b** × **a** is the same ($ab\sin\theta$); also, both of them are perpendicular to the plane of **a** and **b**. But the rotation of the right-handed screw in case of **a** × **b** is from **a** to **b**, whereas in case of **b** × **a** it is from **b** to **a**. This means the two vectors are in opposite directions. We have

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$$

- Another interesting property of a vector product is its behaviour under reflection. Under reflection (i.e. on taking the mirror image) we have $x \to -x, y \to -y$ and $z \to -z$. As a result all the components of a vector change sign and thus $a \to -a,\ b \to -b$. What happens to **a** × **b** under reflection?

$$\mathbf{a}\times\mathbf{b} \to (-\mathbf{a})\times(-\mathbf{b}) = \mathbf{a}\times\mathbf{b}$$

Thus, **a** × **b** does not change sign under reflection.

- Both scalar and vector products are distributive with respect to vector addition. Thus,

$$\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c}$$

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$$

- We may write **c** = **a** × **b** in the component form. For this we first need to obtain some elementary cross products:

(i) **a** × **a** = **0** (**0** is a null vector, i.e. a vector with zero magnitude)

This follows since magnitude of **a** × **a** is $a^2\sin 0° = 0$.

From this follow the results

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}, \ \ \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \mathbf{0}, \ \ \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$$

(ii) $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$

Note that the magnitude of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is $\sin 90^0$ or 1, since $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ both have unit magnitude and the angle between them is 90^0. Thus, $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is a unit vector. A unit vector perpendicular to the plane of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ and related to them by the right hand screw rule is $\hat{\mathbf{k}}$. Hence, the above result. You may verify similarly,

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \ \text{ and } \ \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

From the rule for commutation of the cross product, it follows:

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \ \ \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \ \ \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

Note if $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ occur cyclically in the above vector product relation, the vector product is positive. If $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ do not occur in cyclic order, the vector product is negative.

Now,

$$\mathbf{a} \times \mathbf{b} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}) \times (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}})$$

$$= a_x b_y \hat{\mathbf{k}} - a_x b_z \hat{\mathbf{j}} - a_y b_x \hat{\mathbf{k}} + a_y b_z \hat{\mathbf{i}} + a_z b_x \hat{\mathbf{j}} - a_z b_y \hat{\mathbf{i}}$$

$$= (a_y b_z - a_z b_x)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}}$$

We have used the elementary cross products in obtaining the above relation. The expression for **a** × **b** can be put in a determinant form which is easy to remember.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

> ▶ ***Example 7.4*** *Find the scalar and vector products of two vectors.* $\mathbf{a} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ *and* $\mathbf{b} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$

Answer

$$\mathbf{a} \cdot \mathbf{b} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \cdot (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$
$$= -6 - 4 - 15$$
$$= -25$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix} = 7\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

Note $\mathbf{b} \times \mathbf{a} = -7\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ ◀

7.6 ANGULAR VELOCITY AND ITS RELATION WITH LINEAR VELOCITY

In this section we shall study what is angular velocity and its role in rotational motion. We have seen that every particle of a rotating body moves in a circle. The linear velocity of the particle is related to the angular velocity. The relation between these two quantities involves a vector product which we learnt about in the last section.

Let us go back to Fig. 7.4. As said above, in rotational motion of a rigid body about a fixed axis, every particle of the body moves in a circle,

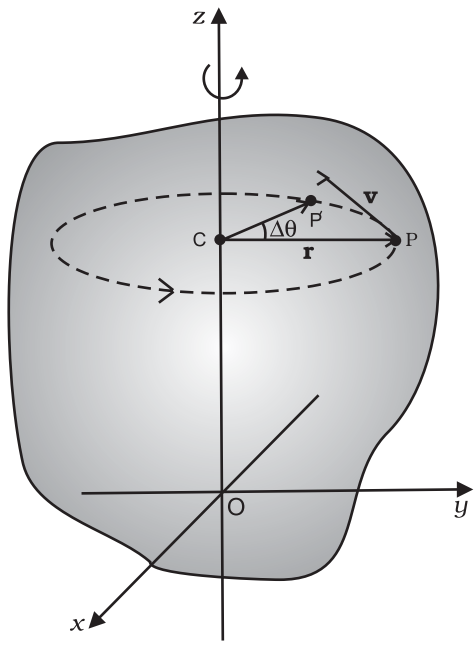


Fig. 7.16 *Rotation about a fixed axis. (A particle (P) of the rigid body rotating about the fixed (z-) axis moves in a circle with centre (C) on the axis.)*

which lies in a plane perpendicular to the axis and has its centre on the axis. In Fig. 7.16 we redraw Fig. 7.4, showing a typical particle (at a point P) of the rigid body rotating about a fixed axis (taken as the *z*-axis). The particle describes

a circle with a centre C on the axis. The radius of the circle is r, the perpendicular distance of the point P from the axis. We also show the linear velocity vector $\mathbf{v}$ of the particle at P. It is along the tangent at P to the circle.

Let P′ be the position of the particle after an interval of time Δt (Fig. 7.16). The angle PCP′ describes the angular displacement $\Delta\theta$ of the particle in time Δt. The average angular velocity of the particle over the interval Δt is $\Delta\theta/\Delta t$. As Δt tends to zero (i.e. takes smaller and smaller values), the ratio $\Delta\theta/\Delta t$ approaches a limit which is the instantaneous angular velocity $d\theta/dt$ of the particle at the position P. We denote the **instantaneous angular velocity** by ω (the Greek letter omega). We know from our study of circular motion that the magnitude of linear velocity v of a particle moving in a circle is related to the angular velocity of the particle ω by the simple relation $v = \omega r$, where r is the radius of the circle.

We observe that at any given instant the relation $v = \omega r$ applies to all particles of the rigid body. Thus for a particle at a perpendicular distance r_i from the fixed axis, the linear velocity at a given instant v_i is given by

$\omega = d\theta/dt$

$$v_i = \omega r_i \tag{7.19}$$

The index i runs from 1 to n, where n is the total number of particles of the body.

For particles on the axis, $r = 0$, and hence $v = \omega r = 0$. Thus, particles on the axis are stationary. This verifies that the axis is *fixed*.

Note that we use the same angular velocity ω for all the particles. **We therefore, refer to $\boldsymbol{\omega}$ as the angular velocity of the whole body**.

We have characterised pure translation of a body by all parts of the body having the same velocity at any instant of time. Similarly, we may characterise pure rotation by all parts of the body having the same angular velocity at any instant of time. Note that this characterisation of the rotation of a rigid body about a fixed axis is **just another way** of saying as in Sec. 7.1 that each particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has the centre on the axis.

In our discussion so far the angular velocity appears to be a scalar. In fact, it is a vector. We shall not justify this fact, but we shall accept it. For rotation about a fixed axis, the angular velocity vector lies along the axis of rotation, and points out in the direction in which a right handed screw would advance, if the head of the screw is rotated with the body. (See Fig. 7.17a).

The magnitude of this vector is referred as above.

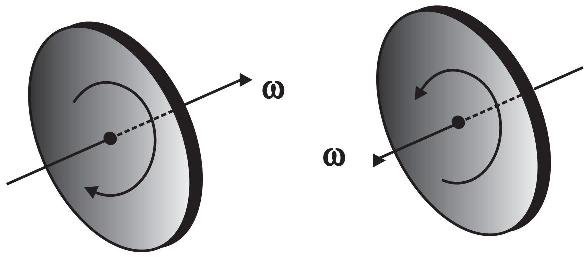


Fig. 7.17 (a) *If the head of a right handed screw rotates with the body, the screw advances in the direction of the angular velocity* $\boldsymbol{\omega}$. *If the sense (clockwise or anticlockwise) of rotation of the body changes, so does the direction of* $\boldsymbol{\omega}$

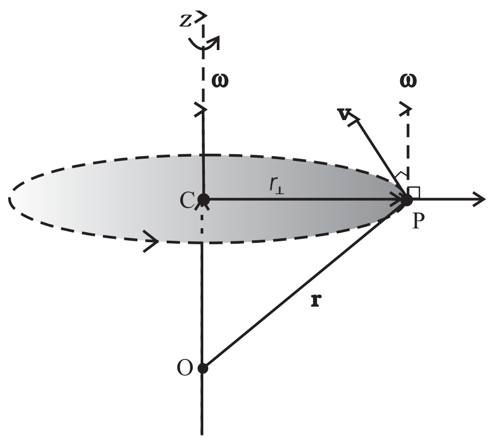


Fig. 7.17 (b) *The angular velocity vector* $\boldsymbol{\omega}$ *is directed along the fixed axis as shown. The linear velocity of the particle at P is* $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. *It is perpendicular to both* $\boldsymbol{\omega}$ *and* $\mathbf{r}$ *and is directed along the tangent to the circle described by the particle.*

We shall now look at what the vector product $\boldsymbol{\omega} \times \mathbf{r}$ corresponds to. Refer to Fig. 7.17(b) which is a part of Fig. 7.16 reproduced to show the path of the particle P. The figure shows the vector $\boldsymbol{\omega}$ directed along the fixed (z-) axis and also the position vector $\mathbf{r} = \mathbf{OP}$ of the particle at P of the rigid body with respect to the origin O. Note that the origin is chosen to be on the axis of rotation.

Now $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{OP} = \boldsymbol{\omega} \times (\mathbf{OC} + \mathbf{CP})$

But $\boldsymbol{\omega} \times \mathbf{OC} = \mathbf{0}$ *as* $\boldsymbol{\omega}$ is along $\mathbf{OC}$

Hence $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{CP}$

The vector $\boldsymbol{\omega} \times \mathbf{CP}$ is perpendicular to $\boldsymbol{\omega}$ i.e. to **CP**, the *z*-axis and also to the radius of the circle described by the particle at P. It is therefore, along the tangent to the circle at P. Also, the magnitude of $\boldsymbol{\omega} \times \mathbf{CP}$ is ω(CP) since $\boldsymbol{\omega}$ and **CP** are perpendicular to each other. We shall denote **CP** by $\mathbf{r}_\perp$ and not by **r**, as we did earlier.

Thus, $\boldsymbol{\omega} \times \mathbf{r}$ is a vector of magnitude $\omega r_\perp$ and is along the tangent to the circle described by the particle at P. The linear velocity vector **v** at P has the same magnitude and direction. Thus,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{7.20}$$

In fact, the relation, Eq. (7.20), holds good even for rotation of a rigid body with one point fixed, such as the rotation of the top [Fig. 7.6(a)]. In this case **r** represents the position vector of the particle with respect to the fixed point taken as the origin.

We note that **for rotation about a fixed axis, the direction of the vector $\boldsymbol{\omega}$ does not change with time. Its magnitude may, however, change from instant to instant. For the more general rotation, both the magnitude and the direction of $\boldsymbol{\omega}$ may change from instant to instant.**

7.6.1 Angular acceleration

You may have noticed that we are developing the study of rotational motion along the lines of the study of translational motion with which we are already familiar. Analogous to the kinetic variables of linear displacement and velocity (**v**) in translational motion, we have angular displacement and angular velocity ($\boldsymbol{\omega}$) in rotational motion. It is then natural to define in rotational motion the concept of angular acceleration in analogy with linear acceleration defined as the time rate of change of velocity in translational motion. We define angular acceleration $\boldsymbol{\alpha}$ as the time rate of change of angular velocity; Thus,

$$\boldsymbol{\alpha} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \tag{7.21}$$

If the axis of rotation is fixed, the direction of $\boldsymbol{\omega}$ and hence, that of $\boldsymbol{\alpha}$ is fixed. In this case the vector equation reduces to a scalar equation

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{7.22}$$

7.7 TORQUE AND ANGULAR MOMENTUM

In this section, we shall acquaint ourselves with two physical quantities which are defined as vector products of two vectors. These as we shall see, are especially important in the discussion of motion of systems of particles, particularly rigid bodies.

7.7.1 Moment of force (Torque)

We have learnt that the motion of a rigid body in general is a combination of rotation and translation. If the body is fixed at a point or along a line, it has only rotational motion. We know that force is needed to change the translational state of a body, i.e. to produce linear acceleration. We may then ask, what is the analogue of force in the case of rotational motion? To look into the question in a concrete situation let us take the example of opening or closing of a door. A door is a rigid body which can rotate about a fixed vertical axis passing through the hinges. What makes the door rotate? It is clear that unless a force is applied the door does not rotate. But any force does not do the job. A force applied to the hinge line cannot produce any rotation at all, whereas a force of given magnitude applied at right angles to the door at its outer edge is most effective in producing rotation. It is not the force alone, but how and where the force is applied is important in rotational motion.

The rotational analogue of force is **moment of force**. It is also referred to as **torque**. (We shall use the words moment of force and torque interchangeably.) We shall first define the moment of force for the special case of a single particle. Later on we shall extend the concept to systems of particles including rigid bodies. We shall also relate it to a change in the state of rotational motion, i.e. is angular acceleration of a rigid body.

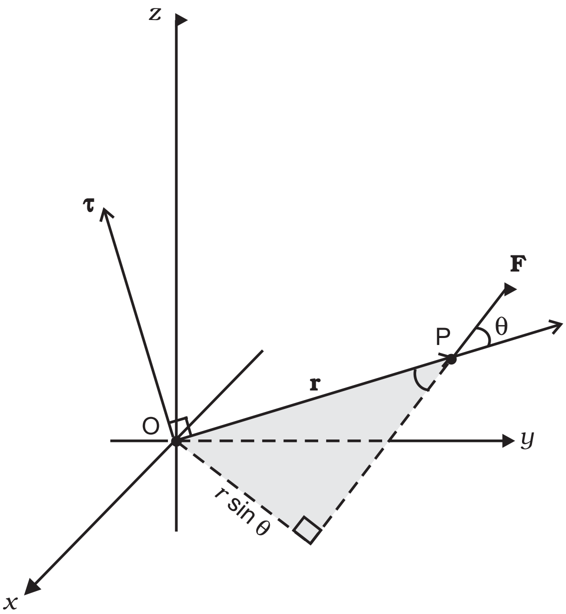


Fig. 7.18 *$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, τ is perpendicular to the plane containing* ***r*** *and* ***F****, and its direction is given by the right handed screw rule.*

If a force acts on a single particle at a point P whose position with respect to the origin O is
$l = r\,p_\perp$ given by the position vector **r** (Fig. 7.18), the moment of the force acting on the particle with respect to the origin O is defined as the vector product

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad (7.23)$$

The moment of force (or torque) is a vector quantity. The symbol $\boldsymbol{\tau}$ stands for the Greek letter **tau**. The magnitude of $\boldsymbol{\tau}$ is

$$\tau = rF\sin\theta \qquad (7.24a)$$

where r is the magnitude of the position vector **r**, i.e. the length OP, F is the magnitude of force **F** and θ is the angle between **r** and **F** as shown.

Moment of force has dimensions M L^2 T^{-2}. Its dimensions are the same as those of work or energy. It is, however, a very different physical quantity than work. Moment of a force is a vector, while work is a scalar. The SI unit of moment of force is Newton-metre (Nm). The magnitude of the moment of force may be written

$$\tau = (r\sin\theta)F = r_\perp F \qquad (7.24b)$$

$$\text{or} \quad \tau = rF\sin\theta = rF_\perp \qquad (7.24c)$$

where $r_\perp = r\sin\theta$ is the perpendicular distance of the line of action of **F** form the origin and $F_\perp (= F\sin\theta)$ is the component of **F** in the direction perpendicular to **r**. Note that $\tau = 0$ if $r = 0$, $F = 0$ or $\theta = 0^0$ or 180^0. Thus, the moment of a force vanishes if either the magnitude of the force is zero, or if the line of action of the force passes through the origin.

One may note that since $\mathbf{r} \times \mathbf{F}$ is a vector product, properties of a vector product of two vectors apply to it. If the direction of **F** is reversed, the direction of the moment of force is reversed. If directions of both **r** and **F** are reversed, the direction of the moment of force remains the same.

7.7.2 Angular momentum of a particle

Just as the moment of a force is the rotational analogue of force, the quantity angular momentum is the rotational analogue of linear momentum. We shall first define angular momentum for the special case of a single particle and look at its usefulness in the context of single particle motion. We shall then extend the definition of angular momentum to systems of particles including rigid bodies.

Like moment of a force, angular momentum is also a vector product. It could also be referred to as moment of (linear) momentum. From this term one could guess how angular momentum is defined.

Consider a particle of mass m and linear momentum **p** at a position **r** relative to the origin O. The angular momentum **l** of the particle with respect to the origin O is defined to be

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} \qquad (7.25a)$$

The magnitude of the angular momentum vector is

$$l = rp\sin\theta \qquad (7.26a)$$

where p is the magnitude of **p** and θ is the angle between **r** and **p**. We may write

$$\text{or} \quad r_\perp p \qquad (7.26b)$$

where $r_\perp$ $(= r\sin\theta)$ is the perpendicular distance of the directional line of **p** from the origin and $p_\perp (= p\sin\theta)$ is the component of **p** in a direction perpendicular to **r**. We expect the angular momentum to be zero ($l = 0$), if the linear momentum vanishes ($p = 0$), if the particle is at the origin ($r = 0$), or if the directional line of **p** passes through the origin $\theta = 0^0$ or 180^0.

The physical quantities, moment of a force and angular momentum, have an important relation between them. It is the rotational analogue of the relation between force and linear momentum. For deriving the relation in the context of a single particle, we differentiate $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ with respect to time,

$$\frac{d\mathbf{l}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$

Applying the product rule for differentiation to the right hand side,

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

Now, the velocity of the particle is $\mathbf{v} = d\mathbf{r}/dt$ and $\mathbf{p} = m\,\mathbf{v}$

Because of this $\frac{d\mathbf{r}}{dt} \times \mathbf{p} = \mathbf{v} \times m\,\mathbf{v} = 0$,

as the vector product of two parallel vectors vanishes. Further, since $d\mathbf{p} / dt = \mathbf{F}$,

$$\mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}$$

Hence $\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \boldsymbol{\tau}$

or $$\frac{d\mathbf{l}}{dt} = \boldsymbol{\tau} \qquad (7.27)$$

Thus, the time rate of change of the angular momentum of a particle is equal to the torque acting on it. This is the rotational analogue of the equation $\mathbf{F} = d\mathbf{p}/dt$, which expresses Newton's second law for the translational motion of a single particle.

Torque and angular momentum for a system of particles

To get the total angular momentum of a system of particles about a given point we need to add vectorially the angular momenta of individual particles. Thus, for a system of n particles,

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2 + \ldots + \mathbf{l}_n = \sum_{i=1}^{n} \mathbf{l}_i$$

The angular momentum of the i^{th} particle is given by

$$\mathbf{l}_i = \mathbf{r}_i \times \mathbf{p}_i$$

where $\mathbf{r}_i$ is the position vector of the i^{th} particle with respect to a given origin and $\mathbf{p} = (m_i\mathbf{v}_i)$ is the linear momentum of the particle. (The particle has mass m_i and velocity $\mathbf{v}_i$) We may write the total angular momentum of a system of particles as

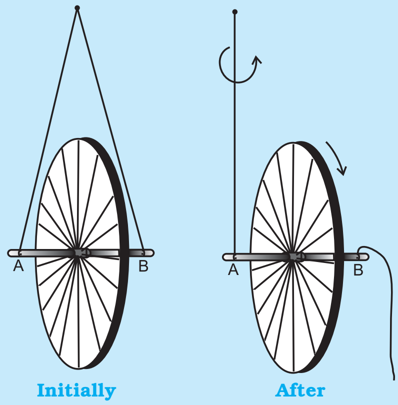
$$\mathbf{L} = \sum \mathbf{l}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i \qquad (7.25b)$$

This is a generalisation of the definition of angular momentum (Eq. 7.25a) for a single particle to a system of particles.

Using Eqs. (7.23) and (7.25b), we get

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\sum \mathbf{l}_i\right) = \sum_i \frac{d\mathbf{l}_i}{dt} = \sum_i \boldsymbol{\tau}_i \qquad (7.28a)$$

An experiment with the bicycle rim



Take a bicycle rim and extend its axle on both sides. Tie two strings at both ends A and B, as shown in the adjoining figure. Hold both the strings together in one hand such that the rim is vertical. If you leave one string, the rim will tilt. Now keeping the rim in vertical position with both the strings in one hand, put the wheel in fast rotation around the axle with the other hand. Then leave one string, say B, from your hand, and observe what happens.

The rim keeps rotating in a vertical plane and the plane of rotation turns around the string A which you are holding. We say that the axis of rotation of the rim or equivalently its angular momentum precesses about the string A.

The rotating rim gives rise to an angular momentum. Determine the direction of this angular momentum. When you are holding the rotating rim with string A, a torque is generated. (We leave it to you to find out how the torque is generated and what its direction is.) The effect of the torque on the angular momentum is to make it precess around an axis perpendicular to both the angular momentum and the torque. Verify all these statements.

where $\boldsymbol{\tau}_i$ is the torque acting on the i^{th} particle:

$$\boldsymbol{\tau}_i = \mathbf{r}_i \times \mathbf{F}_i$$

The force $\mathbf{F}_i$ on the i^{th} particle is the vector sum of external forces $\mathbf{F}_i^{ext}$ acting on the particle and the internal forces $\mathbf{F}_i^{int}$ exerted on it by the other particles of the system. We may therefore separate the contribution of the external and the internal forces to the total torque $\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i$ as

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{ext} + \boldsymbol{\tau}_{int},$$

where $\boldsymbol{\tau}_{ext} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext}$

and $\boldsymbol{\tau}_{int} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{int}$

We shall assume not only Newton's third law, i.e. the forces between any two particles of the system are equal and opposite, but also that these forces are directed along the line joining the two particles. In this case the contribution of the internal forces to the total torque on the system is zero, since the torque resulting from each action-reaction pair of forces is zero. We thus have, $\boldsymbol{\tau}_{int} = \mathbf{0}$ and therefore $\boldsymbol{\tau} = \boldsymbol{\tau}_{ext}$.

Since $\boldsymbol{\tau} = \sum \boldsymbol{\tau}_i$, it follows from Eq. (7.28a) that

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_{ext} \qquad (7.28\ b)$$

Thus, the time rate of the total angular
torques (i.e. the torques due to external forces) acting on the system taken about the same point. Eq. (7.28 b) is the generalisation of the single particle case of Eq. (7.23) to a system of particles. Note that when we have only one particle, there are no internal forces or torques. Eq.(7.28 b) is the rotational analogue of

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{ext} \qquad (7.17)$$

Note that like Eq.(7.17), Eq.(7.28b) holds good for any system of particles, whether it is a rigid body or its individual particles have all kinds of internal motion.

Conservation of angular momentum

If $\boldsymbol{\tau}_{ext} = \mathbf{0}$, Eq. (7.28b) reduces to

$$\frac{d\mathbf{L}}{dt} = 0$$

or $\mathbf{L}$ = constant. (7.29a)

Thus, if the total external torque on a system of particles is zero, then the total angular momentum of the system is conserved, i.e. remains constant. Eq. (7.29a) is equivalent to three scalar equations,

$$L_x = K_1,\ L_y = K_2 \text{ and } L_z = K_3 \qquad (7.29\ b)$$

Here K_1, K_2 and K_3 are constants; L_x, L_y and L_z are the components of the total angular momentum vector $\mathbf{L}$ along the x, y and z axes respectively. The statement that the total angular momentum is conserved means that each of these three components is conserved.

Eq. (7.29a) is the rotational analogue of Eq. (7.18a), i.e. the conservation law of the total linear momentum for a system of particles. Like Eq. (7.18a), it has applications in many practical situations. We shall look at a few of the interesting applications later on in this chapter.

▶ ***Example 7.5*** Find the torque of a force $7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ about the origin. The force acts on a particle whose position vector is $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Answer Here $\mathbf{r} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$

and $\mathbf{F} = 7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$.

$$\boldsymbol{\tau} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 7 & 3 & -5 \end{vmatrix} = (5-3)\hat{\mathbf{i}} - (-5-7)\hat{\mathbf{j}} + (3-(-7))\hat{\mathbf{k}}$$

or $\boldsymbol{\tau} = 2\hat{\mathbf{i}} + 12\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$ ◀

▶ ***Example 7.6*** Show that the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

Answer Let the particle with velocity **v** be at point P at some instant *t*. We want to calculate the angular momentum of the particle about an arbitrary point O.

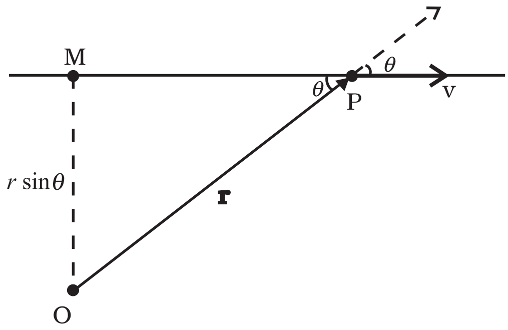


Fig 7.19

The angular momentum is $\mathbf{l} = \mathbf{r} \times m\mathbf{v}$. Its magnitude is $mvr \sin\theta$, where θ is the angle between **r** and **v** as shown in Fig. 7.19. Although the particle changes position with time, the line of direction of **v** remains the same and hence OM = $r \sin\theta$. is a constant.

Further, the direction of **l** is perpendicular to the plane of **r** and **v**. It is into the page of the figure.This direction does not change with time.

Thus, **l** remains the same in magnitude and direction and is therefore conserved. Is there any external torque on the particle? ◀

7.8 EQUILIBRIUM OF A RIGID BODY

We are now going to concentrate on the motion of rigid bodies rather than on the motion of general systems of particles.

We shall recapitulate what effect the external forces have on a rigid body. (Henceforth we shall omit the adjective 'external' because unless stated otherwise, we shall deal with only external forces and torques.) The forces change the translational state of the motion of the rigid body, i.e. they change its total linear momentum in accordance with Eq. (7.17). But this is not the only effect the forces have. The total torque on the body may not vanish. Such a torque changes the rotational state of motion of the rigid body, i.e. it changes the total angular momentum of the body in accordance with Eq. (7.28 b).

A rigid body is said to be in mechanical equilibrium, if both its linear momentum and angular momentum are not changing with time, or equivalently, the body has neither linear acceleration nor angular acceleration. This means

(1) the total force, i.e. the vector sum of the forces, on the rigid body is zero;

$$\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{0} \qquad (7.30a)$$

If the total force on the body is zero, then the total linear momentum of the body does not change with time. Eq. (7.30a) gives the condition for the translational equilibrium of the body.

(2) The total torque, i.e. the vector sum of the torques on the rigid body is zero,

$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \ldots + \boldsymbol{\tau}_n = \sum_{i=1}^{n} \boldsymbol{\tau}_i = \mathbf{0} \qquad (7.30b)$$

If the total torque on the rigid body is zero, the total angular momentum of the body does not change with time. Eq. (7.30 b) gives the condition for the rotational equilibrium of the body.

One may raise a question, whether the rotational equilibrium condition [Eq. 7.30(b)] remains valid, if the origin with respect to which the torques are taken is shifted. One can show that if the translational equilibrium condition [Eq. 7.30(a)] holds for a rigid body, then such a shift of origin does not matter, i.e. the rotational equilibrium condition is independent of the location of the origin about which the torques are taken. Example 7.7 gives a proof of this result in a special case of a couple, i.e. two forces acting on a rigid body in translational equilibrium. The generalisation of this result to *n* forces is left as an exercise.

Eq. (7.30a) and Eq. (7.30b), both, are vector equations. They are equivalent to three scalar equations each. Eq. (7.30a) corresponds to

$$\sum_{i=1}^{n} F_{ix} = 0, \ \sum_{i=1}^{n} F_{iy} = 0 \ \text{and} \ \sum_{i=1}^{n} F_{iz} = 0 \qquad (7.31a)$$

where F_{ix}, F_{iy} and F_{iz} are respectively the *x*, *y* and *z* components of the forces $\mathbf{F}_i$. Similarly, Eq. (7.30b) is equivalent to three scalar equations

$$\sum_{i=1}^{n} \tau_{ix} = 0, \ \sum_{i=1}^{n} \tau_{iy} = 0 \ \text{and} \ \sum_{i=1}^{n} \tau_{iz} = 0 \qquad (7.31b)$$

where τ_{ix}, τ_{iy} and τ_{iz} are respectively the *x*, *y* and *z* components of the torque $\boldsymbol{\tau}_i$.

Eq. (7.31a) and (7.31b) give six independent conditions to be satisfied for mechanical equilibrium of a rigid body. In a number of problems all the forces acting on the body are coplanar. Then we need only three conditions to be satisfied for mechanical equilibrium. Two of these conditions correspond to translational equilibrium; the sum of the components of the forces along any two perpendicular axes in the plane must be zero. The third condition corresponds to rotational equilibrium. The sum of the components of the torques along any axis perpendicular to the plane of the forces must be zero.

The conditions of equilibrium of a rigid body may be compared with those for a particle, which we considered in earlier chapters. Since consideration of rotational motion does not apply to a particle, only the conditions for translational equilibrium (Eq. 7.30 a) apply to a particle. Thus, for equilibrium of a particle the vector sum of all the forces on it must be zero. Since all these forces act on the single particle, they must be concurrent. Equilibrium under concurrent forces was discussed in the earlier chapters.

A body may be in partial equilibrium, i.e., it may be in translational equilibrium and not in rotational equilibrium, or it may be in rotational equilibrium and not in translational equilibrium.

Consider a light (i.e. of negligible mass) rod (AB), at the two ends (A and B) of which two parallel forces both equal in magnitude are applied perpendicular to the rod as shown in Fig. 7.20(a).

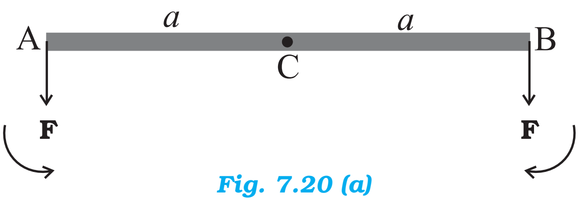


Fig. 7.20 (a)

Let C be the midpoint of AB, CA = CB = a. the moment of the forces at A and B will both be equal in magnitude (aF), but opposite in sense as shown. The net moment on the rod will be zero. The system will be in rotational equilibrium, but it will not be in translational equilibrium; $\sum \mathbf{F} \neq \mathbf{0}$

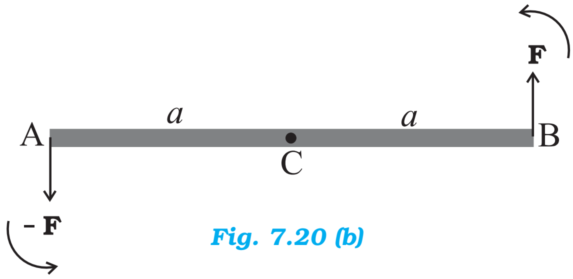


Fig. 7.20 (b)

The force at B in Fig. 7.20(a) is reversed in Fig. 7.20(b). Thus, we have the same rod with two equal and opposite forces applied perpendicular to the rod, one at end A and the other at end B. Here the moments of both the forces are equal, but they are not opposite; they act in the same sense and cause anticlockwise rotation of the rod. The total force on the body is zero; so the body is in translational equilibrium; but it is not in rotational equilibrium. Although the rod is not fixed in any way, it undergoes pure rotation (i.e. rotation without translation).

A pair of equal and opposite forces with different lines of action is known as a **couple**. A couple produces rotation without translation.

When we open the lid of a bottle by turning it, our fingers are applying a couple to the lid [Fig. 7.21(a)]. Another known example is a compass needle in the earth's magnetic field as shown in the Fig. 7.21(b). The earth's magnetic field exerts equal forces on the north and south poles. The force on the North Pole is towards the north, and the force on the South Pole is toward the south. Except when the needle points in the north-south direction; the two forces do not have the same line of action. Thus there is a **couple** acting on the needle due to the earth's magnetic field.

Fig. 7.21(a) *Our fingers apply a couple to turn the lid.*

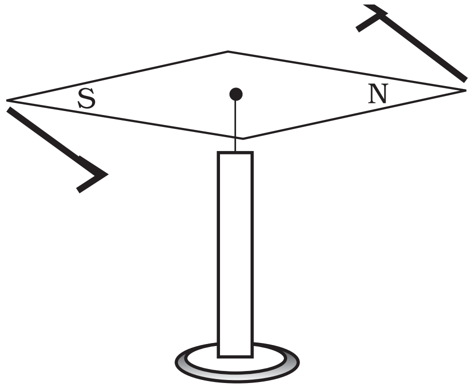


Fig. 7.21(b) *The Earth's magnetic field exerts equal and opposite forces on the poles of a compass needle. These two forces form a couple.*

▶ ***Example 7.7*** Show that moment of a couple does not depend on the point about which you take the moments.

Answer

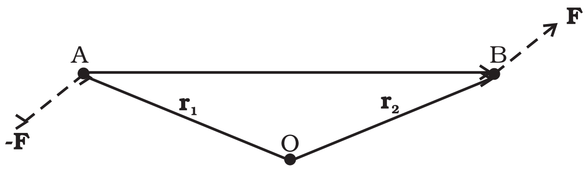


Fig. 7.22

Consider a couple as shown in Fig. 7.22 acting on a rigid body. The forces **F** and -**F** act respectively at points B and A. These points have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to origin O. Let us take the moments of the forces about the origin.

The moment of the couple = sum of the moments of the two forces making the couple

$= \mathbf{r}_1 \times (-\mathbf{F}) + \mathbf{r}_2 \times \mathbf{F}$

$= \mathbf{r}_2 \times \mathbf{F} - \mathbf{r}_1 \times \mathbf{F}$

$= (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}$

But $\mathbf{r}_1 + \mathbf{AB} = \mathbf{r}_2$, and hence $\mathbf{AB} = \mathbf{r}_2 - \mathbf{r}_1$.

The moment of the couple, therefore, is $\mathbf{AB} \times \mathbf{F}$.

Clearly this is independent of the origin, the point about which we took the moments of the forces. ◀

7.8.1 Principle of moments

An ideal lever is essentially a light (i.e. of negligible mass) rod pivoted at a point along its length. This point is called the fulcrum. A see-saw on the children's playground is a typical example of a lever. Two forces F_1 and F_2, parallel to each other and usually perpendicular to the lever, as shown here, act on the lever at distances d_1 and d_2 respectively from the fulcrum as shown in Fig. 7.23.

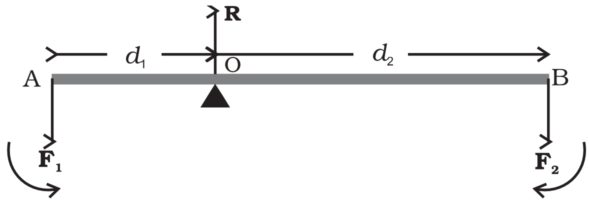


Fig. 7.23

The lever is a system in mechanical equilibrium. Let **R** be the reaction of the support at the fulcrum; **R** is directed opposite to the forces F_1 and F_2. For translational equilibrium,

$$R - F_1 - F_2 = 0 \qquad \text{(i)}$$

For considering rotational equilibrium we take the moments about the fulcrum; the sum of moments must be zero,

$$d_1F_1 - d_2F_2 = 0 \qquad \text{(ii)}$$

Normally the anticlockwise (clockwise) moments are taken to be positive (negative). Note R acts at the fulcrum itself and has zero moment about the fulcrum.

In the case of the lever force F_1 is usually some weight to be lifted. It is called the *load* and its distance from the fulcrum d_1 is called the *load arm*. Force F_2 is the *effort* applied to lift the load; distance d_2 of the effort from the fulcrum is the *effort arm*.

Eq. (ii) can be written as

$$d_1F_1 = d_2F_2 \qquad \text{(7.32a)}$$

or load arm × load = effort arm × effort

The above equation expresses the principle of moments for a lever. Incidentally the ratio F_1/F_2 is called the Mechanical Advantage (M.A.);

$$\text{M.A.} = \frac{F_1}{F_2} = \frac{d_2}{d_1} \qquad \text{(7.32b)}$$

If the effort arm d_2 is larger than the load arm, the mechanical advantage is greater than one. Mechanical advantage greater than one means that a small effort can be used to lift a large load. There are several examples of a lever around you besides the see-saw. The beam of a balance is a lever. Try to find more such

examples and identify the fulcrum, the effort and effort arm, and the load and the load arm of the lever in each case.

You may easily show that the principle of moment holds even when the parallel forces F_1 and F_2 are not perpendicular, but act at some angle, to the lever.

7.8.2 Centre of gravity

Many of you may have the experience of balancing your notebook on the tip of a finger. Figure 7.24 illustrates a similar experiment that you can easily perform. Take an irregular-shaped cardboard and a narrow tipped object like a pencil. You can locate by trial and error a point G on the cardboard where it can be balanced on the tip of the pencil. (The cardboard remains horizontal in this position.) This point of balance is the centre of gravity (CG) of the cardboard. The tip of the pencil provides a vertically upward force due to which the cardboard is in mechanical equilibrium. As shown in the Fig. 7.24, the reaction of the tip is equal and opposite to $M\mathbf{g}$, the total weight of (i.e., the force of gravity on) the cardboard and hence the cardboard is in translational equilibrium. It is also in rotational equilibrium; if it were not so, due to the unbalanced torque it would tilt and fall. There are torques on the card board due to the forces of gravity like $m_1\mathbf{g}$, $m_2\mathbf{g}$ …. etc, acting on the individual particles that make up the cardboard.

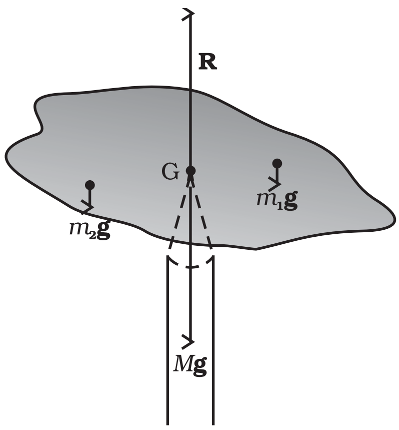


Fig. 7.24 *Balancing a cardboard on the tip of a pencil. The point of support, G, is the centre of gravity.*

The CG of the cardboard is so located that the total torque on it due to the forces $m_1\mathbf{g}$, $m_2\mathbf{g}$ …. etc. is zero.

If $\mathbf{r}_i$ is the position vector of the *i*th particle of an extended body with respect to its CG, then the torque about the CG, due to the force of gravity on the particle is $\boldsymbol{\tau}_i = \mathbf{r}_i \times m_i\mathbf{g}$. The total gravitational torque about the CG is zero, i.e.

$$\boldsymbol{\tau}_g = \sum \boldsymbol{\tau}_i = \sum \mathbf{r}_i \times m_i\mathbf{g} = \mathbf{0} \qquad (7.33)$$

We may therefore, define the CG of a body as that point where the total gravitational torque on the body is zero.

We notice that in Eq. (7.33), $\mathbf{g}$ is the same for all particles, and hence it comes out of the summation. This gives, since $\mathbf{g}$ is nonzero, $\sum m_i\mathbf{r}_i = \mathbf{0}$. Remember that the position vectors ($\mathbf{r}_i$) are taken with respect to the CG. Now, in accordance with the reasoning given below Eq. (7.4a) in Sec. 7.2, if the sum is zero, the origin must be the centre of mass of the body. Thus, the centre of gravity of the body coincides with the centre of mass. We note that this is true because the body being small, $\mathbf{g}$ does not

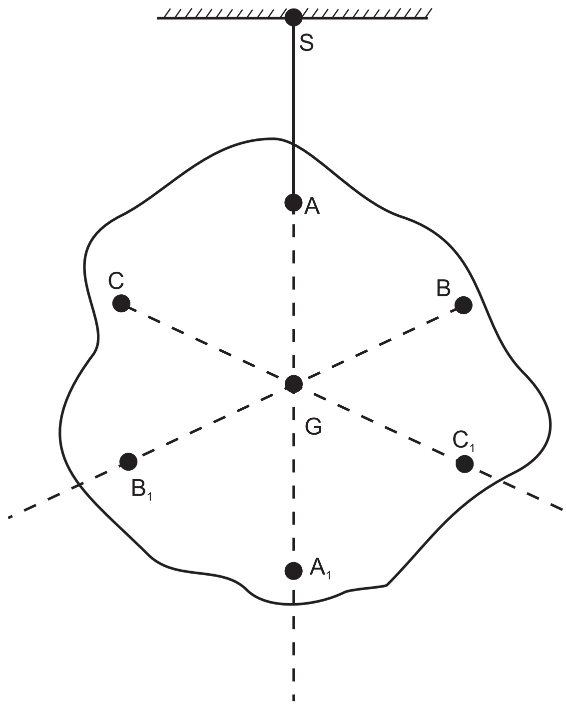


Fig. 7.25 *Determining the centre of gravity of a body of irregular shape. The centre of gravity G lies on the vertical AA_1 through the point of suspension of the body A.*

vary from one point of the body to the other. If the body is so extended that **g** varies from part to part of the body, then the centre of gravity and centre of mass will not coincide. Basically, the two are different concepts. The centre of mass has nothing to do with gravity. It depends only on the distribution of mass of the body.

In Sec. 7.2 we found out the position of the centre of mass of several regular, homogeneous objects. Obviously the method used there gives us also the centre of gravity of these bodies, if they are small enough.

Figure 7.25 illustrates another way of determining the CG of an regular shaped body like a cardboard. If you suspend the body from some point like A, the vertical line through A passes through the CG. We mark the vertical AA_1. We then suspend the body through other points like B and C. The intersection of the verticals gives the CG. Explain why the method works. Since the body is small enough, the method allows us to determine also its centre of mass.

▶ ***Example 7.8*** A metal bar 70 cm long and 4.00 kg in mass supported on two knife-edges placed 10 cm from each end. A 6.00 kg weight is suspended at 30 cm from one end. Find the reactions at the knife-edges. (Assume the bar to be of uniform cross section and homogeneous.)

Answer

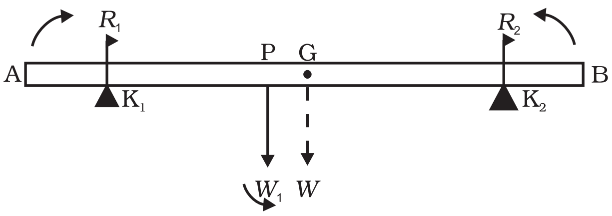


Fig. 7.26

Figure 7.26 shows the rod AB, the positions of the knife edges K_1 and K_2 , the centre of gravity of the rod at G and the suspended weight at P.

Note the weight of the rod W acts at its centre of gravity G. The rod is uniform in cross section and homogeneous; hence G is at the centre of the rod; AB = 70 cm. AG = 35 cm, AP = 30 cm, PG = 5 cm, AK_1 = BK_2 = 10 cm and K_1G = K_2G = 25 cm. Also, W= weight of the rod = 4.00 kg and W_1= suspended weight = 6.00 kg; R_1 and R_2 are the normal reactions of the support at the knife edges.

For translational equilibrium of the rod,

$R_1+R_2-W_1-W = 0$ (i)

Note W_1 and W act vertically down and R_1 and R_2 act vertically up.

For considering rotational equilibrium, we take moments of the forces. A convenient point to take moments about is G. The moments of R_2 and W_1 are anticlockwise (+ve), whereas the moment of R_1 is clockwise (-ve).

For rotational equilibrium,

$-R_1 (K_1G) + W_1 (PG) + R_2 (K_2G) = 0$ (ii)

It is given that $W = 4.00g$ N and $W_1 = 6.00g$ N, where g = acceleration due to gravity. We take $g = 9.8$ m/s².

With numerical values inserted, from (i)

$R_1 + R_2 - 4.00g - 6.00g = 0$

or $R_1 + R_2 = 10.00g$ N (iii)

= 98.00 N

From (ii) $-0.25 R_1 + 0.05 W_1 + 0.25 R_2 = 0$

or $R_2 - R_1 = 1.2g$ N = 11.76 N (iv)

From (iii) and (iv), R_1 = 54.88 N,

R_2 = 43.12 N

Thus the reactions of the support are about 55 N at K_1 and 43 N at K_2. ◀

▶ ***Example 7.9*** A 3m long ladder weighing 20 kg leans on a frictionless wall. Its feet rest on the floor 1 m from the wall as shown in Fig.7.27. Find the reaction forces of the wall and the floor.

Answer

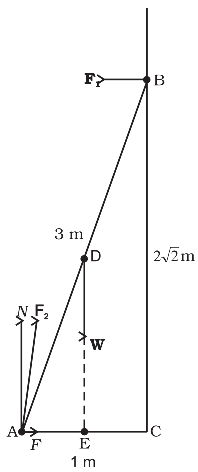


Fig. 7.27

The ladder AB is 3 m long, its foot A is at distance AC = 1 m from the wall. From

Pythagoras theorem, BC = $2\sqrt{2}$ m. The forces on the ladder are its weight W acting at its centre of gravity D, reaction forces F_1 and F_2 of the wall and the floor respectively. Force F_1 is perpendicular to the wall, since the wall is frictionless. Force F_2 is resolved into two components, the normal reaction N and the force of friction F. Note that F prevents the ladder from sliding away from the wall and is therefore directed toward the wall.

For translational equilibrium, taking the forces in the vertical direction,

$$N - W = 0 \qquad \text{(i)}$$

Taking the forces in the horizontal direction,

$$F - F_1 = 0 \qquad \text{(ii)}$$

For rotational equilibrium, taking the moments of the forces about A,

$$2\sqrt{2}\,F_1 - (1/2)\ W = 0 \qquad \text{(iii)}$$

Now $W = 20\ g = 20 \times 9.8\ \text{N} = 196.0\ \text{N}$

From (i) $N = 196.0$

From (iii) $F_1 = W/4\sqrt{2} = 196.0/4\sqrt{2} = 34.6\,\text{N}$

From (ii) $F = F_1 = 34.6\,\text{N}$

$$F_2 = \sqrt{F^2 + N^2} = 199.0\,\text{N}$$

The force F_2 makes an angle α with the horizontal,

$$\tan\alpha = N/F = 4\sqrt{2}\ , \quad \alpha = \tan^{-1}(4\sqrt{2}) \approx 80^\circ$$ ◀

7.9 MOMENT OF INERTIA

We have already mentioned that we are developing the study of rotational motion parallel to the study of translational motion with which we are familiar. We have yet to answer one major question in this connection. **What is the analogue of mass in rotational motion?** We shall attempt to answer this question in the present section. To keep the discussion simple, we shall consider rotation about a fixed axis only. Let us try to get an expression for *the* **kinetic energy of a rotating body**. We know that for a body rotating about a fixed axis, each particle of the body moves in a circle with linear velocity given by Eq. (7.19). (Refer to Fig. 7.16). For a particle at a distance from the axis, the linear velocity is $\upsilon_i = r_i\omega$. The kinetic energy of motion of this particle is

$$k_i = \frac{1}{2}m_i\upsilon_i^2 = \frac{1}{2}m_i r_i^2\omega^2$$

where m_i is the mass of the particle. The total kinetic energy K of the body is then given by the sum of the kinetic energies of individual particles,

$$K = \sum_{i=1}^{n} k_i = \frac{1}{2}\sum_{i=1}^{n}(m_i r_i^2\omega^2)$$

Here n is the number of particles in the body. Note ω is the same for all particles. Hence, taking ω out of the sum,

$$K = \frac{1}{2}\omega^2\left(\sum_{i=1}^{n} m_i r_i^2\right)$$

We define a new parameter characterising the rigid body, called the moment of inertia I, given by

$$I = \sum_{i=1}^{n} m_i r_i^2 \qquad (7.34)$$

With this definition,

$$K = \frac{1}{2}I\omega^2 \qquad (7.35)$$

Note that the parameter I is independent of the magnitude of the angular velocity. It is a characteristic of the rigid body and the axis about which it rotates.

Compare Eq. (7.35) for the kinetic energy of a rotating body with the expression for the kinetic energy of a body in linear (translational) motion,

$$K = \frac{1}{2}m\,\upsilon^2$$

Here m is the mass of the body and υ is its velocity. We have already noted the analogy between angular velocity ω (in respect of rotational motion about a fixed axis) and linear velocity υ (in respect of linear motion). It is then evident that the parameter, moment of inertia I, is the desired rotational analogue of mass. In rotation (about a fixed axis), the moment of inertia plays a similar role as mass does in linear motion.

We now apply the definition Eq. (7.34), to calculate the moment of inertia in two simple cases.

(a) Consider a thin ring of radius R and mass M, rotating in its own plane around its

centre with angular velocity ω Each mass element of the ring is at a distance R from the axis, and moves with a speed $R\omega$ The kinetic energy is therefore,

$$K = \frac{1}{2}Mv^2 = \frac{1}{2}MR^2\omega^2$$

Comparing with Eq. (7.35) we get $I = MR^2$ for the ring.

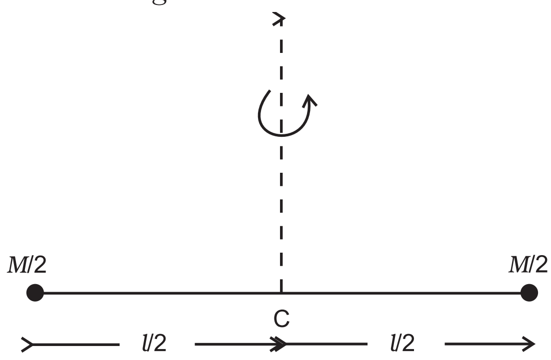


Fig. 7.28 *A light rod of length l with a pair of masses rotating about an axis through the centre of mass of the system and perpendicular to the rod. The total mass of the system is M.*

(b) Next, take a rigid massless rod of length l with a pair of small masses, rotating about an axis through the centre of mass perpendicular to the rod (Fig. 7.28). Each mass $M/2$ is at a distance $l/2$ from the axis. The moment of inertia of the masses is therefore given by

$$(M/2)\,(l/2)^2 + (M/2)(l/2)^2$$

Thus, for the pair of masses, rotating about the axis through the centre of mass perpendicular to the rod

$$I = Ml^2 / 4$$

Table 7.1 gives the moment of inertia of various familiar regular shaped solids about specific axes.

As the mass of a body resists a change in its state of linear motion, it is a measure of its inertia in linear motion. Similarly, as the moment of inertia about a given axis of rotation resists a change in its rotational motion, it can be regarded as a measure of rotational inertia of the body; it is a measure of the way in which different parts of the body are distributed at different distances from the axis. Unlike the mass of a body, the moment of inertia is not a fixed quantity but depends on the orientation and position of the axis of rotation with respect to the body as a whole. As a measure of the way in which the mass of a rotating rigid body is distributed with respect to the axis of rotation, we can define a new parameter, the **radius of gyration**. It is related to the moment of inertia and the total mass of the body.

Notice from the Table 7.1 that in all cases, we can write $I = Mk^2$, where k has the dimension of length. For a rod, about the perpendicular axis at its midpoint, $k^2 = L^2/12$, **i.e.** $k^2 = L/\sqrt{12}$. Similarly, $k = R/2$ for the circular disc about its diameter. The length k is a geometric property of the body and axis of rotation. It is called the **radius of gyration**. **The radius of gyration of a body about an axis** may be defined as the distance from the axis of a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

Thus, the moment of inertia of a rigid body depends on the mass of the body, its shape and size; distribution of mass about the axis of rotation, and the position and orientation of the axis of rotation.

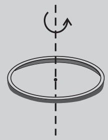
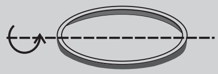
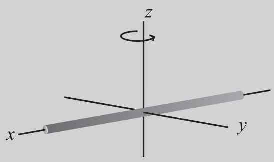
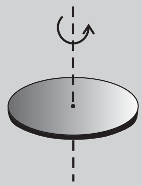
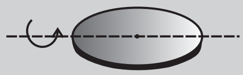
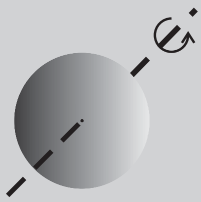
From the definition, Eq. (7.34), we can infer that the dimensions of moments of inertia we ML^2 and its SI units are kg m^2.

The property of this extremely important quantity I as a measure of rotational inertia of the body has been put to a great practical use. The machines, such as steam engine and the automobile engine, etc., that produce rotational motion have a disc with a large moment of inertia, called a **flywheel**. Because of its large moment of inertia, the flywheel resists the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motions, thereby ensuring a smooth ride for the passengers on the vehicle.

7.10 THEOREMS OF PERPENDICULAR AND PARALLEL AXES

These are two useful theorems relating to moment of inertia. We shall first discuss the theorem of perpendicular axes and its simple yet instructive application in working out the moments of inertia of some regular-shaped bodies.

Table 7.1 Moments of Inertia of some regular shaped bodies about specific axes

Z	Body	Axis	Figure	*I*
(1)	Thin circular ring, radius *R*	Perpendicular to plane, at centre		$M R^2$
(2)	Thin circular ring, radius *R*	Diameter		$M R^2/2$
(3)	Thin rod, length L	Perpendicular to rod, at mid point		$M L^2/12$
(4)	Circular disc, radius *R*	Perpendicular to disc at centre		$M R^2/2$
(5)	Circular disc, radius *R*	Diameter		$M R^2/4$
(6)	Hollow cylinder, radius *R*	Axis of cylinder		$M R^2$
(7)	Solid cylinder, radius *R*	Axis of cylinder		$M R^2/2$
(8)	Solid sphere, radius *R*	Diameter		$2 M R^2/5$

Theorem of perpendicular axes

This theorem is applicable to bodies which are planar. In practice this means the theorem applies to flat bodies whose thickness is very small compared to their other dimensions (e.g. length, breadth or radius). Fig. 7.29 illustrates the theorem. It states that **the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.**

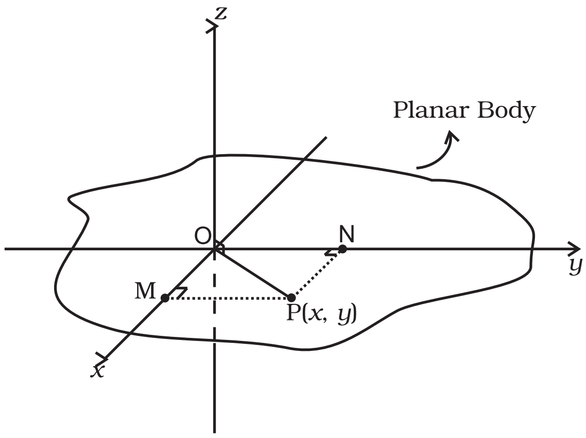


Fig. 7.29 *Theorem of perpendicular axes applicable to a planar body; x and y axes are two perpendicular axes in the plane and the z-axis is perpendicular to the plane.*

The figure shows a planar body. An axis perpendicular to the body through a point O is taken as the *z*-axis. Two mutually perpendicular axes lying in the plane of the body and concurrent with *z*-axis, i.e. passing through O, are taken as the *x* and *y*-axes. The theorem states that

$$I_z = I_x + I_y \tag{7.36}$$

Let us look at the usefulness of the theorem through an example.

▶ **Example 7.10** What is the moment of inertia of a disc about one of its diameters?

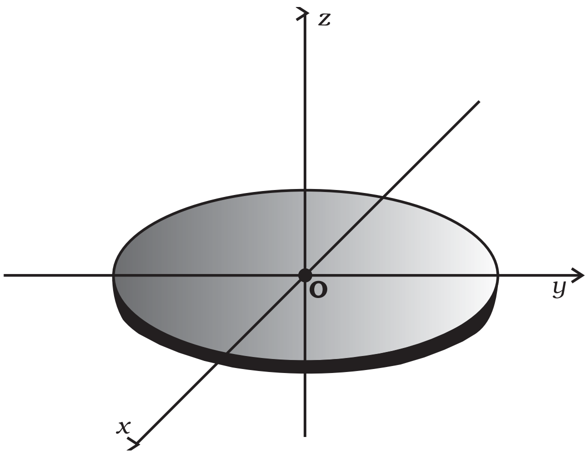


Fig. 7.30 *M.I. of a disc about a diameter, given its M.I. about the perpendicular axis through its centre.*

Answer We assume the moment of inertia of the disc about an axis perpendicular to it and through its centre to be known; it is $MR^2/2$, where M is the mass of the disc and *R* is its radius (Table 7.1)

The disc can be considered to be a planar body. Hence the theorem of perpendicular axes is applicable to it. As shown in Fig. 7.30, we take three concurrent axes through the centre of the disc, O as the *x,y,z* axes; *x* and *y*-axes lie in the plane of the disc and *z* is perpendicular to it. By the theorem of perpendicular axes,

$$I_z = I_x + I_y$$

Now, *x* and *y* axes are along two diameters of the disc, and by symmetry the moment of inertia of the disc is the same about any diameter. Hence

$$I_x = I_y$$

and $\quad I_z = 2I_x$

But $\quad I_z = MR^2/2$

So finally, $\quad I_x = I_z/2 = MR^2/4$

Thus the moment of inertia of a disc about any of its diameter is $MR^2/4$. ◀

Find similarly the moment of inertia of a ring about any of its diameter. Will the theorem be applicable to a solid cylinder?

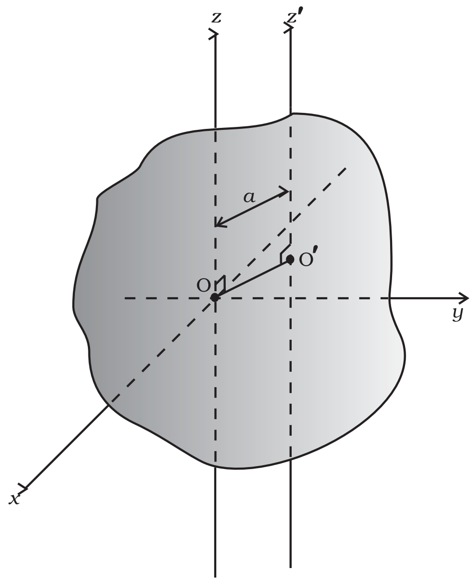


Fig.7.31 *The theorem of parallel axes The z and z′ axes are two parallel axes separated by a distance a; O is the centre of mass of the body,* ***OO′*** *= a.*

7.10.1 Theorem of parallel axes

This theorem is applicable to a body of any shape. It allows to find the moment of inertia of a body about any axis, given the moment of inertia of the body about a parallel axis through the centre of mass of the body. We shall only state this theorem and not give its proof. We shall, however, apply it to a few simple situations which will be enough to convince us about the usefulness of the theorem. The theorem may be stated as follows:

The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes. As shown in the Fig. 7.31, z and z' are two parallel axes separated by a distance a. The z-axis passes through the centre of mass O of the rigid body. Then according to the theorem of parallel axes

$$I_{z'} = I_z + Ma^2 \qquad (7.37)$$

where I_z and $I_{z'}$ are the moments of inertia of the body about the z and z' axes respectively, M is the total mass of the body and a is the perpendicular distance between the two parallel axes.

Example 7.11 What is the moment of inertia of a rod of mass M, length l about an axis perpendicular to it through one end?

Answer For the rod of mass M and length l, $I = Ml^2/12$. Using the parallel axes theorem, $I' = I + Ma^2$ with $a = l/2$ we get,

$$I' = M\frac{l^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

We can check this independently since I is half the moment of inertia of a rod of mass $2M$ and length $2l$ about its midpoint,

$$I' = 2M.\frac{4l^2}{12} \times \frac{1}{2} = \frac{Ml^2}{3}$$ ◀

Example 7.12 What is the moment of inertia of a ring about a tangent to the circle of the ring?

Answer

The tangent to the ring in the plane of the ring is parallel to one of the diameters of the ring. The distance between these two parallel axes is R, the radius of the ring. Using the parallel axes theorem,

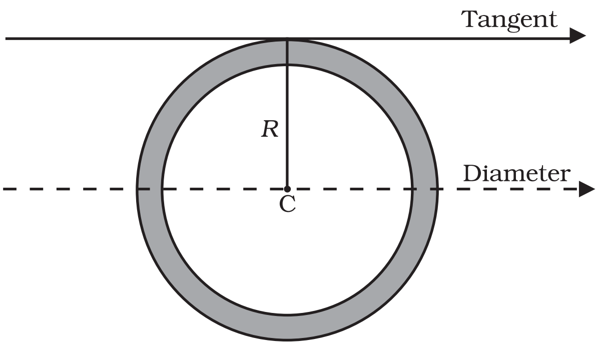


Fig. 7.32

$$I_{tangent} = I_{dia} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$ ◀

7.11 KINEMATICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

We have already indicated the analogy between rotational motion and translational motion. For example, the angular velocity $\boldsymbol{\omega}$ plays the same role in rotation as the linear velocity $\mathbf{v}$ in translation. We wish to take this analogy further. In doing so we shall restrict the discussion only to rotation about fixed axis. This case of motion involves only one degree of freedom, i.e., needs only one independent variable to describe the motion. This in translation corresponds to linear motion. This section is limited only to kinematics. We shall turn to dynamics in later sections.

We recall that for specifying the angular displacement of the rotating body we take any particle like P (Fig.7.33) of the body. Its angular displacement θ in the plane it moves is the angular displacement of the whole body; θ is measured from a fixed direction in the plane of motion of P, which we take to be the x' - axis, chosen parallel to the x-axis. Note, as shown, the axis of rotation is the z – axis and the plane of the motion of the particle is the x - y plane. Fig. 7.33 also shows θ_0, the angular displacement at $t = 0$.

We also recall that the angular velocity is the time rate of change of angular displacement, $\boldsymbol{\omega} = d\theta/dt$. Note since the axis of rotation is

fixed, there is no need to treat angular velocity as a vector. Further, the angular acceleration, $\boldsymbol{\alpha}$ = d$\boldsymbol{\omega}$/dt.

The kinematical quantities in rotational motion, angular displacement (θ), angular velocity (ω) and angular acceleration (α) respectively correspond to kinematic quantities in linear motion, displacement (x), velocity (v) and acceleration (a). We know the kinematical equations of linear motion with uniform (i.e. constant) acceleration:

$$v = v_0 + at \quad \text{(a)}$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \quad \text{(b)}$$

$$v^2 = v_0^2 + 2ax \quad \text{(c)}$$

where x_0 = initial displacement and v_0= initial velocity. The word 'initial' refers to values of the quantities at t = 0

The corresponding kinematic equations for rotational motion with uniform angular acceleration are:

$$\omega = \omega_0 + \alpha t \quad (7.38)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \quad (7.39)$$

$$\text{and } \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \quad (7.40)$$

where θ_0= initial angular displacement of the rotating body, and ω_0 = initial angular velocity of the body.

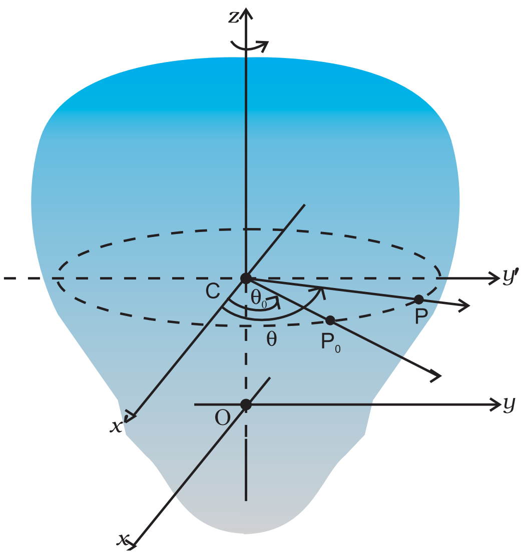


Fig.7.33 *Specifying the angular position of a rigid body.*

▶ ***Example 7.13*** Obtain Eq. (7.38) from first principles.

Answer The angular acceleration is uniform, hence

$$\frac{d\omega}{dt} = \alpha = constant \quad \text{(i)}$$

Integrating this equation,

$$\omega = \int \alpha \, dt + c$$

$$= \alpha t + c \quad \text{(as } \alpha \text{ is constant)}$$

At $t = 0$, $\omega = \omega_0$ (given)

From (i) we get at $t = 0$, $\omega = c = \omega_0$

Thus, $\omega = \alpha t + \omega_0$ as required.

With the definition of $\omega = d\theta/dt$ we may integrate Eq. (7.38) to get Eq. (7.39). This derivation and the derivation of Eq. (7.40) is left as an exercise.

▶ ***Example 7.14*** The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform? (ii) How many revolutions does the engine make during this time?

Answer

(i) We shall use $\omega = \omega_0 + \alpha t$

ω_0 = initial angular speed in rad/s

= $2\pi \times$ angular speed in rev/s

$$= \frac{2\pi \times \text{angular speed in rev/min}}{60 \text{ s/min}}$$

$$= \frac{2\pi \times 1200}{60} \text{rad/s}$$

$= 40\pi$ rad/s

Similarly ω = final angular speed in rad/s

$$= \frac{2\pi \times 3120}{60} \text{rad/s}$$

$= 2\pi \times 52$ rad/s

$= 104\ \pi$ rad/s

∴ Angular acceleration

$$\alpha = \frac{\omega - \omega_0}{t} = 4\ \pi \text{ rad/s}^2$$

The angular acceleration of the engine = 4π rad/s^2

(ii) The angular displacement in time t is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= (40\pi \times 16 + \frac{1}{2} \times 4\pi \times 16^2) \text{ rad}$$

$$= (640\pi + 512\pi) \text{ rad}$$

$= 1152\pi$ rad

Number of revolutions = $\frac{1152\pi}{2\pi} = 576$ ◀

7.12 DYNAMICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

Table 7.2 lists quantities associated with linear motion and their analogues in rotational motion. We have already compared kinematics of the two motions. Also, we know that in rotational motion moment of inertia and torque play the same role as mass and force respectively in linear motion. Given this we should be able to guess what the other analogues indicated in the table are. For example, we know that in linear motion, work done is given by $F\,dx$, in rotational motion about a fixed axis it should be $\tau d\theta$, since we already know the correspondence $dx \to d\theta$ and $F \to \tau$. It is, however, necessary that these correspondences are established on sound dynamical considerations. This is what we now turn to.

Before we begin, we note *a* **simplification that arises in the case of rotational motion about a fixed axis**. Since the axis is fixed, only those components of torques, which are along the direction of the fixed axis need to be considered in our discussion. Only these components can cause the body to rotate about the axis. A component of the torque perpendicular to the axis of rotation will tend to turn the axis from its position. We specifically assume that there will arise necessary forces of constraint to cancel the effect of the perpendicular components of the (external) torques, so that the fixed position of the axis will be maintained. The perpendicular components of the torques, therefore need not be taken into account. This means that for our calculation of torques on a rigid body:

(1) We need to consider only those forces that lie in planes perpendicular to the axis. Forces which are parallel to the axis will give torques perpendicular to the axis and need not be taken into account.

(2) We need to consider only those components of the position vectors which are perpendicular to the axis. Components of position vectors along the axis will result in torques perpendicular to the axis and need not be taken into account.

Work done by a torque

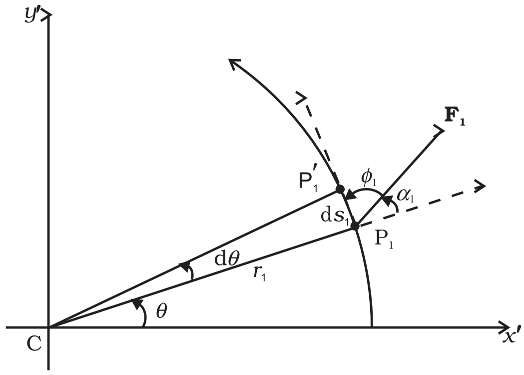


Fig. 7.34 *Work done by a force* $\mathbf{F}_1$ *acting on a particle of a body rotating about a fixed axis; the particle describes a circular path with centre C on the axis; arc* $P_1P'_1$ *(*ds_1*) gives the displacement of the particle.*

Figure 7.34 shows a cross-section of a rigid body rotating about a fixed axis, which is taken as the *z*-axis (perpendicular to the plane of the page; see Fig. 7.33). As said above we need to consider only those forces which lie in planes perpendicular to the axis. Let $\mathbf{F}_1$ be one such typical force acting as shown on a particle of the body at point P_1 with its line of action in a plane perpendicular to the axis. For convenience we call this to be the $x'-y'$ plane (coincident with the plane of the page). The particle at P_1 describes a circular path of radius r_1 with centre C on the axis; $CP_1 = r_1$.

In time Δt, the point moves to the position P_1'. The displacement of the particle $d\mathbf{s}_1$, therefore, has magnitude $ds_1 = r_1 d\theta$ and direction tangential at P_1 to the circular path as shown. Here $d\theta$ is the angular displacement of the particle, $d\theta = \angle P_1CP_1'$. The work done by the force on the particle is

$$dW_1 = \mathbf{F}_1 . d\mathbf{s}_1 = F_1 ds_1 \cos\phi_1 = F_1(r_1\, d\theta)\sin\alpha_1$$

where ϕ_1 is the angle between $\mathbf{F}_1$ and the tangent

Table 7.2 Comparison of Translational and Rotational Motion

	Linear Motion	Rotational Motion about a Fixed Axis
1	Displacement x	Angular displacement θ
2	Velocity $v = \mathrm{d}x/\mathrm{d}t$	Angular velocity $\omega = \mathrm{d}\theta/\mathrm{d}t$
3	Acceleration $a = \mathrm{d}v/\mathrm{d}t$	Angular acceleration $\alpha = \mathrm{d}\omega/\mathrm{d}t$
4	Mass M	Moment of inertia I
5	Force $F = Ma$	Torque $\tau = I\,\alpha$
6	Work $dW = F\,\mathrm{d}s$	Work $W = \tau\ d\theta$
7	Kinetic energy $K = Mv^2/2$	Kinetic energy $K = I\omega^2/2$
8	Power $P = F\,v$	Power $P = \tau\omega$
9	Linear momentum $p = Mv$	Angular momentum $L = I\omega$

at P_1 and α_1 is the angle between $\mathbf{F}_1$ and the radius vector $\mathbf{OP}_1$; $\phi_1 + \alpha_1 = 90°$.

The torque due to $\mathbf{F}_1$ about the origin is $\mathbf{OP}_1 \times \mathbf{F}_1$. Now $\mathbf{OP}_1 = \mathbf{OC} + \mathbf{OP}_1$. [Refer to Fig. 7.17(b).] Since $\mathbf{OC}$ is along the axis, the torque resulting from it is excluded from our consideration. The effective torque due to $\mathbf{F}_1$ is $\boldsymbol{\tau}_1 = \mathbf{CP} \times \mathbf{F}_1$; it is directed along the axis of rotation and has a magnitude $\tau_1 = r_1 F_1 \sin\alpha$, Therefore,

$$\mathrm{d}W_1 = \tau_1 \mathrm{d}\theta$$

If there are more than one forces acting on the body, the work done by all of them can be added to give the total work done on the body. Denoting the magnitudes of the torques due to the different forces as $\tau_1, \tau_2, \ldots$ etc,

$$\mathrm{d}W = (\tau_1 + \tau_2 + \ldots)\mathrm{d}\theta$$

Remember, the forces giving rise to the torques act on different particles, but the angular displacement $\mathrm{d}\theta$ is the same for all particles. Since all the torques considered are parallel to the fixed axis, the magnitude τ of the total torque is just the algebraic sum of the magnitudes of the torques, i.e., $\tau = \tau_1 + \tau_2 + \ldots$.. We, therefore, have

$$\mathrm{d}W = \tau \mathrm{d}\theta \qquad (7.41)$$

This expression gives the work done by the total (external) torque τ which acts on the body rotating about a fixed axis. Its similarity with the corresponding expression

$$\mathrm{d}W = F\,\mathrm{d}s$$

for linear (translational) motion is obvious.

Dividing both sides of Eq. (7.41) by $\mathrm{d}t$ gives

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \tau\frac{\mathrm{d}\theta}{\mathrm{d}t} = \tau\omega$$

or $P = \tau\omega$ (7.42)

This is the instantaneous power. Compare this expression for power in the case of rotational motion about a fixed axis with the expression for power in the case of linear motion,

$P = Fv$

In a perfectly rigid body there is no internal motion. The work done by external torques is therefore, not dissipated and goes on to increase the kinetic energy of the body. The rate at which work is done on the body is given by Eq. (7.42). This is to be equated to the rate at which kinetic energy increases. The rate of increase of kinetic energy is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{I\omega^2}{2}\right) = I\frac{(2\omega)}{2}\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

We assume that the moment of inertia does not change with time. This means that the mass of the body does not change, the body remains rigid and also the axis does not change its position with respect to the body.

Since $\alpha = \mathrm{d}\omega/\mathrm{d}t$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{I\omega^2}{2}\right) = I\,\omega\,\alpha$$

Equating rates of work done and of increase in kinetic energy,

$$\tau\omega = I\,\omega\,\alpha$$

$$\tau = I\alpha \qquad (7.43)$$

Eq. (7.43) is similar to Newton's second law for linear motion expressed symbolically as

$F = ma$

Just as force produces acceleration, torque produces angular acceleration in a body. The angular acceleration is directly proportional to the applied torque and is inversely proportional to the moment of inertia of the body. Eq.(7.43) can be called Newton's second law for rotation about a fixed axis.

Example 7.15 A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Fig. 7.35. The flywheel is mounted on a horizontal axle with frictionless bearings.

(a) Compute the angular acceleration of the wheel.

(b) Find the work done by the pull, when 2m of the cord is unwound.

(c) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.

(d) Compare answers to parts (b) and (c).

Answer

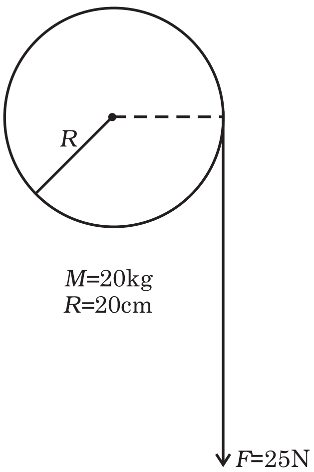


Fig. 7.35

(a) We use $I\alpha = \tau$

the torque $\tau = FR$

$= 25 \times 0.20$ Nm (as $R = 0.20$m)

$= 5.0$ Nm

I = M. I. of flywheel about its axis $= \dfrac{MR^2}{2}$

$= \dfrac{20.0 \times (0.2)^2}{2} = 0.4$ kg m^2

α = angular acceleration

$= 5.0$ N m/0.4 kg m^2 = 12.35 s^{-2}

(b) Work done by the pull unwinding 2m of the cord

= 25 N × 2m = 50 J

(c) Let ω be the final angular velocity. The kinetic energy gained $= \dfrac{1}{2} I\omega^2$,

since the wheel starts from rest. Now,

$$\omega^2 = \omega_0^2 + 2\alpha\theta, \quad \omega_0 = 0$$

The angular displacement θ = length of unwound string / radius of wheel

= 2m/0.2 m = 10 rad

$$\omega^2 = 2 \times 12.5 \times 10.0 = 250\,(\text{rad/s})^2$$

$$\therefore K.E. \text{ gained} = \frac{1}{2} \times 0.4 \times 250 = 50 \text{ J}$$

(d) The answers are the same, i.e. the kinetic energy gained by the wheel = work done by the force. There is no loss of energy due to friction. ◀

7.13 ANGULAR MOMENTUM IN CASE OF ROTATION ABOUT A FIXED AXIS

We have studied in section 7.7, the angular momentum of a system of particles. We already know from there that the time rate of total angular momentum of a system of particles about a point is equal to the total external torque on the system taken about the same point. When the total external torque is zero, the total angular momentum of the system is conserved.

We now wish to study the angular momentum in the special case of rotation about a fixed axis. The general expression for the total angular momentum of the system is

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \qquad (7.25b)$$

We first consider the angular momentum of a typical particle of the rotating rigid body. We then sum up the contributions of individual particles to get **L** of the whole body.

For a typical particle **l** = **r** × **p**. As seen in the last section **r** = **OP** = **OC** + **CP** [Fig. 7.17(b)].

With $\mathbf{p} = m\,\mathbf{v}$,

$$\mathbf{l} = (\mathbf{OC} \times m\,\mathbf{v}) + (\mathbf{CP} \times m\,\mathbf{v})$$

The magnitude of the linear velocity $\mathbf{v}$ of the particle at P is given by $v = \omega r_\perp$ where $r_\perp$ is the length of CP or the perpendicular distance of P from the axis of rotation. Further, $\mathbf{v}$ is tangential at P to the circle which the particle describes. Using the right-hand rule one can check that $\mathbf{CP} \times \mathbf{v}$ is parallel to the fixed axis. The unit vector along the fixed axis (chosen as the z-axis) is $\hat{\mathbf{k}}$. Hence

$$\mathbf{CP} \times m\,\mathbf{v} = r_\perp (mv)\hat{\mathbf{k}}$$

$$= mr_\perp^2 \omega \hat{\mathbf{k}} \quad (\text{since } v = \omega r_\perp)$$

Similarly, we can check that $\mathbf{OC} \times \mathbf{v}$ is perpendicular to the fixed axis. Let us denote the part of $\mathbf{l}$ along the fixed axis (i.e. the z-axis) by $\mathbf{l}_z$, then

$$\mathbf{l}_z = \mathbf{CP} \times m\,\mathbf{v} = mr_\perp^2 \omega \hat{\mathbf{k}}$$

and $\mathbf{l} = \mathbf{l}_z + \mathbf{OC} \times m\,\mathbf{v}$

We note that $\mathbf{l}_z$ is parallel to the fixed axis, but $\mathbf{l}$ is not. In general, for a particle, the angular momentum $\mathbf{l}$ is not along the axis of rotation, i.e. for a particle, $\mathbf{l}$ and $\boldsymbol{\omega}$ are not necessarily parallel. Compare this with the corresponding fact in translation. For a particle, $\mathbf{p}$ and $\mathbf{v}$ are always parallel to each other.

For computing the total angular momentum of the whole rigid body, we add up the contribution of each particle of the body.

Thus $\quad \mathbf{L} = \sum \mathbf{l}_i = \sum l_{iz} + \sum \mathbf{OC}_i \times m_i \mathbf{v}_i$

We denote by $\mathbf{L}_\perp$ and $\mathbf{L}_z$ the components of $\mathbf{L}$ respectively perpendicular to the z-axis and along the z-axis;

$$\mathbf{L}_\perp = \sum \mathbf{OC}_i \times m_i \mathbf{v}_i \qquad (7.44a)$$

where m_i and $\mathbf{v}_i$ are respectively the mass and the velocity of the i^{th} particle and C_i is the centre of the circle described by the particle;

and $$\mathbf{L}_z = \sum \mathbf{l}_{iz} = \left(\sum_i m_i r_i^2 \right) \omega \hat{\mathbf{k}}$$

or $$\mathbf{L}_z = I\omega\hat{\mathbf{k}} \qquad (7.44b)$$

The last step follows since the perpendicular distance of the i^{th} particle from the axis is r_i; and by definition the moment of inertia of the body about the axis of rotation is $I = \sum m_i r_i^2$.

Note $\quad \mathbf{L} = \mathbf{L}_z + \mathbf{L}_\perp \qquad (7.44c)$

The rigid bodies which we have mainly considered in this chapter are symmetric about the axis of rotation, i.e. the axis of rotation is one of their symmetry axes. For such bodies, for a given $\mathbf{OC}_i$, for every particle which has a velocity $\mathbf{v}_i$, there is another particle of velocity $-\mathbf{v}_i$ located diametrically opposite on the circle with centre C_i described by the particle. Together such pairs will contribute zero to $\mathbf{L}_\perp$ and as a result for symmetric bodies $\mathbf{L}_\perp$ is zero, and hence

$$\mathbf{L} = \mathbf{L}_z = I\omega\hat{\mathbf{k}} \qquad (7.44d)$$

For bodies, which are not symmetric about the axis of rotation, $\mathbf{L}$ is not equal to $\mathbf{L}_z$ and hence $\mathbf{L}$ does not lie along the axis of rotation.

Referring to table 7.1, can you tell in which cases $\mathbf{L} = \mathbf{L}_z$ will not apply?

Let us differentiate Eq. (7.44b). Since $\hat{\mathbf{k}}$ is a fixed (constant) vector, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L}_z) = \left(\frac{\mathrm{d}}{\mathrm{d}t}(I\,\omega) \right) \hat{\mathbf{k}}$$

Now, Eq. (7.28b) states

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}$$

As we have seen in the last section, only those components of the external torques which are along the axis of rotation, need to be taken into account, when we discuss rotation about a fixed axis. This means we can take $\boldsymbol{\tau} = \tau\hat{\mathbf{k}}$.

Since $\mathbf{L} = \mathbf{L}_z + \mathbf{L}_\perp$ and the direction of $\mathbf{L}_z$ (vector $\hat{\mathbf{k}}$) is fixed, it follows that for rotation about a fixed axis,

$$\frac{\mathrm{d}\mathbf{L}_z}{\mathrm{d}t} = \tau\hat{\mathbf{k}} \qquad (7.45a)$$

and $$\frac{\mathrm{d}\mathbf{L}_\perp}{\mathrm{d}t} = 0 \qquad (7.45b)$$

Thus, for rotation about a fixed axis, the component of angular momentum perpendicular to the fixed axis is constant. As $\mathbf{L}_z = I\omega\hat{\mathbf{k}}$, we get from Eq. (7.45a),

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\omega) = \tau \qquad (7.45c)$$

If the moment of inertia I does not change with time,

$$\frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\alpha$$

and we get from Eq. (7.45c),

$$\tau = I\alpha \quad (7.43)$$

We have already derived this equation using the work - kinetic energy route.

7.13.1 Conservation of angular momentum

We are now in a position to revisit the principle of conservation of angular momentum in the context of rotation about a fixed axis. From Eq. (7.45c), if the external torque is zero,

$$L_z = I\omega = \text{constant} \quad (7.46)$$

For symmetric bodies, from Eq. (7.44d), L_z may be replaced by L .(L and L_z are respectively the magnitudes of $\mathbf{L}$ and $\mathbf{L}_z$.)

This then is the required form, for fixed axis rotation, of Eq. (7.29a), which expresses the general law of conservation of angular momentum of a system of particles. Eq. (7.46) applies to many situations that we come across in daily life. You may do this experiment with your friend. Sit on a swivel chair with your arms folded and feet not resting on, i.e., away from, the ground. Ask your friend to rotate the chair rapidly. While the chair is rotating with considerable angular speed stretch your arms horizontally. What happens? Your angular speed is reduced. If you bring back your arms closer to your body, the angular speed increases again. This is a situation where the principle of conservation of angular momentum is applicable. If friction in the rotational mechanism is neglected, there is no external torque about the axis of rotation of the chair and hence $I\omega$ is constant. Stretching the arms increases I about the axis of rotation, resulting in decreasing the angular speed ω. Bringing the arms closer to the body has the opposite effect.

A circus acrobat and a diver take advantage of this principle. Also, skaters and classical, Indian or western, dancers performing a pirouette on the toes of one foot display 'mastery' over this principle. Can you explain?

7.14 ROLLING MOTION

One of the most common motions observed in daily life is the rolling motion. All wheels used in transportation have rolling motion. For specificness we shall begin with the case of a disc, but the result will apply to any rolling body rolling on a level surface. We shall assume that the disc rolls without slipping. This means that at any instant of time the bottom of the disc which is in contact with the surface is at rest on the surface.

Fig 7.36 (a) *A demonstration of conservation of angular momentum. A girl sits on a swivel chair and stretches her arms/ brings her arms closer to the body.*

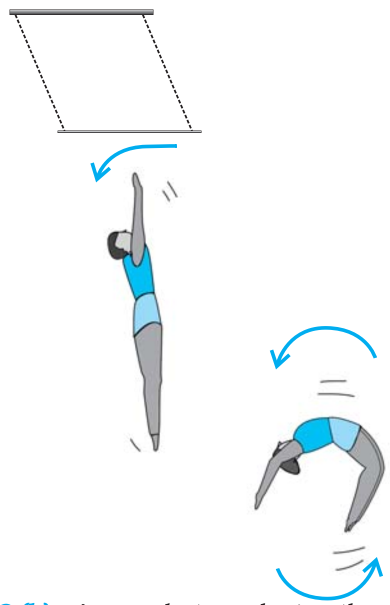


Fig 7.36 (b) *An acrobat employing the principle of conservation of angular momentum in her performance.*

We have remarked earlier that rolling motion is a combination of rotation and translation. We know that the translational motion of a system of particles is the motion of its centre of mass.

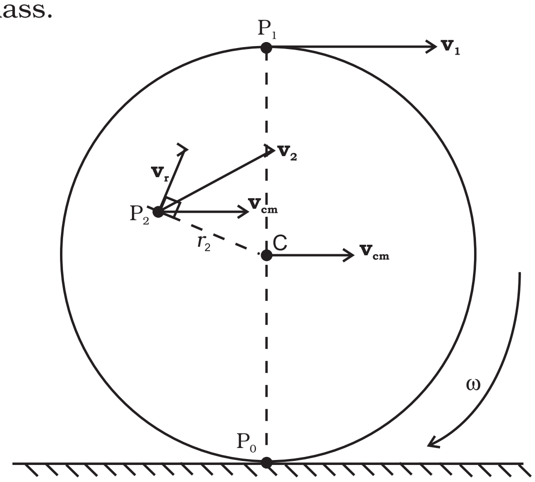


Fig. 7.37 *The rolling motion (without slipping) of a disc on a level surface. Note at any instant, the point of contact P_0 of the disc with the surface is at rest; the centre of mass of the disc moves with velocity, v_{cm}. The disc rotates with angular velocity about its axis which passes through C; $v_{cm} = R\omega$, where R is the radius of the disc.*

Let $\mathbf{v}_{cm}$ be the velocity of the centre of mass and therefore the translational velocity of the disc. Since the centre of mass of the rolling disc is at its geometric centre C (Fig. 7. 37), $\mathbf{v}_{cm}$ is the velocity of C. It is parallel to the level surface. The rotational motion of the disc is about its symmetry axis, which passes through C. Thus, the velocity of any point of the disc, like P_0, P_1 or P_2, consists of two parts, one is the translational velocity $\mathbf{v}_{cm}$ and the other is the linear velocity $\mathbf{v}_r$ on account of rotation. The magnitude of $\mathbf{v}_r$ is $v_r = r\omega$, where ω is the angular velocity of the rotation of the disc about the axis and *r* is the distance of the point from the axis (i.e. from C). The velocity $\mathbf{v}_r$ is directed perpendicular to the radius vector of the given point with respect to C. In Fig. 7.37, the velocity of the point P_2 ($\mathbf{v}_2$) and its components $\mathbf{v}_r$ and $\mathbf{v}_{cm}$ are shown; $\mathbf{v}_r$ here is perpendicular to CP_2. It is easy to show that $\mathbf{v}_z$ is perpendicular to the line P_0P_2. Therefore the line passing through P_0 and parallel to **ω** is called the instantaneous axis of rotation.

At P_o, the linear velocity, $\mathbf{v}_r$, due to rotation is directed exactly opposite to the translational velocity $\mathbf{v}_{cm}$. Further the magnitude of $\mathbf{v}_r$ here is $R\omega$, where *R* is the radius of the disc. The condition that P_o is instantaneously at rest requires $v_{cm} = R\omega$. Thus for the disc the condition for rolling without slipping is

(7.47)

Incidentally, this means that the velocity of point P_1 at the top of the disc ($\mathbf{v}_1$) has a magnitude $v_{cm} + R\omega$ or 2 v_{cm} and is directed parallel to the level surface. The condition (7.47) applies to all rolling bodies.

7.14.1 Kinetic Energy of Rolling Motion

Our next task will be to obtain an expression for the kinetic energy of a rolling body. The kinetic energy of a rolling body can be separated into kinetic energy of translation and kinetic energy of rotation. This is a special case of a general result for a system of particles, according to which the kinetic energy of a system of particles (*K*) can be separated into the kinetic energy of motion of the centre of mass (translation) ($MV^2/2$) and kinetic energy of rotational motion about the centre of mass of the system of particles (K'). Thus,

$$K = K' + MV^2/2 \tag{7.48}$$

We assume this general result (see Exercise 7.31), and apply it to the case of rolling motion. In our notation, the kinetic energy of the centre of mass, i.e., the kinetic energy of translation, of the rolling body is $mv^2_{cm}/2$, where *m* is the mass of the body and v_{cm} is the centre of the mass velocity. Since the motion of the rolling body about the centre of mass is rotation, K' represents the kinetic energy of rotation of the body; $K' = I\omega^2/2$, where *I* is the moment of inertia about the appropriate axis, which is the symmetry axis of the rolling body. The kinetic energy of a rolling body, therefore, is given by

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2_{cm} \tag{7.49a}$$

Substituting $I = mk^2$ where k = the corresponding radius of gyration of the body and $v_{cm} = R\,\omega$, we get

$$K = \frac{1}{2}\frac{mk^2v^2_{cm}}{R^2} + \frac{1}{2}mv^2_{cm}$$

or $$K = \frac{1}{2}mv^2_{cm}\left(1 + \frac{k^2}{R^2}\right) \tag{7.49b}$$

Equation (7.49b) applies to any rolling body: a disc, a cylinder, a ring or a sphere.

▶ ***Example 7.16*** Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity?

Answer We assume conservation of energy of the rolling body, i.e. there is no loss of energy due to friction etc. The potential energy lost by the body in rolling down the inclined plane (= mgh) must, therefore, be equal to kinetic energy gained. (See Fig.7.38) Since the bodies start from rest the kinetic energy gained is equal to the final kinetic energy of the bodies. From

Eq. (7.49b), $K = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$, where v is the final velocity of (the centre of mass of) the body. Equating K and mgh,

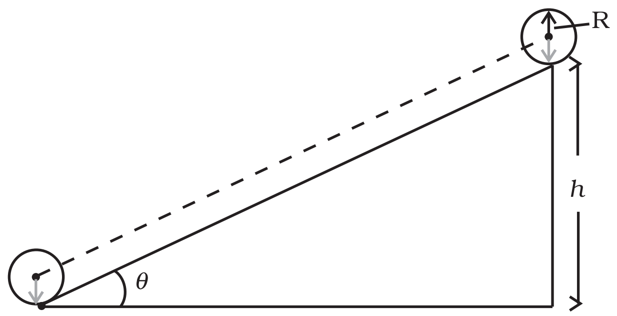


Fig.7.38

$$mgh = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$$

$$\text{or } v^2 = \left(\frac{2gh}{1+k^2/R^2}\right)$$

Note is independent of the mass of the rolling body;

For a ring, $k^2 = R^2$

$$v_{ring} = \sqrt{\frac{2gh}{1+1}},$$

$$= \sqrt{gh}$$

For a solid cylinder $k^2 = R^2/2$

$$v_{disc} = \sqrt{\frac{2gh}{1+1/2}}$$

$$= \sqrt{\frac{4gh}{3}}$$

For a solid sphere $k^2 = 2R^2/5$

$$v_{sphere} = \sqrt{\frac{2gh}{1+2/5}}$$

$$= \sqrt{\frac{10gh}{7}}$$

From the results obtained it is clear that among the three bodies the sphere has the greatest and the ring has the least velocity of the centre of mass at the bottom of the inclined plane.

Suppose the bodies have the same mass. Which body has the greatest rotational kinetic energy while reaching the bottom of the inclined plane? ◀

SUMMARY

1. Ideally, a rigid body is one for which the distances between different particles of the body do not change, even though there are forces on them.
2. A rigid body fixed at one point or along a line can have only rotational motion. A rigid body not fixed in some way can have either pure translation or a combination of translation and rotation.
3. In rotation about a fixed axis, every particle of the rigid body moves in a circle which lies in a plane perpendicular to the axis and has its centre on the axis. Every Point in the rotating rigid body has the same angular velocity at any instant of time.
4. In pure translation, every particle of the body moves with the same velocity at any instant of time.
5. Angular velocity is a vector. Its magnitude is $\omega = d\theta/dt$ and it is directed along the axis of rotation. For rotation about a fixed axis, this vector $\boldsymbol{\omega}$ has a fixed direction.

6. The vector or cross product of two vector **a** and **b** is a vector written as **a** × **b**. The magnitude of this vector is $ab\sin\theta$ and its direction is given by the right handed screw or the right hand rule.
7. The linear velocity of a particle of a rigid body rotating about a fixed axis is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, where **r** is the position vector of the particle with respect to an origin along the fixed axis. The relation applies even to more general rotation of a rigid body with one point fixed. In that case **r** is the position vector of the particle with respect to the fixed point taken as the origin.
8. The centre of mass of a system of particles is defined as the point whose position vector is

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M}$$

9. Velocity of the centre of mass of a system of particles is given by $\mathbf{V} = \mathbf{P}/M$, where **P** is the linear momentum of the system. The centre of mass moves as if all the mass of the system is concentrated at this point and all the external forces act at it. If the total external force on the system is zero, then the total linear momentum of the system is constant.

10. The angular momentum of a system of n particles about the origin is

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i$$

The torque or moment of force on a system of n particles about the origin is

$$\boldsymbol{\tau} = \sum_{1} \mathbf{r}_i \times \mathbf{F}_i$$

The force $\mathbf{F}_i$ acting on the i^{th} particle includes the external as well as internal forces. Assuming Newton's third law and that forces between any two particles act along the line joining the particles, we can show $\boldsymbol{\tau}_{int} = \mathbf{0}$ and

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_{ext}$$

11. A rigid body is in mechanical equilibrium if
    (1) it is in translational equilibrium, i.e., the total external force on it is zero : $\sum \mathbf{F}_i = \mathbf{0}$, and
    (2) it is in rotational equilibrium, i.e. the total external torque on it is zero : $\sum \boldsymbol{\tau}_i = \sum \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0}$.
12. The centre of gravity of an extended body is that point where the total gravitational torque on the body is zero.
13. The moment of intertia of a rigid body about an axis is defined by the formula $I = \sum m_i r_i^2$ where r_i is the perpendicular distance of the ith point of the body from the axis. The kinetic energy of rotation is $K = \frac{1}{2} I\omega^2$.
14. The theorem of parallel axes: $I'_z = I_z + Ma^2$, allows us to determine the moment of intertia of a rigid body about an axis as the sum of the moment of inertia of the body about a parallel axis through its centre of mass and the product of mass and square of the perpendicular distance between these two axes.

15. Rotation about a fixed axis is directly analogous to linear motion in respect of kinematics and dynamics.
16. The angular acceleration of a rigid body rotating about a fixed axis is given by $I\alpha = \tau$. If the external torque τ is zero, the component of angular momentum about the fixed axis $I\omega$ of such a rotating body is constant.
17. For rolling motion without slipping $v_{cm} = R\omega$ where v_{cm} is the velocity of translation (i.e. of the centre of mass), R is the radius and m is the mass of the body. The kinetic energy of such a rolling body is the sum of kinetic energies of translation and rotation:
$$K = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I\omega^2 .$$

Quantity	Symbols	Dimensions	Units	Remarks
Angular velocity	ω	$[T^{-1}]$	rad s^{-1}	$v = \omega \times r$
Angular momentum	L	$[ML^2T^{-1}]$	J s	$L = r \times p$
Torque	τ	$[ML^2T^{-2}]$	N m	$\tau = r \times F$
Moment of inertia	I	$[ML^2]$	kg m^2	$I = \sum m_i r_i^2$

POINTS TO PONDER

1. To determine the motion of the centre of mass of a system no knowledge of internal forces of the system is required. For this purpose we need to know only the external forces on the body.
2. Separating the motion of a system of particles as, i.e. the motion of the centre of mass transnational motion of the system and motion about (i.e. relative to) the centre of mass of the system is a useful technique in dynamics of a system of particles. One example of this technique is separating the kinetic energy of a system of particles K as the kinetic energy of the system about its centre of mass K′ and the kinetic energy of the centre of mass $MV^2/2$,
$$K = K' + MV^2/2$$
3. Newton's Second Law for finite sized bodies (or systems of particles) is based in Newton's Second Law and also Newton's Third Law for particles.
4. To establish that the time rate of change of the total angular momentum of a system of particles is the total external torque in the system, we need not only Newton's second law for particles, but also Newton's third law with the provision that the forces between any two particle act along the line joining the particles.
5. The vanishing of the total external force and the vanishing of the total external torque are independent conditions. We can have one without the other. In a couple, total external force is zero but total torque is non-zero.
6. The total torque on a system is independent of the origin if the total external force is zero.
7. The centre of gravity of a body coincides with its centre of mass only if the gravitational field does not vary from one part of the body to the other.
8. The angular momentum **L** and the angular velocity $\boldsymbol{\omega}$ are not necessarily parallel vectors. However, for the simpler situations discussed in this chapter when rotation is about a fixed axis which is an axis of symmetry of the rigid body, the relation $\mathbf{L} = I\boldsymbol{\omega}$ holds good, where I is the moment of the inertia of the body about the rotation axis.

EXERCISES

7.1 Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body ?

7.2 In the HC1 molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

7.3 A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system ?

7.4 Show that the area of the triangle contained between the vectors **a** and **b** is one half of the magnitude of **a** × **b**.

7.5 Show that **a**·(**b** × **c**) is equal in magnitude to the volume of the parallelepiped formed on the three vectors , **a**, **b** and **c**.

7.6 Find the components along the x, y, z axes of the angular momentum **l** of a particle, whose position vector is **r** with components x, y, z and momentum is **p** with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane the angular momentum has only a z-component.

7.7 Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two particle system is the same whatever be the point about which the angular momentum is taken.

7.8 A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig.7.39. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.

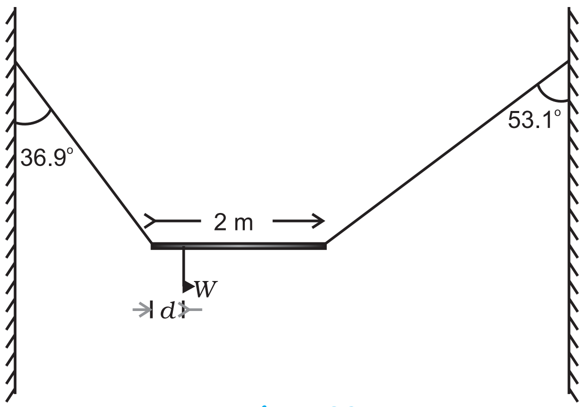


Fig. 7.39

7.9 A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

7.10 (a) Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be $2MR^2/5$, where M is the mass of the sphere and R is the radius of the sphere.

(b) Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $MR^2/4$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.

7.11 Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time.

7.12 A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

7.13 (a) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value ? Assume that the turntable rotates without friction.

(b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

7.14 A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N ? What is the linear acceleration of the rope ? Assume that there is no slipping.

7.15 To maintain a rotor at a uniform angular speed or 200 rad s^{-1}, an engine needs to transmit a torque of 180 N m. What is the power required by the engine ? (Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

7.16 From a uniform disk of radius R, a circular hole of radius $R/2$ is cut out. The centre of the hole is at $R/2$ from the centre of the original disc. Locate the centre of gravity of the resulting flat body.

7.17 A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

7.18 A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination. (a) Will it reach the bottom with the same speed in each case? (b) Will it take longer to roll down one plane than the other? (c) If so, which one and why?

7.19 A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it?

7.20 The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m^2 about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

7.21 A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5 m/s.
(a) How far will the cylinder go up the plane?
(b) How long will it take to return to the bottom?

Additional Exercises

7.22 As shown in Fig.7.40, the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied half way up. A weight 40 kg is suspended from a point F, 1.2 m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take g = 9.8 m/s^2)
(Hint: Consider the equilibrium of each side of the ladder separately.)

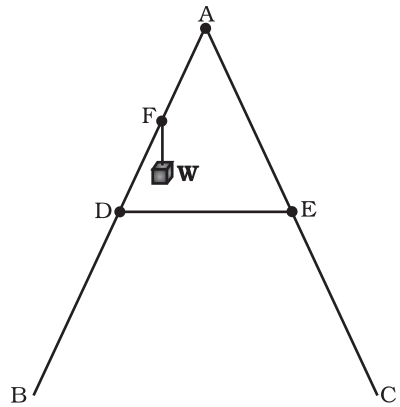


Fig.7.40

7.23 A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90cm to 20cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m^2.
(a) What is his new angular speed? (Neglect friction.)
(b) Is kinetic energy conserved in the process? If not, from where does the change come about?

7.24 A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.
(Hint: The moment of inertia of the door about the vertical axis at one end is $ML^2/3$.)

7.25 Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident. (a) What is the angular speed of the two-disc system? (b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$.

7.26 (a) Prove the theorem of perpendicular axes.

(Hint : Square of the distance of a point (x, y) in the x–y plane from an axis through the origin perpendicular to the plane is x^2+y^2).

(b) Prove the theorem of parallel axes.

(Hint : If the centre of mass is chosen to be the origin $\sum m_i \mathbf{r}_i = 0$).

7.27 Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by

$$v^2 = \frac{2gh}{\left(1 + k^2/R^2\right)}$$

using dynamical consideration (i.e. by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

7.28 A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in Fig. 7.41? Will the disc roll in the direction indicated ?

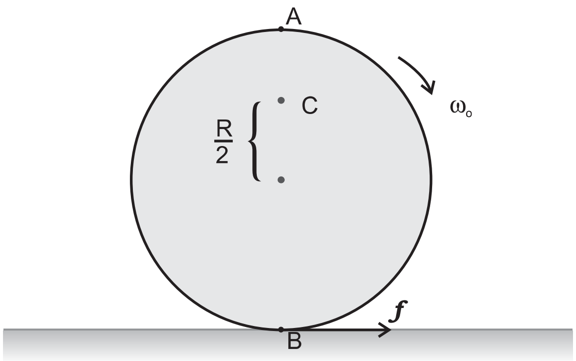


Fig. 7.41

7.29 Explain why friction is necessary to make the disc in Fig. 7.41 roll in the direction indicated.
(a) Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.
(b) What is the force of friction after perfect rolling begins ?

7.30 A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10 π rad s^{-1}. Which of the two will start to roll earlier ? The co-efficient of kinetic friction is μ_k = 0.2.

7.31 A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30^o. The co-efficient of static friction μ_s = 0.25.
(a) How much is the force of friction acting on the cylinder ?
(b) What is the work done against friction during rolling ?
(c) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly ?

$\mathbf{L} = \mathbf{L}' + \mathbf{R} \times M\mathbf{V}$

7.32 Read each statement below carefully, and state, with reasons, if it is true or false;
(a) During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.
(b) The instantaneous speed of the point of contact during rolling is zero.
(c) The instantaneous acceleration of the point of contact during rolling is zero.
(d) For perfect rolling motion, work done against friction is zero.
(e) A wheel moving down a perfectly *frictionless* inclined plane will undergo slipping (not rolling) motion.

7.33 Separation of Motion of a system of particles into motion of the centre of mass and motion about the centre of mass :
(a) Show $\mathbf{p} = \mathbf{p}'_i + m_i\mathbf{V}$

where $\mathbf{p}_i$ is the momentum of the *i*th particle (of mass m_i) and $\mathbf{p}'_i = m_i \mathbf{v}'_i$. Note $\mathbf{v}'_i$ is the velocity of the *i*th particle relative to the centre of mass.

Also, prove using the definition of the centre of mass $\sum \mathbf{p}'_i = 0$

(b) Show $K = K' + ½MV^2$

where K is the total kinetic energy of the system of particles, K' is the total kinetic energy of the system when the particle velocities are taken with respect to the centre of mass and $MV^2/2$ is the kinetic energy of the translation of the system as a whole (i.e. of the centre of mass motion of the system). The result has been used in Sec. 7.14.

(c) Show

where $\mathbf{L}' = \sum \mathbf{r}'_i \times \mathbf{p}'_i$ is the angular momentum of the system about the centre of mass with velocities taken relative to the centre of mass. Remember $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}$; rest of the notation is the

standard notation used in the chapter. Note $\mathbf{L}'$ and $M\mathbf{R} \times \mathbf{V}$ can be said to be angular momenta, respectively, about and of the centre of mass of the system of particles.

(d) Show $\dfrac{d\mathbf{L}'}{dt} = \sum \mathbf{r}_i' \times \dfrac{d\mathbf{p}'}{dt}$

Further, show that

$$\frac{d\mathbf{L}'}{dt} = \boldsymbol{\tau}'_{ext}$$

where $\boldsymbol{\tau}'_{ext}$ is the sum of all external torques acting on the system about the centre of mass.

(Hint : Use the definition of centre of mass and Newton's Third Law. Assume the internal forces between any two particles act along the line joining the particles.)

CHAPTER EIGHT

GRAVITATION

8.1 INTRODUCTION

Early in our lives, we become aware of the tendency of all material objects to be attracted towards the earth. Anything thrown up falls down towards the earth, going uphill is lot more tiring than going downhill, raindrops from the clouds above fall towards the earth and there are many other such phenomena. Historically it was the Italian Physicist Galileo (1564-1642) who recognised the fact that all bodies, irrespective of their masses, are accelerated towards the earth with a constant acceleration. It is said that he made a public demonstration of this fact. To find the truth, he certainly did experiments with bodies rolling down inclined planes and arrived at a value of the acceleration due to gravity which is close to the more accurate value obtained later.

A seemingly unrelated phenomenon, observation of stars, planets and their motion has been the subject of attention in many countries since the earliest of times. Observations since early times recognised stars which appeared in the sky with positions unchanged year after year. The more interesting objects are the planets which seem to have regular motions against the background of stars. The earliest recorded model for planetary motions proposed by Ptolemy about 2000 years ago was a 'geocentric' model in which all celestial objects, stars, the sun and the planets, all revolved around the earth. The only motion that was thought to be possible for celestial objects was motion in a circle. Complicated schemes of motion were put forward by Ptolemy in order to describe the observed motion of the planets. The planets were described as moving in circles with the center of the circles themselves moving in larger circles. Similar theories were also advanced by Indian astronomers some 400 years later. However a more elegant model in which the Sun was the center around which the planets revolved – the 'heliocentric' model – was already mentioned by Aryabhatta (5th century A.D.) in his treatise. A thousand years later, a Polish monk named Nicolas

Copernicus (1473-1543) proposed a definitive model in which the planets moved in circles around a fixed central sun. His theory was discredited by the church, but notable amongst its supporters was Galileo who had to face prosecution from the state for his beliefs.

It was around the same time as Galileo, a nobleman called Tycho Brahe (1546-1601) hailing from Denmark, spent his entire lifetime recording observations of the planets with the naked eye. His compiled data were analysed later by his assistant Johannes Kepler (1571-1640). He could extract from the data three elegant laws that now go by the name of Kepler's laws. These laws were known to Newton and enabled him to make a great scientific leap in proposing his universal law of gravitation.

8.2 KEPLER'S LAWS

The three laws of Kepler can be stated as follows:

1. Law of orbits : All planets move in elliptical orbits with the Sun situated at one of the foci

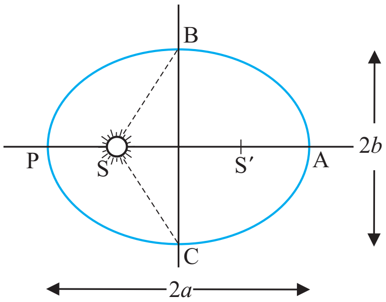


Fig. 8.1(a) *An ellipse traced out by a planet around the sun. The closest point is P and the farthest point is A, P is called the perihelion and A the aphelion. The semimajor axis is half the distance AP.*

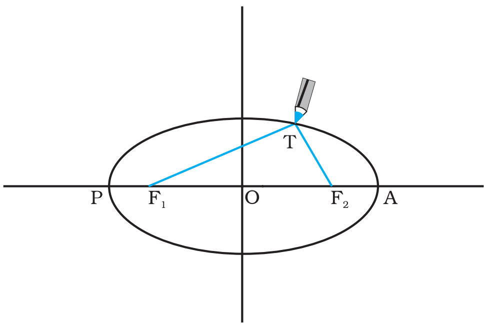


Fig. 8.1(b) *Drawing an ellipse. A string has its ends fixed at F_1 and F_2. The tip of a pencil holds the string taut and is moved around.*

of the ellipse (Fig. 8.1a). This law was a deviation from the Copernican model which allowed only circular orbits. The ellipse, of which the circle is a special case, is a closed curve which can be drawn very simply as follows.

Select two points F_1 and F_2. Take a length of a string and fix its ends at F_1 and F_2 by pins. With the tip of a pencil stretch the string taut and then draw a curve by moving the pencil keeping the string taut throughout.(Fig. 8.1(b)) The closed curve you get is called an ellipse. Clearly for any point T on the ellipse, the sum of the distances from F_1 and F_2 is a constant. F_1, F_2 are called the focii. Join the points F_1 and F_2 and extend the line to intersect the ellipse at points P and A as shown in Fig. 8.1(b). The midpoint of the line PA is the centre of the ellipse O and the length PO = AO is called the semi-major axis of the ellipse. For a circle, the two focii merge into one and the semi-major axis becomes the radius of the circle.

2. Law of areas : The line that joins any planet to the sun sweeps equal areas in equal intervals of time (Fig. 8.2). This law comes from the observations that planets appear to move slower when they are farther from the sun than when they are nearer.

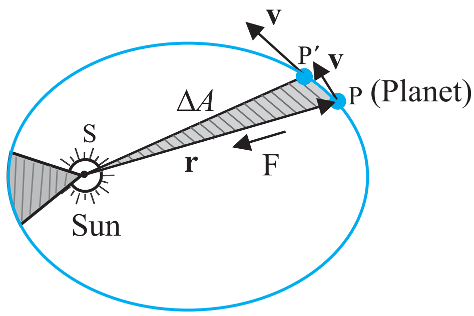


Fig. 8.2 *The planet P moves around the sun in an elliptical orbit. The shaded area is the area ΔA swept out in a small interval of time Δt.*

3. Law of periods : The square of the time period of revolution of a planet is proportional to the cube of the semi-major axis of the ellipse traced out by the planet.

The table below gives the approximate time periods of revolution of nine planets around the sun along with values of their semi-major axes.

Table 1 Data from measurement of planetary motions given below confirm Kepler's Law of Periods

a ≡ Semi-major axis in units of 10^{10} m.
T ≡ Time period of revolution of the planet in years(y).
Q ≡ The quotient (T^2/a^3) in units of 10^{-34} y^2 m^{-3}.

Planet	a	T	Q
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

The law of areas can be understood as a consequence of conservation of angular momentum whch is valid for any central force . A central force is such that the force on the planet is along the vector joining the sun and the planet. Let the sun be at the origin and let the position and momentum of the planet be denoted by **r** and **p** respectively. Then the area swept out by the planet of mass m in time interval Δt is (Fig. 8.2) $\Delta\mathbf{A}$ given by

$$\Delta\mathbf{A} = \tfrac{1}{2}\ (\mathbf{r} \times \mathbf{v}\Delta t) \qquad (8.1)$$

Hence

$$\Delta\mathbf{A}/\Delta t = \tfrac{1}{2}\ (\mathbf{r} \times \mathbf{p})/\text{m}, \ (\text{since } \mathbf{v} = \mathbf{p}/\text{m})$$

$$= \mathbf{L}\ /\ (2\ \text{m}) \qquad (8.2)$$

where **v** is the velocity, **L** is the angular momentum equal to (**r** × **p**). For a central force, which is directed along **r**, **L** is a constant as the planet goes around. Hence, $\Delta\mathbf{A}/\Delta t$ is a constant according to the last equation. This is the law of areas. Gravitation is a central force and hence the law of areas follows.

Johannes Kepler (1571–1630) was a scientist of German origin. He formulated the three laws of planetary motion based on the painstaking observations of Tycho Brahe and coworkers. Kepler himself was an assistant to Brahe and it took him sixteen long years to arrive at the three planetary laws. He is also known as the founder of geometrical optics, being the first to describe what happens to light after it enters a telescope.

Example 8.1 Let the speed of the planet at the perihelion P in Fig. 8.1(a) be v_P and the Sun-planet distance SP be r_P. Relate $\{r_P, v_P\}$ to the corresponding quantities at the aphelion $\{r_A, v_A\}$. Will the planet take equal times to traverse BAC and CPB ?

Answer The magnitude of the angular momentum at P is $L_p = m_p r_p v_p$, since inspection tells us that $\mathbf{r}_p$ and $\mathbf{v}_p$ are mutually perpendicular. Similarly, $L_A = m_p r_A v_A$. From angular momentum conservation

$$m_p r_p v_p = m_p r_A v_A$$

or

$$\frac{v_p}{v_A} = \frac{r_A}{r_p}$$

Since $r_A > r_p$, $v_p > v_A$.

The area $SBAC$ bounded by the ellipse and the radius vectors SB and SC is larger than SBPC in Fig. 8.1. From Kepler's second law, equal areas are swept in equal times. Hence the planet will take a longer time to traverse BAC than CPB. ◀

8.3 UNIVERSAL LAW OF GRAVITATION

Legend has it that observing an apple falling from a tree, Newton was inspired to arrive at an universal law of gravitation that led to an explanation of terrestrial gravitation as well as of Kepler's laws. Newton's reasoning was that the moon revolving in an orbit of radius R_m was subject to a centripetal acceleration due to earth's gravity of magnitude

$$a_m = \frac{V^2}{R_m} = \frac{4\pi^2 R_m}{T^2} \qquad (8.3)$$

where V is the speed of the moon related to the time period T by the relation $V = 2\pi R_m / T$. The time period T is about 27.3 days and R_m was already known then to be about 3.84×10^8m. If we substitute these numbers in equation (8.3), we get a value of a_m much smaller than the value of acceleration due to gravity g on the surface of the earth, arising also due to earth's gravitational attraction.

Central Forces

We know the time rate of change of the angular momentum of a single particle about the origin is

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F}$$

The angular momentum of the particle is conserved, if the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ due to the force **F** on it vanishes. This happens either when **F** is zero or when **F** is along **r**. We are interested in forces which satisfy the latter condition. Central forces satisfy this condition. A 'central' force is always directed towards or away from a fixed point, i.e., along the position vector of the point of application of the force with respect to the fixed point. (See Figure below.) Further, the magnitude of a central force F depends on r, the distance of the point of application of the force from the fixed point; $F = F(r)$.

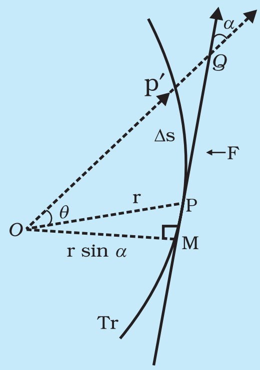
In the motion under a central force the angular momentum is always conserved. Two important results follow from this:

(1) The motion of a particle under the central force is always confined to a plane.

(2) The position vector of the particle with respect to the centre of the force (i.e. the fixed point) has a constant areal velocity. In other words the position vector sweeps out equal areas in equal times as the particle moves under the influence of the central force.

Try to prove both these results. You may need to know that the areal velocity is given by : $\mathrm{d}A/\mathrm{d}t = \frac{1}{2}\, r\, v \sin \alpha$.

An immediate application of the above discussion can be made to the motion of a planet under the gravitational force of the sun. For convenience the sun may be taken to be so heavy that it is at rest. The gravitational force of the sun on the planet is directed towards the sun. This force also satisfies the requirement $F = F(r)$, since $F = G\, m_1 m_2/r^2$ where m_1 and m_2 are respectively the masses of the planet and the sun and G is the universal constant of gravitation. The two results (1) and (2) described above, therefore, apply to the motion of the planet. In fact, the result (2) is the well-known second law of Kepler.



Tr is the trajectory of the particle under the central force. At a position P, the force is directed along **OP**, *O is the centre of the force taken as the origin. In time Δt, the particle moves from* P *to P′, arc PP′ $= \Delta s = v\, \Delta t$. The tangent PQ at P to the trajectory gives the direction of the velocity at P. The area swept in Δt is the area of sector POP′ $\approx (r \sin \alpha)$ PP′ /2 = $(r\, v \sin \alpha)\, \Delta t/2$.)*

This clearly shows that the force due to earth's gravity decreases with distance. If one assumes that the gravitational force due to the earth decreases in proportion to the inverse square of the distance from the center of the earth, we will have $a_m \alpha R_m^{-2}$; $g \alpha R_E^{-2}$ and we get

$$\frac{g}{a_m} = \frac{R_m^2}{R_E^2} ; \; 3600 \qquad (8.4)$$

in agreement with a value of g ; 9.8 m s^{-2} and the value of a_m from Eq. (8.3). These observations led Newton to propose the following Universal Law of Gravitation :

Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The quotation is essentially from Newton's famous treatise called 'Mathematical Principles of Natural Philosophy' (Principia for short).

Stated Mathematically, Newton's gravitation law reads : The force **F** on a point mass m_2 due to another point mass m_1 has the magnitude

$$|\mathbf{F}| = G \frac{m_1 \; m_2}{r^2} \qquad (8.5)$$

Equation (8.5) can be expressed in vector form as

$$\mathbf{F} = G \frac{m_1 \; m_2}{r^2} \left(-\hat{\mathbf{r}}\right) = -G \frac{m_1 \; m_2}{r^2} \hat{\mathbf{r}}$$

$$= -G \frac{m_1 \; m_2}{|\mathbf{r}|^3} \hat{\mathbf{r}}$$

where G is the universal gravitational constant, $\hat{\mathbf{r}}$ is the unit vector from m_1 to m_2 and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ as shown in Fig. 8.3.

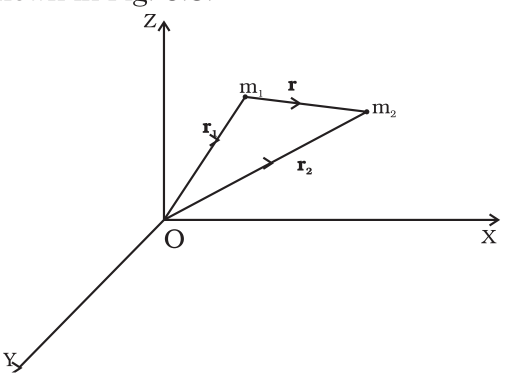


Fig. 8.3 Gravitational force on m_1 due to m_2 is along **r** *where the vector* **r** *is* $(\mathbf{r}_2 - \mathbf{r}_1)$.

The gravitational force is attractive, i.e., the force **F** is along – **r**. The force on point mass m_1 due to m_2 is of course – **F** by Newton's third law. Thus, the gravitational force $\mathbf{F}_{12}$ on the body 1 due to 2 and $\mathbf{F}_{21}$ on the body 2 due to 1 are related as $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Before we can apply Eq. (8.5) to objects under consideration, we have to be careful since the law refers to **point** masses whereas we deal with extended objects which have finite size. . If we have a collection of point masses,the force on any one of them is the vector sum of the gravitational forces exerted by the other point masses as shown in Fig 8.4.

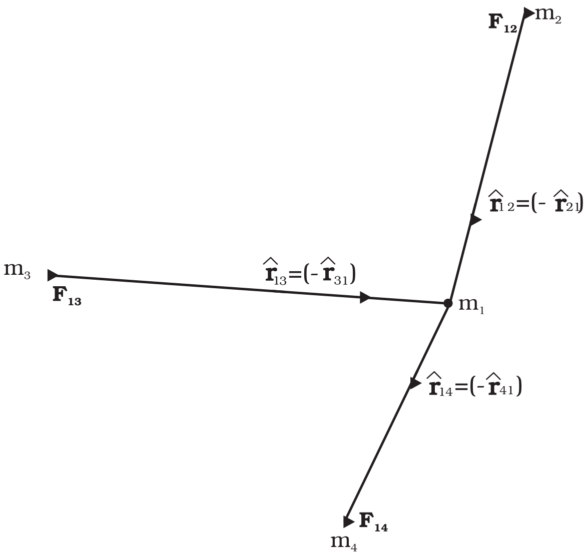


Fig. 8.4 Gravitational force on point mass m_1 is the vector sum of the gravitational forces exerted by m_2, m_3 and m_4.

The total force on m_1 is

$$\mathbf{F}_1 = \frac{Gm_2 \, m_1}{r_{21}^2} \hat{\mathbf{r}}_{21} + \frac{Gm_3 \, m_1}{r_{31}^2} \hat{\mathbf{r}}_{31} + \frac{Gm_4 \, m_1}{r_{41}^2} \hat{\mathbf{r}}_{41}$$

▶ ***Example 8.2*** Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC.
(a) What is the force acting on a mass 2*m* placed at the centroid G of the triangle?
(b) What is the force if the mass at the vertex A is doubled ?
Take AG = BG = CG = 1m (see Fig. 8.5)

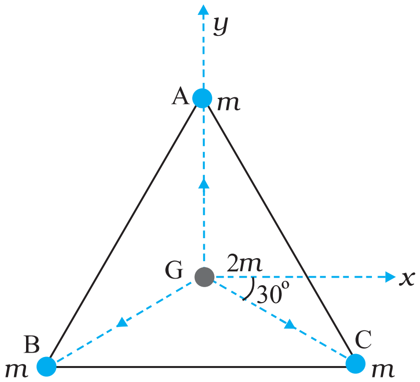


Fig. 8.5 *Three equal masses are placed at the three vertices of the Δ ABC. A mass 2m is placed at the centroid G.*

Answer (a) The angle between GC and the positive x-axis is 30° and so is the angle between GB and the negative x-axis. The individual forces in vector notation are

$$\mathbf{F}_{GA} = \frac{Gm(2m)}{1}\hat{\mathbf{j}}$$

$$\mathbf{F}_{GB} = \frac{Gm(2m)}{1}\left(-\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right)$$

$$\mathbf{F}_{GC} = \frac{Gm(2m)}{1}\left(+\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right)$$

From the principle of superposition and the law of vector addition, the resultant gravitational force $\mathbf{F}_R$ on $(2m)$ is

$$\mathbf{F}_R = \mathbf{F}_{GA} + \mathbf{F}_{GB} + \mathbf{F}_{GC}$$

$$\mathbf{F}_R = 2Gm^2\,\hat{\mathbf{j}} + 2Gm^2\left(-\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right)$$

$$+\,2Gm^2\left(\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right) = 0$$

Alternatively, one expects on the basis of symmetry that the resultant force ought to be zero.

(b) By symmetry the x-component of the force cancels out. The y-component survives.

$$\mathbf{F}_R = 4Gm^2\hat{\mathbf{j}} - 2Gm^2\hat{\mathbf{j}} = 2Gm^2\hat{\mathbf{j}}$$ ◀

For the gravitational force between an extended object (like the earth) and a point mass, Eq. (8.5) is not directly applicable. Each point mass in the extended object will exert a force on the given point mass and these force will not all be in the same direction. We have to add up these forces vectorially for all the point masses in the extended object to get the total force. This is easily done using calculus. For two special cases, a simple law results when you do that :

(1) **The force of attraction between a hollow spherical shell of uniform density and a point mass situated outside is just as if the entire mass of the shell is concentrated at the centre of the shell.** Qualitatively this can be understood as follows: Gravitational forces caused by the various regions of the shell have components along the line joining the point mass to the centre as well as along a direction prependicular to this line. The components prependicular to this line cancel out when summing over all regions of the shell leaving only a resultant force along the line joining the point to the centre. The magnitude of this force works out to be as stated above.

Newton's Principia

Kepler had formulated his third law by 1619. The announcement of the underlying universal law of gravitation came about seventy years later with the publication in 1687 of Newton's masterpiece **Philosophiae Naturalis Principia Mathematica**, often simply called the **Principia**.

Around 1685, Edmund Halley (after whom the famous Halley's comet is named), came to visit Newton at Cambridge and asked him about the nature of the trajectory of a body moving under the influence of an inverse square law. Without hesitation Newton replied that it had to be an ellipse, and further that he had worked it out long ago around 1665 when he was forced to retire to his farm house from Cambridge on account of a plague outbreak. Unfortunately, Newton had lost his papers. Halley prevailed upon Newton to produce his work in book form and agreed to bear the cost of publication. Newton accomplished this feat in eighteen months of superhuman effort. The **Principia** is a singular scientific masterpiece and in the words of Lagrange it is "the greatest production of the human mind." The Indian born astrophysicist and Nobel laureate S. Chandrasekhar spent ten years writing a treatise on the **Principia**. His book, Newton's **Principia for the Common Reader** brings into sharp focus the beauty, clarity and breath taking economy of Newton's methods.

(2) **The force of attraction due to a hollow spherical shell of uniform density, on a point mass situated inside it is zero.**
Qualitatively, we can again understand this result. Various regions of the spherical shell attract the point mass inside it in various directions. These forces cancel each other completely.

8.4 THE GRAVITATIONAL CONSTANT

The value of the gravitational constant G entering the Universal law of gravitation can be determined experimentally and this was first done by English scientist Henry Cavendish in 1798. The apparatus used by him is schematically shown in figure.8.6

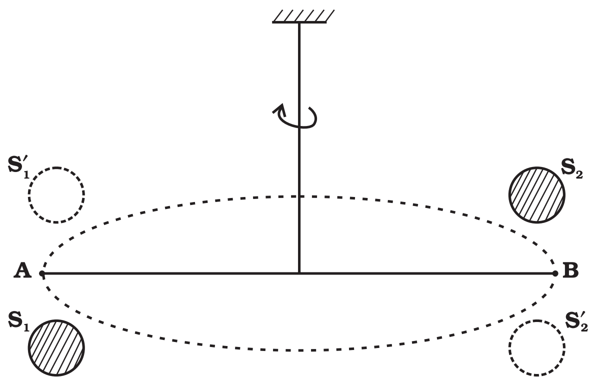


Fig. 8.6 *Schematic drawing of Cavendish's experiment. S_1 and S_2 are large spheres which are kept on either side (shown shades) of the masses at A and B. When the big spheres are taken to the other side of the masses (shown by dotted circles), the bar AB rotates a little since the torque reverses direction. The angle of rotation can be measured experimentally.*

The bar AB has two small lead spheres attached at its ends. The bar is suspended from a rigid support by a fine wire. Two large lead spheres are brought close to the small ones but on opposite sides as shown. The big spheres attract the nearby small ones by equal and opposite force as shown. There is no net force on the bar but only a torque which is clearly equal to F times the length of the bar,where F is the force of attraction between a big sphere and its neighbouring small sphere. Due to this torque, the suspended wire gets twisted till such time as the restoring torque of the wire equals the gravitational torque . If θ is the angle of twist of the suspended wire, the restoring torque is proportional to θ, equal to $\tau\theta$. Where τ is the restoring couple per unit angle of twist. τ can be measured independently e.g. by applying a known torque and measuring the angle of twist. The gravitational force between the spherical balls is the same as if their masses are concentrated at their centers. Thus if d is the separation between the centers of the big and its neighbouring small ball, M and m their masses, the gravitational force between the big sphere and its neighouring small ball is.

$$F = G\frac{Mm}{d^2} \tag{8.6}$$

If L is the length of the bar AB , then the torque arising out of F is F multiplied by L. At equilibrium, this is equal to the restoring torque and hence

$$G\frac{Mm}{d^2}L = \tau\,\theta \tag{8.7}$$

Observation of θ thus enables one to calculate G from this equation.

Since Cavendish's experiment, the measurement of G has been refined and the currently accepted value is

$$G = 6.67\times10^{-11}\ \text{N m}^2/\text{kg}^2 \tag{8.8}$$

8.5 ACCELERATION DUE TO GRAVITY OF THE EARTH

The earth can be imagined to be a sphere made of a large number of concentric spherical shells with the smallest one at the centre and the largest one at its surface. A point outside the earth is obviously outside all the shells. Thus, all the shells exert a gravitational force at the point outside just as if their masses are concentrated at their common centre according to the result stated in the last section. The total mass of all the shells combined is just the mass of the earth. Hence, at a point outside the earth, the gravitational force is just as if its entire mass of the earth is concentrated at its center.

For a point inside the earth, the situation is different. This is illustrated in Fig. 8.7.

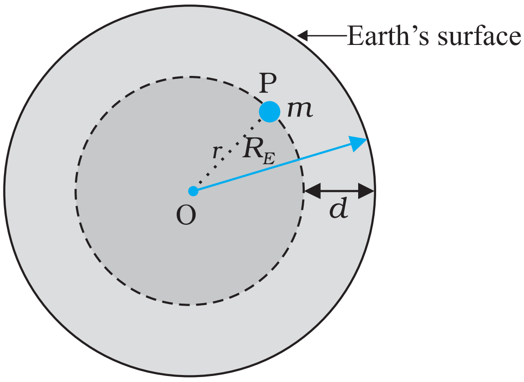


Fig. 8.7 *The mass m is in a mine located at a depth d below the surface of the Earth of mass M_E and radius R_E. We treat the Earth to be spherically symmetric.*

Again consider the earth to be made up of concentric shells as before and a point mass m situated at a distance r from the centre. The point P lies outside the sphere of radius r. For the shells of radius greater than r, the point P lies inside. Hence according to result stated in the last section, they exert no gravitational force on mass m kept at P. The shells with radius $\leq$r make up a sphere of radius r for which the point P lies on the surface. This smaller sphere therefore exerts a force on a mass m at P as if its mass m_r is concentrated at the center. Thus the force on the mass m at P has a magnitude

$$F = \frac{Gm\,(m_r)}{r^2} \qquad (8.9)$$

We assume that the entire earth is of uniform density and hence its mass is $M_E = \frac{4\pi}{3} R_E^3\,\rho$

where M_E is the mass of the earth R_E is its radius and ρ is the density. On the other hand the mass of the sphere M_r of radius r is $\frac{4\pi}{3}\rho\, r^3$ and hence

$$F = G\,m\left(\frac{4\pi}{3}\rho\right)\frac{r^3}{r^2} = G\,m\left(\frac{M_E}{R_E^3}\right)\frac{r^3}{r^2}$$

$$= \frac{G\,m\,M_E}{{R_E}^3}\,r \qquad (8.10)$$

If the mass m is situated on the surface of earth, then r = R_E and the gravitational force on it is, from Eq. (8.10)

$$F = G\,\frac{M_E m}{R_E^2} \qquad (8.11)$$

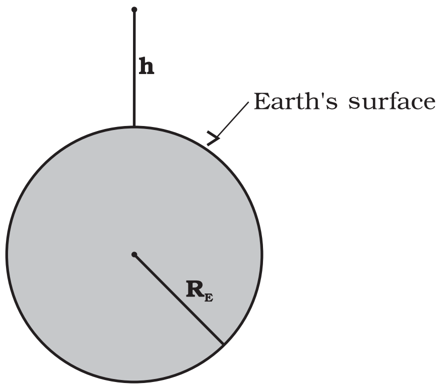
The acceleration experienced by the mass m, which is usually denoted by the symbol g is related to F by Newton's 2nd law by relation F = mg. Thus

$$g = \frac{F}{m} = \frac{GM_E}{R_E^2} \qquad (8.12)$$

Acceleration g is readily measurable. R_E is a known quantity. The measurement of G by Cavendish's experiment (or otherwise), combined with knowledge of g and R_E enables one to estimate M_E from Eq. (8.12). This is the reason why there is a popular statement regarding Cavendish : "Cavendish weighed the earth".

8.6 ACCELERATION DUE TO GRAVITY BELOW AND ABOVE THE SURFACE OF EARTH

Consider a point mass m at a height h above the surface of the earth as shown in Fig. 8.8(a). The radius of the earth is denoted by R_E. Since this point is outside the earth,



(a)

Fig. 8.8 (a) *g at a height h above the surface of the earth.*

its distance from the centre of the earth is $(R_E + h)$. If $F(h)$ denoted the magnitude of the force on the point mass m , we get from Eq. (8.5) :

$$F(h) = \frac{GM_E m}{(R_E + h)^2} \qquad (8.13)$$

The acceleration experienced by the point mass is $F(h)/m \equiv g(h)$ and we get

$$g(h) = \frac{F(h)}{m} = \frac{GM_E}{(R_E + h)^2} \quad (8.14)$$

This is clearly less than the value of g on the surface of earth : $g = \frac{GM_E}{R_E^2}$. For $h << R_E$, we can expand the RHS of Eq. (8.14) :

$$g(h) = \frac{GM}{R_E^2(1 + h/R_E)^2} = g(1 + h/R_E)^{-2}$$

For $\frac{h}{R_E} << 1$, using binomial expression,

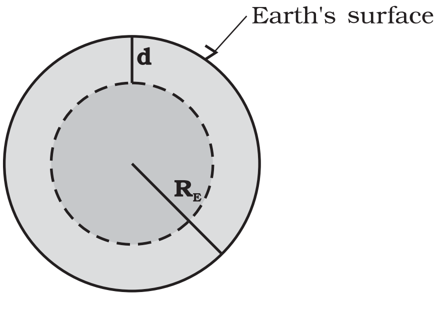
$$g(h) \cong g\left(1 - \frac{2h}{R_E}\right) . \quad (8.15)$$

Equation (8.15) thus tells us that for small heights h above the value of g decreases by a factor $(1 - 2h/R_E)$.

Now, consider a point mass m at a depth d below the surface of the earth (Fig. 8.8(b)), so that its distance from the center of the earth is $(R_E - d)$ as shown in the figure. The earth can be thought of as being composed of a smaller sphere of radius $(R_E - d)$ and a spherical shell of thickness d. The force on m due to the outer shell of thickness d is zero because the result quoted in the previous section. As far as the smaller sphere of radius $(R_E - d)$ is concerned, the point mass is outside it and hence according to the result quoted earlier, the force due to this smaller sphere is just as if the entire mass of the smaller sphere is concentrated at the centre. If M_s is the mass of the smaller sphere, then,

$$M_s / M_E = (R_E - d)^3 / R_E^3 \quad (8.16)$$

Since mass of a sphere is proportional to be cube of its radius.



(b)

Fig. 8.8 (b) *g at a depth d. In this case only the smaller sphere of radius $(R_E–d)$ contributes to g.*

Thus the force on the point mass is

$$F(d) = G M_s m / (R_E - d)^2 \quad (8.17)$$

Substituting for M_s from above , we get

$$F(d) = G M_E m (R_E - d) / R_E^3 \quad (8.18)$$

and hence the acceleration due to gravity at a depth d,

$$g(d) = \frac{F(d)}{m} \text{ is}$$

$$g(d) = \frac{F(d)}{m} = \frac{GM_E}{R_E^3}(R_E - d)$$

$$= g\frac{R_E - d}{R_E} = g(1 - d/R_E) \quad (8.19)$$

Thus, as we go down below earth's surface, the acceleration due gravity decreases by a factor $(1 - d/R_E)$. The remarkable thing about acceleration due to earth's gravity is that it is maximum on its surface decreasing whether you go up or down.

8.7 GRAVITATIONAL POTENTIAL ENERGY

We had discussed earlier the notion of potential energy as being the energy stored in the body at its given position. If the position of the particle changes on account of forces acting on it, then the change in its potential energy is just the amount of work done on the body by the force. As we had discussed earlier, forces for which the work done is independent of the path are the conservative forces.

The force of gravity is a conservative force and we can calculate the potential energy of a body arising out of this force, called the gravitational potential energy. Consider points close to the surface of earth, at distances from the surface much smaller than the radius of the earth. In such cases, the force of gravity is practically a constant equal to mg, directed towards the center of the earth. If we consider a point at a height h_1 from the surface of the earth and another point vertically above it at a height h_2 from the surface, the work done in lifting the particle of mass m from the first to the second position is denoted by W_{12}

$$W_{12} = \text{Force} \times \text{displacement}$$

$$= mg(h_2 - h_1) \quad (8.20)$$

If we associate a potential energy $W(h)$ at a point at a height h above the surface such that

$$W(h) = mgh + W_o \tag{8.21}$$

(where W_o = constant) ;
then it is clear that

$$W_{12} = W(h_2) - W(h_1) \tag{8.22}$$

The work done in moving the particle is just the difference of potential energy between its final and initial positions.Observe that the constant W_o cancels out in Eq. (8.22). Setting $h = 0$ in the last equation, we get $W(h = 0) = W_o$. $h = 0$ means points on the surface of the earth. Thus, W_o is the potential energy on the surface of the earth.

If we consider points at arbitrary distance from the surface of the earth, the result just derived is not valid since the assumption that the gravitational force mg is a constant is no longer valid. However, from our discussion we know that a point outside the earth, the force of gravitation on a particle directed towards the center of the earth is

$$F = \frac{G M_E m}{r^2} \tag{8.23}$$

where M_E = mass of earth, m = mass of the particle and r its distance from the center of the earth. If we now calculate the work done in lifting a particle from $r = r_1$ to $r = r_2$ $(r_2 > r_1)$ along a vertical path, we get instead of Eq. (8.20)

$$W_{12} = \int_{r_1}^{r_2} \frac{G M m}{r^2} \mathrm{d}r$$

$$= -G M_E m \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \tag{8.24}$$

In place of Eq. (8.21), we can thus associate a potential energy $W(r)$ at a distance r, such that

$$W(r) = -\frac{G M_E m}{r} + W_1 , \tag{8.25}$$

valid for $r > R$,

so that once again $W_{12} = W(r_2) - W(r_1)$. Setting r = infinity in the last equation, we get W (r = infinity) = W_1 . Thus, W_1 is the potential energy at infinity. One should note that only the difference of potential energy between two points has a definite meaning from Eqs. (8.22) and (8.24). One conventionally sets W_1 equal to zero, so that the potential energy at a point is just the amount of work done in displacing the particle from infinity to that point.

We have calculated the potential energy at a point of a particle due to gravitational forces on it due to the earth and it is proportional to the mass of the particle. The gravitational potential due to the gravitational force of the earth is defined as the potential energy of a particle of unit mass at that point. From the earlier discussion, we learn that the gravitational potential energy associated with two particles of masses m_1 and m_2 separated by distance by a distance r is given by

$$V = -\frac{G m_1 m_2}{r} \quad \text{(if we choose } V = 0 \text{ as } r \to \infty)$$

It should be noted that an isolated system of particles will have the total potential energy that equals the sum of energies (given by the above equation) for all possible pairs of its constituent particles. This is an example of the application of the superposition principle.

▶ ***Example 8.3*** Find the potential energy of a system of four particles placed at the vertices of a square of side l. Also obtain the potential at the centre of the square.

Answer Consider four masses each of mass m at the corners of a square of side l; See Fig. 8.9. We have four mass pairs at distance l and two diagonal pairs at distance $\sqrt{2}\, l$

Hence,

$$W(r) = -4 \frac{G m^2}{l} - 2 \frac{G m^2}{\sqrt{2}\, l}$$

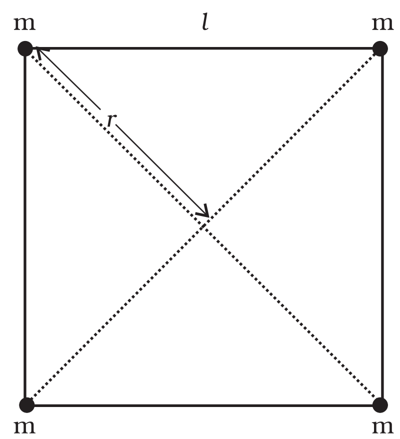


Fig. 8.9

$$= -\frac{2\,G\,m^2}{l}\left(2+\frac{1}{\sqrt{2}}\right) = -5.41\frac{G\,m^2}{l}$$

The gravitational potential at the centre of the square $\left(r = \sqrt{2}\,l/2\right)$ is

$$U(r) = -4\sqrt{2}\,\frac{G\,m}{l}.$$

8.8 ESCAPE SPEED

If a stone is thrown by hand, we see it falls back to the earth. Of course using machines we can shoot an object with much greater speeds and with greater and greater initial speed, the object scales higher and higher heights. A natural query that arises in our mind is the following: 'can we throw an object with such high initial speeds that it does not fall back to the earth?'

The principle of conservation of energy helps us to answer this question. Suppose the object did reach infinity and that its speed there was V_f. The energy of an object is the sum of potential and kinetic energy. As before W_1 denotes that gravitational potential energy of the object at infinity. The total energy of the projectile at infinity then is

$$E\,(\infty) = W_1 + \frac{mV_f^2}{2} \qquad (8.26)$$

If the object was thrown initially with a speed V_i from a point at a distance $(h+R_E)$ from the center of the earth (R_E = radius of the earth), its energy initially was

$$E\,(h+R_E) = \frac{1}{2}mV_i^2 - \frac{GmM_E}{(h+R_E)} + W_1 \qquad (8.27)$$

By the principle of energy conservation Eqs. (8.26) and (8.27) must be equal. Hence

$$\frac{mV_i^2}{2} - \frac{GmM_E}{(h+R_E)} = \frac{mV_f^2}{2} \qquad (8.28)$$

The R.H.S. is a positive quantity with a minimum value zero hence so must be the L.H.S. Thus, an object can reach infinity as long as V_i is such that

$$\frac{mV_i^2}{2} - \frac{GmM_E}{(h+R_E)} \geq 0 \qquad (8.29)$$

The minimum value of V_i corresponds to the case when the L.H.S. of Eq. (8.29) equals zero. Thus, the minimum speed required for an object to reach infinity (i.e. escape from the earth) corresponds to

$$\frac{1}{2}m\left(V_i^2\right)_{min} = \frac{GmM_E}{h+R_E} \qquad (8.30)$$

If the object is thrown from the surface of the earth, h=0, and we get

$$\left(V_i\right)_{min} = \sqrt{\frac{2GM_E}{R_E}} \qquad (8.31)$$

Using the relation $g = GM_E/R_E^2$, we get

$$\left(V_i\right)_{min} = \sqrt{2gR_E} \qquad (8.32)$$

Using the value of g and R_E, numerically $(V_i)_{min} \approx$ 11.2 km/s. This is called the escape speed, sometimes loosely called the escape velocity.

Equation (8.32) applies equally well to an object thrown from the surface of the moon with g replaced by the acceleration due to Moon's gravity on its surface and r_E replaced by the radius of the moon. Both are smaller than their values on earth and the escape speed for the moon turns out to be 2.3 km/s, about five times smaller. This is the reason that moon has no atmosphere. Gas molecules if formed on the surface of the moon having velocities larger than this will escape the gravitational pull of the moon.

Example 8.4 Two uniform solid spheres of equal radii *R*, but mass *M* and 4 *M* have a center to centre separation 6 *R*, as shown in Fig. 8.10. The two spheres are held fixed. A projectile of mass *m* is projected from the surface of the sphere of mass *M* directly towards the centre of the second sphere. Obtain an expression for the minimum speed v of the projectile so that it reaches the surface of the second sphere.

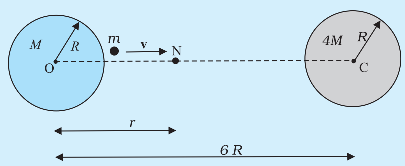


Fig. 8.10

Answer The projectile is acted upon by two mutually opposing gravitational forces of the two

spheres. The neutral point N (see Fig. 8.10) is defined as the position where the two forces cancel each other exactly. If ON = r, we have

$$\frac{G\,M\,m}{r^2} = \frac{4\,G\,M\,m}{(6R-r)^2}$$

$(6R - r)^2 = 4r^2$

$6R - r = \pm 2r$

$r = 2R$ or $-6R$.

The neutral point $r = -6R$ does not concern us in this example. Thus ON = r = $2R$. It is sufficient to project the particle with a speed which would enable it to reach N. Thereafter, the greater gravitational pull of $4M$ would suffice. The mechanical energy at the surface of M is

$$E_i = \frac{1}{2}m\,v^2 - \frac{G\,M\,m}{R} - \frac{4\,G\,M\,m}{5\,R}.$$

At the neutral point N, the speed approaches zero. The mechanical energy at N is purely potential.

$$E_N = -\frac{G\,M\,m}{2\,R} - \frac{4\,G\,M\,m}{4\,R}.$$

From the principle of conservation of mechanical energy

$$\frac{1}{2}v^2 - \frac{GM}{R} - \frac{4GM}{5R} = -\frac{GM}{2R} - \frac{GM}{R}$$

or

$$v^2 = \frac{2\,G\,M}{R}\left(\frac{4}{5} - \frac{1}{2}\right)$$

$$v = \left(\frac{3\,G\,M}{5\,R}\right)^{1/2}$$ ◀

A point to note is that the speed of the projectile is zero at N, but is nonzero when it strikes the heavier sphere 4 M. The calculation of this speed is left as an exercise to the students.

8.9 EARTH SATELLITES

Earth satellites are objects which revolve around the earth. Their motion is very similar to the motion of planets around the Sun and hence Kepler's laws of planetary motion are equally applicable to them. In particular, their orbits around the earth are circular or elliptic. Moon is the only natural satellite of the earth with a near circular orbit with a time period of approximately 27.3 days which is also roughly equal to the rotational period of the moon about its own axis. Since, 1957, advances in technology have enabled many countries including India to launch artificial earth satellites for practical use in fields like telecommunication, geophysics and meteorology.

We will consider a satellite in a circular orbit of a distance $(R_E + h)$ from the centre of the earth, where R_E = radius of the earth. If m is the mass of the satellite and V its speed, the centripetal force required for this orbit is

$$\text{F(centripetal)} = \frac{mV^2}{(R_E + h)} \qquad (8.33)$$

directed towards the center. This centripetal force is provided by the gravitational force, which is

$$\text{F(gravitation)} = \frac{G\,m\,M_E}{(R_E + h)^2} \qquad (8.34)$$

Where M_E is the mass of the earth.

Equating R.H.S of Eqs. (8.33) and (8.34) and cancelling out m, we get

$$V^2 = \frac{G\,M_E}{(R_E + h)} \qquad (8.35)$$

Thus V decreases as h increases. From equation (8.35),the speed V for $h = 0$ is

$$V^2\ (h = 0) = GM/R_E = gR_E \qquad (8.36)$$

where we have used the relation $g = GM/R_E^2$. In every orbit, the satellite traverses a distance $2\pi(R_E + h)$ with speed V. It's time period T therefore is

$$T = \frac{2\pi(R_E + h)}{V} = \frac{2\pi(R_E + h)^{3/2}}{\sqrt{G\,M_E}} \qquad (8.37)$$

on substitution of value of V from Eq. (8.35). Squaring both sides of Eq. (8.37), we get

$$T^2 = k\ (R_E + h)^3 \quad (\text{where } k = 4\,\pi^2 / GM_E) \qquad (8.38)$$

which is Kepler's law of periods, as applied to motion of satellites around the earth. For a satellite very close to the surface of earth h can be neglected in comparison to R_E in Eq. (8.38). Hence, for such satellites, T is T_o, where

$$T_0 = 2\pi\sqrt{R_E / g} \qquad (8.39)$$

If we substitute the numerical values $g \simeq 9.8$ m s^{-2} and R_E = 6400 km., we get

$$T_0 = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}}\ \text{s}$$

Which is approximately 85 minutes.

▶ **Example 8.5** The planet Mars has two moons, phobos and delmos. (i) phobos has a period 7 hours, 39 minutes and an orbital radius of 9.4×10^3 km. Calculate the mass of mars. (ii) Assume that earth and mars move in circular orbits around the sun, with the martian orbit being 1.52 times the orbital radius of the earth. What is the length of the martian year in days ?

Answer (i) We employ Eq. (8.38) with the sun's mass replaced by the martian mass M_m

$$T^2 = \frac{4\pi^2}{GM_m} R^3$$

$$M_m = \frac{4\pi^2}{G} \frac{R^3}{T^2}$$

$$= \frac{4 \times (3.14)^2 \times (9.4)^3 \times 10^{18}}{6.67 \times 10^{-11} \times (459 \times 60)^2}$$

$$M_m = \frac{4 \times (3.14)^2 \times (9.4)^3 \times 10^{18}}{6.67 \times (4.59 \times 6)^2 \times 10^{-5}}$$

$= 6.48 \times 10^{23}$ kg.

(ii) Once again Kepler's third law comes to our aid,

$$\frac{T_M^2}{T_E^2} = \frac{R_{MS}^3}{R_{ES}^3}$$

where R_{MS} is the mars -sun distance and R_{ES} is the earth-sun distance.

$\therefore T_M = (1.52)^{3/2} \times 365$

$= 684$ days

We note that the orbits of all planets except Mercury, Mars and Pluto are very close to being circular. For example, the ratio of the semi-minor to semi-major axis for our Earth is, $b/a = 0.99986$. ◀

▶ **Example 8.6** **Weighing the Earth** : You are given the following data: $g = 9.81$ ms^{-2}, $R_E = 6.37 \times 10^6$ m, the distance to the moon $R = 3.84 \times 10^8$ m and the time period of the moon's revolution is 27.3 days. Obtain the mass of the Earth M_E in two different ways.

Answer From Eq. (8.12) we have

$$M_E = \frac{g R_E^2}{G}$$

$$= \frac{9.81 \times \left(6.37 \times 10^6\right)^2}{6.67 \times 10^{-11}}$$

$= 5.97 \times 10^{24}$ kg.

The moon is a satellite of the Earth. From the derivation of Kepler's third law [see Eq. (8.38)]

$$T^2 = \frac{4\pi^2 R^3}{G M_E}$$

$$M_E = \frac{4\pi^2 R^3}{G T^2}$$

$$= \frac{4 \times 3.14 \times 3.14 \times (3.84)^3 \times 10^{24}}{6.67 \times 10^{-11} \times (27.3 \times 24 \times 60 \times 60)^2}$$

$= 6.02 \times 10^{24}$ kg

Both methods yield almost the same answer, the difference between them being less than 1%. ◀

▶ **Example 8.7** Express the constant k of Eq. (8.38) in days and kilometres. Given $k = 10^{-13}$ s^2 m^{-3}. The moon is at a distance of 3.84×10^5 km from the earth. Obtain its time-period of revolution in days.

Answer Given

$k = 10^{-13}$ s^2 m^{-3}

$$= 10^{-13}\left[\frac{1}{(24 \times 60 \times 60)^2} \text{d}^2\right]\left[\frac{1}{(1/1000)^3 \text{km}^3}\right]$$

$= 1.33 \times 10^{-14}$ d^2 km^{-3}

Using Eq. (8.38) and the given value of k, the time period of the moon is

$T^2 = (1.33 \times 10^{-14})(3.84 \times 10^5)^3$

$T = 27.3$ d ◀

Note that Eq. (8.38) also holds for elliptical orbits if we replace (R_E+h) by the semi-major axis of the ellipse. The earth will then be at one of the foci of this ellipse.

8.10 ENERGY OF AN ORBITING SATELLITE

Using Eq. (8.35), the kinetic energy of the satellite in a circular orbit with speed v is

$$KgE = \frac{1}{2} m v^2$$

$$= \frac{Gm M_E}{2(R_E + h)}, \quad (8.40)$$

Considering gravitational potential energy at infinity to be zero, the potential energy at distance (R_e+h) from the center of the earth is

$$P.E = -\frac{G\,m\,M_E}{(R_E + h)} \qquad (8.41)$$

The K.E is positive whereas the P.E is negative. However, in magnitude the K.E. is half the P.E, so that the total E is

$$E = K.E + P.E = -\frac{G\,m\,M_E}{2(R_E + h)} \qquad (8.42)$$

The total energy of an circularly orbiting satellite is thus negative, with the potential energy being negative but twice is magnitude of the positive kinetic energy.

When the orbit of a satellite becomes elliptic, both the *K.E.* and *P.E.* vary from point to point. The total energy which remains constant is negative as in the circular orbit case. This is what we expect, since as we have discussed before if the total energy is positive or zero, the object escapes to infinity. Satellites are always at finite distance from the earth and hence their energies cannot be positive or zero.

Example 8.8 A 400 kg satellite is in a circular orbit of radius $2R_E$ about the Earth. How much energy is required to transfer it to a circular orbit of radius $4R_E$? What are the changes in the kinetic and potential energies ?

Answer Initially,

$$E_i = -\frac{G\,M_E\,m}{4\,R_E}$$

While finally

$$E_f = -\frac{G\,M_E\,m}{8\,R_E}$$

The change in the total energy is

$\Delta E = E_f - E_i$

$$= \frac{G\,M_E\,m}{8\,R_E} = \left(\frac{G\,M_E}{R_E^2}\right)\frac{m\,R_E}{8}$$

$$\Delta E = \frac{g\,m\,R_E}{8} = \frac{9.81 \times 400 \times 6.37 \times 10^6}{8} = 3.13 \times 10^9\,\text{J}$$

The kinetic energy is reduced and it mimics ΔE, namely, $\Delta K = K_f - K_i = -3.13 \times 10^9$ J.

The change in potential energy is twice the change in the total energy, namely

$\Delta V = V_f - V_i = -6.25 \times 10^9$ J

8.11 GEOSTATIONARY AND POLAR SATELLITES

An interesting phenomenon arises if in we arrange the value of ($R_E + h$) such that *T* in Eq. (8.37) becomes equal to 24 hours. If the circular orbit is in the equatorial plane of the earth, such a satellite, having the same period as the period of rotation of the earth about its own axis would appear stationery viewed from a point on earth. The ($R_E + h$) for this purpose works out to be large as compared to R_E :

$$R_E + h = \left(\frac{T^2\,G\,M_E}{4\pi^2}\right)^{1/3} \qquad (8.43)$$

and for *T* = 24 hours, *h* works out to be 35800 km. which is much larger than R_E. Satellites in a circular orbits around the earth in the

India's Leap into Space

India entered the space age with the launching of the low orbit satellite Aryabhatta in 1975. In the first few years of its programme the launch vehicles were provided by the erstwhile Soviet Union. Indigenous launch vehicles were employed in the early 1980's to send the Rohini series of satellites into space. The programme to send polar satellites into space began in late 1980's. A series of satellites labelled IRS (Indian Remote Sensing Satellites) have been launched and this programme is expected to continue in future. The satellites have been employed for surveying, weather prediction and for carrying out experiments in space. The INSAT (Indian National Satellite) series of satellites were designed and made operational for communications and weather prediction purposes beginning in 1982. European launch vehicles have been employed in the INSAT series. India tested its geostationary launch capability in 2001 when it sent an experimental communications satellite (GSAT-1) into space. In 1984 Rakesh Sharma became the first Indian astronaut. The Indian Space Research Organisation (ISRO) is the umbrella organisation that runs a number of centre. Its main lauch centre at Sriharikota (SHAR) is 100 km north of Chennai. The National Remote Sensing Agency (NRSA) is near Hyderabad. Its national centre for research in space and allied sciences is the Physical Research Laboratory (PRL) at Ahmedabad.

equatorial plane with T = 24 hours are called Geostationery Satellites. Clearly, since the earth rotates with the same period, the satellite would appear fixed from any point on earth. It takes very powerful rockets to throw up a satellite to such large heights above the earth but this has been done in view of the several benefits of many practical applications.

It is known that electromagnetic waves above a certain frequency are not reflected from ionosphere. Radio waves used for radio broadcast which are in the frequency range 2 MHz to 10 MHz, are below the critical frequency. They are therefore reflected by the ionosphere. Thus radio waves broadcast from an antenna can be received at points far away where the direct wave fail to reach on account of the curvature of the earth. Waves used in television broadcast or other forms of communication have much higher frequencies and thus cannot be received beyond the line of sight. A Geostationery satellite, appearing fixed above the broadcasting station can however receive these signals and broadcast them back to a wide area on earth. The INSAT group of satellites sent up by India are one such group of Geostationary satellites widely used for telecommunications in India.

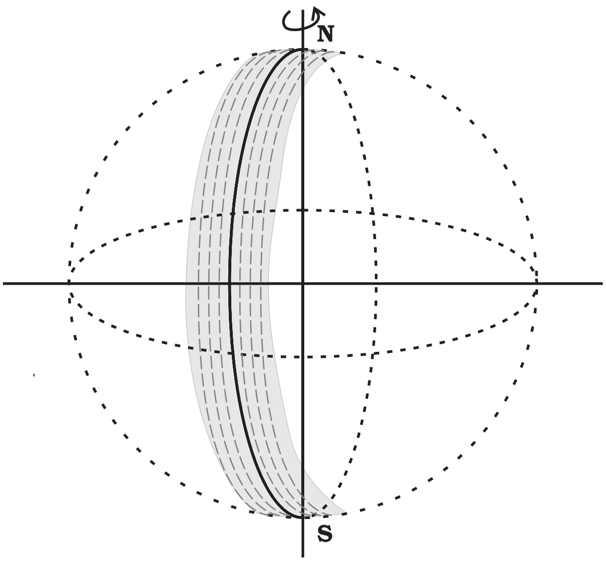


Fig. 8.11 *A Polar satellite. A strip on earth's surface (shown shaded) is visible from the satellite during one cycle. For the next revolution of the satellite, the earth has rotated a little on its axis so that an adjacent strip becomes visible.*

Another class of satellites are called the Polar satellites (Fig. 8.11). These are low altitude ($h \approx$ 500 to 800 km) satellites, but they go around the poles of the earth in a north-south direction whereas the earth rotates around its axis in an east-west direction. Since its time period is around 100 minutes it crosses any altitude many times a day. However, since its height h above the earth is about 500-800 km, a camera fixed on it can view only small strips of the earth in one orbit. Adjacent strips are viewed in the next orbit, so that in effect the whole earth can be viewed strip by strip during the entire day. These satellites can view polar and equatorial regions at close distances with good resolution. Information gathered from such satellites is extremely useful for remote sensing, meterology as well as for environmental studies of the earth.

8.12 WEIGHTLESSNESS

Weight of an object is the force with which the earth attracts it. We are conscious of our own weight when we stand on a surface, since the surface exerts a force opposite to our weight to keep us at rest. The same principle holds good when we measure the weight of an object by a spring balance hung from a fixed point e.g. the ceiling. The object would fall down unless it is subject to a force opposite to gravity. This is exactly what the spring exerts on the object. This is because the spring is pulled down a little by the gravitational pull of the object and in turn the spring exerts a force on the object vertically upwards.

Now, imagine that the top end of the balance is no longer held fixed to the top ceiling of the room. Both ends of the spring as well as the object move with identical acceleration g. The spring is not stretched and does not exert any upward force on the object which is moving down with acceleration g due to gravity. The reading recorded in the spring balance is zero since the spring is not stretched at all. If the object were a human being, he or she will not feel his weight since there is no upward force on him. Thus, when an object is in free fall, it is weightless and this phenomenon is usually called the phenomenon of weightlessness.

In a satellite around the earth, every part and parcel of the satellite has an acceleration towards the center of the earth which is exactly

the value of earth's acceleration due to gravity at that position. Thus in the satellite everything inside it is in a state of free fall. This is just as if we were falling towards the earth from a height. Thus, in a manned satellite, people inside experience no gravity. Gravity for us defines the vertical direction and thus for them there are no horizontal or vertical directions, all directions are the same. Pictures of astronauts floating in a satellite reflect show this fact.

SUMMARY

1. Newton's law of universal gravitation states that the gravitational force of attraction between any two particles of masses m_1 and m_2 separated by a distance r has the magnitude

$$F = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant, which has the value 6.672×10^{-11} N m^2 kg^{-2}.

2. If we have to find the resultant gravitational force acting on the particle m due to a number of masses M_1, M_2,….M_n etc. we use the principle of superposition. Let F_1, F_2,….F_n be the individual forces due to M_1, M_2,….M_n, each given by the law of gravitation. From the principle of superposition each force acts independently and uninfluenced by the other bodies. The resultant force F_R is then found by vector addition

$$F_R = F_1 + F_2 + \ldots\ldots + F_n = \sum_{i=1}^{n} \mathbf{F}_i$$

where the symbol 'Σ' stands for summation.

3. Kepler's laws of planetary motion state that
   (a) All planets move in elliptical orbits with the Sun at one of the focal points
   (b) The radius vector drawn from the sun to a planet sweeps out equal areas in equal time intervals. This follows from the fact that the force of gravitation on the planet is central and hence angular momentum is conserved.
   (c) The square of the orbital period of a planet is proportional to the cube of the semi-major axis of the elliptical orbit of the planet

   The period T and radius R of the circular orbit of a planet about the Sun are related by

$$T^2 = \left(\frac{4\pi^2}{G\,M_s}\right)R^3$$

   where M_s is the mass of the Sun. Most planets have nearly circular orbits about the Sun. For elliptical orbits, the above equation is valid if R is replaced by the semi-major axis, a.

4. The acceleration due to gravity.
   (a) at a height h above the Earth's surface

$$g(h) = \frac{G\,M_E}{(R_E + h)^2}$$

$$\approx \frac{G\,M_E}{R_E^2}\left(1 - \frac{2h}{R_E}\right) \quad \text{for } h << R_E$$

$$g(h) = g(0)\left(1 - \frac{2h}{R_E}\right) \quad \text{where } g(0) = \frac{G\,M_E}{R_E^2}$$

(b) at depth d below the Earth's surface is

$$g(d) = \frac{G\,M_E}{R_E^2}\left(1 - \frac{d}{R_E}\right) = g(0)\left(1 - \frac{d}{R_E}\right)$$

5. The gravitational force is a conservative force, and therefore a potential energy function can be defined. The *gravitational potential energy* associated with two particles separated by a distance r is given by

$$V = -\frac{G\,m_1\,m_2}{r}$$

where V is taken to be zero at $r \to \infty$. The total potential energy for a system of particles is the sum of energies for all pairs of particles, with each pair represented by a term of the form given by above equation. This prescription follows from the principle of superposition.

6. If an isolated system consists of a particle of mass m moving with a speed v in the vicinity of a massive body of mass M, the total mechanical energy of the particle is given by

$$E = \frac{1}{2}m\,v^2 - \frac{G\,M\,m}{r}$$

That is, the total mechanical energy is the sum of the kinetic and potential energies. The total energy is a constant of motion.

7. If m moves in a circular orbit of radius a about M, where $M >> m$, the total energy of the system is

$$E = -\frac{G\,M\,m}{2a}$$

with the choice of the arbitrary constant in the potential energy given in the point 5., above. The total energy is negative for any bound system, that is, one in which the orbit is closed, such as an elliptical orbit. The kinetic and potential energies are

$$K = \frac{G\,M\,m}{2a}$$

$$V = -\frac{G\,M\,m}{a}$$

8. The escape speed from the surface of the Earth is

$$v_e = \sqrt{\frac{2\,G\,M_E}{R_E}} = \sqrt{2gR_E}$$

and has a value of 11.2 km s^{-1}.

9. If a particle is outside a uniform spherical shell or solid sphere with a spherically symmetric internal mass distribution, the sphere attracts the particle as though the mass of the sphere or shell were concentrated at the centre of the sphere.

10. If a particle is inside a uniform spherical shell, the gravitational force on the particle is zero. If a particle is inside a homogeneous solid sphere, the force on the particle acts toward the centre of the sphere. This force is exerted by the spherical mass interior to the particle.

11. A geostationary (geosynchronous communication) satellite moves in a circular orbit in the equatorial plane at a approximate distance of 4.22×10^4 km from the Earth's centre.

Physical Quantity	Symbol	Dimensions	Unit	Remarks
Gravitational Constant	G	$[M^{-1} L^3 T^{-2}]$	$N\ m^2\ kg^{-2}$	6.67×10^{-11}
Gravitational Potential Energy	$V(\mathbf{r})$	$[M\ L^2 T^{-2}]$	J	$-\frac{GMm}{r}$ (scalar)
Gravitational Potential	$U(\mathbf{r})$	$[L^2 T^{-2}]$	$J\ kg^{-1}$	$-\frac{GM}{r}$ (scalar)
Gravitational Intensity	**E** or **g**	$[LT^{-2}]$	$m\ s^{-2}$	$\frac{GM}{r^2}\hat{\mathbf{r}}$ (vector)

POINTS TO PONDER

1. In considering motion of an object under the gravitational influence of another object the following quantities are conserved:
   (a) Angular momentum
   (b) Total mechanical energy
   Linear momentum is **not** conserved
2. Angular momentum conservation leads to Kepler's second law. However, it is not special to the inverse square law of gravitation. It holds for any central force.
3. In Kepler's third law (see Eq. (8.1) and $T^2 = K_S R^3$. The constant K_S is the same for all planets in circular orbits. This applies to satellites orbiting the Earth [(Eq. (8.38)].
4. An astronaut experiences weightlessness in a space satellite. This is not because the gravitational force is small at that location in space. It is because both the astronaut and the satellite are in "free fall" towards the Earth.
5. The *gravitational potential energy* associated with two particles separated by a distance r is given by
   $$V = -\frac{G m_1 m_2}{r} + \text{constant}$$
   The constant can be given any value. The simplest choice is to take it to be zero. With this choice
   $$V = -\frac{G m_1 m_2}{r}$$
   This choice implies that $V \to 0$ as $r \to \infty$. Choosing location of zero of the gravitational energy is the same as choosing the arbitrary constant in the potential energy. Note that the gravitational force is not altered by the choice of this constant.
6. The total mechanical energy of an object is the sum of its kinetic energy (which is always positive) and the potential energy. Relative to infinity (i.e. if we presume that the potential energy of the object at infinity is zero), the gravitational potential energy of an object is negative. The total energy of a satellite is negative.
7. The commonly encountered expression mgh for the potential energy is actually an approximation to the difference in the gravitational potential energy discussed in the point 6, above.
8. Although the gravitational force between two particles is central, the force between two finite rigid bodies is not necessarily along the line joining their centre of mass. For a spherically symmetric body however the force on a particle external to the body is as if the mass is concentrated at the centre and this force is therefore central.
9. The gravitational force on a particle inside a spherical shell is zero. However, (unlike a metallic shell which shields electrical forces) the shell does not shield other bodies outside it from exerting gravitational forces on a particle inside. *Gravitational shielding is not possible.*

EXERCISES

8.1 Answer the following :

(a) You can shield a charge from electrical forces by putting it inside a hollow conductor. Can you shield a body from the gravitational influence of nearby matter by putting it inside a hollow sphere or by some other means ?

(b) An astronaut inside a small space ship orbiting around the earth cannot detect gravity. If the space station orbiting around the earth has a large size, can he hope to detect gravity ?

(c) If you compare the gravitational force on the earth due to the sun to that due to the moon, you would find that the Sun's pull is greater than the moon's pull. (you can check this yourself using the data available in the succeeding exercises). However, the tidal effect of the moon's pull is greater than the tidal effect of sun. Why ?

8.2 Choose the correct alternative :

(a) Acceleration due to gravity increases/decreases with increasing altitude.

(b) Acceleration due to gravity increases/decreases with increasing depth (assume the earth to be a sphere of uniform density).

(c) Acceleration due to gravity is independent of mass of the earth/mass of the body.

(d) The formula $-G\,Mm(1/r_2 - 1/r_1)$ is more/less accurate than the formula $mg(r_2 - r_1)$ for the difference of potential energy between two points r_2 and r_1 distance away from the centre of the earth.

8.3 Suppose there existed a planet that went around the sun twice as fast as the earth. What would be its orbital size as compared to that of the earth ?

8.4 Io, one of the satellites of Jupiter, has an orbital period of 1.769 days and the radius of the orbit is 4.22×10^8 m. Show that the mass of Jupiter is about one-thousandth that of the sun.

8.5 Let us assume that our galaxy consists of 2.5×10^{11} stars each of one solar mass. How long will a star at a distance of 50,000 ly from the galactic centre take to complete one revolution ? Take the diameter of the Milky Way to be 10^5 ly.

8.6 Choose the correct alternative:

(a) If the zero of potential energy is at infinity, the total energy of an orbiting satellite is negative of its kinetic/potential energy.

(b) The energy required to launch an orbiting satellite out of earth's gravitational influence is more/less than the energy required to project a stationary object at the same height (as the satellite) out of earth's influence.

8.7 Does the escape speed of a body from the earth depend on (a) the mass of the body, (b) the location from where it is projected, (c) the direction of projection, (d) the height of the location from where the body is launched?

8.8 A comet orbits the sun in a highly elliptical orbit. Does the comet have a constant (a) linear speed, (b) angular speed, (c) angular momentum, (d) kinetic energy, (e) potential energy, (f) total energy throughout its orbit? Neglect any mass loss of the comet when it comes very close to the Sun.

8.9 Which of the following symptoms is likely to afflict an astronaut in space (a) swollen feet, (b) swollen face, (c) headache, (d) orientational problem.

In the following two exercises, choose the correct answer from among the given ones:

8.10 The gravitational intensity at the centre of a hemispherical shell of uniform mass density has the direction indicated by the arrow (see Fig 8.12) (i) a, (ii) b, (iii) c, (iv) 0.

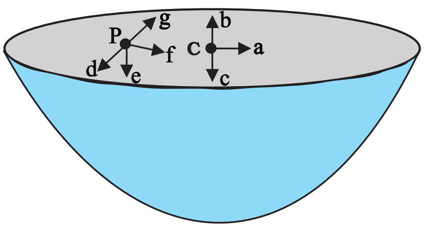


Fig. 8.12

8.11 For the above problem, the direction of the gravitational intensity at an arbitrary point P is indicated by the arrow (i) d, (ii) e, (iii) f, (iv) g.

8.12 A rocket is fired from the earth towards the sun. At what distance from the earth's centre is the gravitational force on the rocket zero ? Mass of the sun = 2×10^{30} kg, mass of the earth = 6×10^{24} kg. Neglect the effect of other planets etc. (orbital radius = 1.5×10^{11} m).

8.13 How will you 'weigh the sun', that is estimate its mass? The mean orbital radius of the earth around the sun is 1.5×10^{8} km.

8.14 A saturn year is 29.5 times the earth year. How far is the saturn from the sun if the earth is 1.50×10^{8} km away from the sun ?

8.15 A body weighs 63 N on the surface of the earth. What is the gravitational force on it due to the earth at a height equal to half the radius of the earth ?

8.16 Assuming the earth to be a sphere of uniform mass density, how much would a body weigh half way down to the centre of the earth if it weighed 250 N on the surface ?

8.17 A rocket is fired vertically with a speed of 5 km s^{-1} from the earth's surface. How far from the earth does the rocket go before returning to the earth ? Mass of the earth = 6.0×10^{24} kg; mean radius of the earth = 6.4×10^{6} m; $G = 6.67\times10^{-11}$ N m^2 kg^{-2}.

8.18 The escape speed of a projectile on the earth's surface is 11.2 km s^{-1}. A body is projected out with thrice this speed. What is the speed of the body far away from the earth? Ignore the presence of the sun and other planets.

8.19 A satellite orbits the earth at a height of 400 km above the surface. How much energy must be expended to rocket the satellite out of the earth's gravitational influence? Mass of the satellite = 200 kg; mass of the earth = 6.0×10^{24} kg; radius of the earth = 6.4×10^{6} m; $G = 6.67\times10^{-11}$ N m^2 kg^{-2}.

8.20 Two stars each of one solar mass (= 2×10^{30} kg) are approaching each other for a head on collision. When they are a distance 10^{9} km, their speeds are negligible. What is the speed with which they collide ? The radius of each star is 10^{4} km. Assume the stars to remain undistorted until they collide. (Use the known value of G).

8.21 Two heavy spheres each of mass 100 kg and radius 0.10 m are placed 1.0 m apart on a horizontal table. What is the gravitational force and potential at the mid point of the line joining the centres of the spheres ? Is an object placed at that point in equilibrium? If so, is the equilibrium stable or unstable ?

Additional Exercises

8.22 As you have learnt in the text, a geostationary satellite orbits the earth at a height of nearly 36,000 km from the surface of the earth. What is the potential due to earth's gravity at the site of this satellite ? (Take the potential energy at infinity to be zero). Mass of the earth = 6.0×10^{24} kg, radius = 6400 km.

8.23 A star 2.5 times the mass of the sun and collapsed to a size of 12 km rotates with a speed of 1.2 rev. per second. (Extremely compact stars of this kind are known as neutron stars. Certain stellar objects called pulsars belong to this category). Will an object placed on its equator remain stuck to its surface due to gravity ? (mass of the sun = 2×10^{30} kg).

8.24 A spaceship is stationed on Mars. How much energy must be expended on the spaceship to launch it out of the solar system ? Mass of the space ship = 1000 kg; mass of the sun = 2×10^{30} kg; mass of mars = 6.4×10^{23} kg; radius of mars = 3395 km; radius of the orbit of mars = 2.28×10^{8} km; $G = 6.67\times10^{-11}$ N m^2 kg^{-2}.

8.25 A rocket is fired 'vertically' from the surface of mars with a speed of 2 km s^{-1}. If 20% of its initial energy is lost due to martian atmospheric resistance, how far will the rocket go from the surface of mars before returning to it ? Mass of mars = 6.4×10^{23} kg; radius of mars = 3395 km; $G = 6.67\times10^{-11}$ N m^2 kg^{-2}.

APPENDICES

APPENDIX A 1
THE GREEK ALPHABET

Alpha	A	α	Iota	I	ι	Rho	P	ρ
Beta	B	β	Kappa	K	κ	Sigma	Σ	σ
Gamma	Γ	γ	Lambda	Λ	λ	Tau	T	τ
Delta	Δ	δ	Mu	M	μ	Upsilon	ϒ	υ
Epsilon	E	ε	Nu	N	ν	Phi	Φ	ϕ, φ
Zeta	Z	ς	Xi	Ξ	ξ	Chi	X	χ
Eta	H	η	Omicron	O	ο	Psi	Ψ	ψ
Theta	Θ	θ	Pi	Π	π	Omega	Ω	ω

APPENDIX A 2
COMMON SI PREFIXES AND SYMBOLS FOR MULTIPLES AND SUB-MULTIPLES

Multiple			Sub-Multiple		
Factor	**Prefix**	**Symbol**	**Factor**	**Prefix**	**symbol**
10^{18}	Exa	E	10^{-18}	atto	a
10^{15}	Peta	P	10^{-15}	femto	f
10^{12}	Tera	T	10^{-12}	pico	p
10^{9}	Giga	G	10^{-9}	nano	n
10^{6}	Mega	M	10^{-6}	micro	μ
10^{3}	kilo	k	10^{-3}	milli	m
10^{2}	Hecto	h	10^{-2}	centi	c
10^{1}	Deca	da	10^{-1}	deci	d

APPENDIX A 3
SOME IMPORTANT CONSTANTS

Name	Symbol	Value
Speed of light in vacuum	c	2.9979×10^{8} m s^{-1}
Charge of electron	e	1.602×10^{-19} C
Gravitational constant	G	6.673×10^{-11} N m^{2} kg^{-2}
Planck constant	h	6.626×10^{-34} J s
Boltzmann constant	k	1.381×10^{-23} J K^{-1}
Avogadro number	N_A	6.022×10^{23} mol^{-1}
Universal gas constant	R	8.314 J mol^{-1}K^{-1}
Mass of electron	m_e	9.110×10^{-31} kg
Mass of neutron	m_n	1.675×10^{-27} kg
Mass of proton	m_p	1.673×10^{-27} kg
Electron-charge to mass ratio	e/m_e	1.759×10^{11} C/kg
Faraday constant	F	9.648×10^{4} C/mol
Rydberg constant	R	1.097×10^{7} m^{-1}
Bohr radius	a_0	5.292×10^{-11} m
Stefan-Boltzmann constant	σ	5.670×10^{-8} W m^{-2} K^{-4}
Wien's Constant	b	2.898×10^{-3} m K
Permittivity of free space	ε_0 $1/4\pi\,\varepsilon_0$	8.854×10^{-12} C^{2} N^{-1}m^{-2} 8.987×10^{9} N m^{2} C^{-2}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ T m A^{-1} $\cong 1.257 \times 10^{-6}$ Wb A^{-1} m^{-1}

Other useful constants

Name	Symbol	Value
Mechanical equivalent of heat	J	4.186 J cal^{-1}
Standard atmospheric pressure	1 atm	1.013×10^{5}Pa
Absolute zero	0 K	−273.15 °C
Electron volt	1 eV	1.602×10^{-19}J
Unified Atomic mass unit	1 u	1.661×10^{-27} kg
Electron rest energy	mc^2	0.511 MeV
Energy equivalent of 1 u	1 uc^2	931.5 MeV
Volume of ideal gas(0 °C and 1atm)	V	22.4 L mol^{-1}
Acceleration due to gravity (sea level, at equator)	g	9.78049 m s^{-2}

APPENDIX A 4
CONVERSION FACTORS

Conversion factors are written as equations for simplicity.

Length

1 km = 0.6215 mi

1mi = 1.609 km

1m = 1.0936 yd = 3.281 ft = 39.37 in

1 in = 2.54 cm

1 ft = 12 in = 30.48 cm

1 yd = 3ft = 91.44 cm

1 lightyear = 1 ly = 9.461×10^{15}m

1 Å= 0.1nm

Area

1 $m^2 = 10^4 cm^2$

1km^2 = 0.3861 mi^2 = 247.1 acres

1 in^2= 6.4516 cm^2

1ft^2= $9.29 \times 10^{-2} m^2$

1 m^2= 10.76 ft^2

1 acre = 43,560 ft^2

1 mi^2= 460 acres = 2.590 km^2

Volume

1$m^3 = 10^6 cm^3$

1 L = 1000 cm^3 = 10^{-3} m^3

1 gal = 3.786 L

1 gal = 4 qt = 8 pt = 128 oz = 231 in^3

1 in^3 = 16.39 cm^3

1ft^3 = 1728 in^3 = 28.32 L = 2.832×10^4 cm^3

Speed

1 km h^{-1} = 0.2778 m s^{-1} = 0.6215 mi h^{-1}

1mi h^{-1} = 0.4470 m s^{-1} = 1.609 km h^{-1}

1mi h^{-1} = 1.467 ft s^{-1}

Magnetic Field

1 G = 10^{-4} T

1 T = 1 Wb m^{-2} = 10^4 G

Angle and Angular Speed

π rad = 180°

1 rad = 57.30°

1° = 1.745×10^{-2} rad

1 rev min^{-1} = 0.1047 rad s^{-1}

1 rad s^{-1} = 9.549 rev min^{-1}

Mass

1 kg = 1000 g

1 tonne = 1000 kg = 1 Mg

1 u = 1.6606×10^{-27} kg

1 kg = 6.022×10^{26} u

1 slug = 14.59 kg

1 kg = 6.852×10^{-2} slug

1 u = 931.50 MeV/c^2

Density

1 g cm^{-3} = 1000 kg m^{-3} = 1 kg L^{-1}

Force

1 N = 0.2248 lbf = 10^5 dyn

1 lbf = 4.4482 N

1 kgf = 2.2046 lbf

Time

1 h = 60 min = 3.6 ks

1 d = 24 h = 1440 min = 86.4 ks

1y = 365.24 d = 31.56 Ms

Pressure

1 Pa = 1 N m^{-2}

1 bar = 100 kPa

1 atm = 101.325 kPa = 1.01325 bar

1atm = 14.7 lbf/in^2 = 760 mm Hg

= 29.9 in Hg = 33.8 ft H_2O

1 lbf in^{-2} = 6.895 kPa

1 torr = 1mm Hg = 133.32 Pa

Energy

1 kW h = 3.6 MJ

1 cal = 4.186 J

1ft lbf = 1.356 J = 1.286 × 10^{-3} Btu

1 L atm = 101.325 J

1 L atm = 24.217 cal

1 Btu = 778 ft lb = 252 cal = 1054.35 J

1 eV = 1.602 × 10^{-19}J

1 u c^2 = 931.50 MeV

1 erg = 10^{-7}J

Power

1 horsepower (hp) = 550 ft lbf/s

= 745.7 W

1 Btu min^{-1} = 17.58 W

1 W = 1.341 × 10^{-3} hp

= 0.7376 ft lbf/s

Thermal Conductivity

1 W m^{-1} K^{-1} = 6.938 Btu in/hft^2 °F

1 Btu in/hft^2 °F = 0.1441 W/m K

APPENDIX A 5
MATHEMATICAL FORMULAE

Geometry

Circle of radius r: circumference = $2\pi r$;

area = πr^2

Sphere of radius r: area = $4\pi r^2$;

volume = $\frac{4}{3}\pi r^3$

Right circular cylinder of radius r and height h: area = $2\pi r^2 + 2\pi r h$;

volume = $\pi r^2 h$;

Triangle of base a and altitude h.

area = $\frac{1}{2} a h$

Quadratic Formula

If $ax^2 + bx + c = 0$,

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2}$

Trigonometric Functions of Angle θ

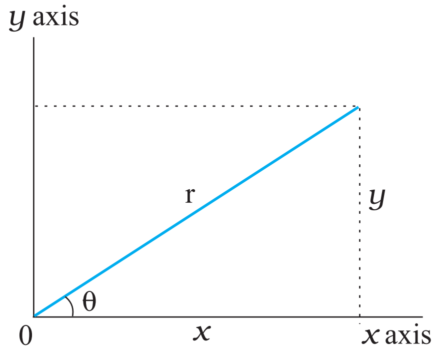


Fig. A 5.1

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

$\sec\theta = \frac{r}{x}$ $\csc\theta = \frac{r}{y}$

Pythagorean Theorem

In this right triangle, $a^2 + b^2 = c^2$

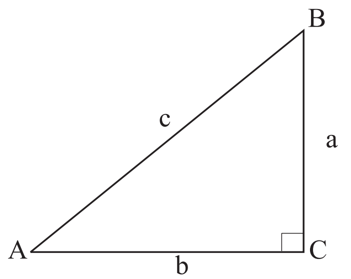


Fig. A 5.2

Triangles

Angles are A, B, C

Opposite sides are a, b, c

Angles $A + B + C = 180^0$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$c^2 = a^2 + b^2 - 2ab \cos C$

Exterior angle $D = A + C$

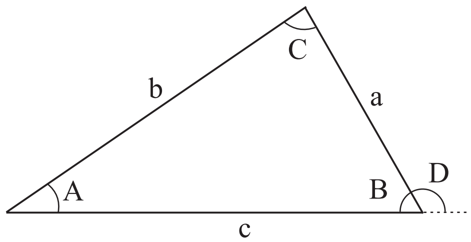


Fig. A 5.3

Mathematical Signs and Symbols

= equals
≅ equals approximately
~ is the order of magnitude of
≠ is not equal to
≡ is identical to, is defined as
> is greater than (>> is much greater than)
< is less than (<< is much less than)
≥ is greater than or equal to (or, is no less than)
≤ is less than or equal to (or, is no more than)
± plus or minus
∝ is proportional to
Σ the sum of
$\bar{x}$ or $<x>$ or x_{av} the average value of x

Trigonometric Identities

$\sin(90^0 - \theta) = \cos\theta$

$\cos(90^0 - \theta) = \sin\theta$

$\sin\theta / \cos\theta = \tan\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta - \tan^2\theta = 1$

$\csc^2\theta - \cot^2\theta = 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \mp \beta)$

$\cos\alpha + \cos\beta$

$= 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

$\cos\alpha - \cos\beta$

$= -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$

Binomial Theorem

$$(1 \pm x)^n = 1 \pm \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + ...(x^2 < 1)$$

$$(1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n+1)x^2}{2!} + ...(x^2 < 1)$$

Exponential Expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Logarithmic Expansion

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - ..... \quad (|x|<1)$$

Trigonometric Expansion
(θ in radians)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - .....$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - .....$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} - .....$$

Products of Vectors

Let $\hat{i}, \hat{j}$ and $\hat{k}$ be unit vectors in the x, y and z directions. Then

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1, \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0, \quad \hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$$

Any vector **a** with components a_x, a_y, and a_z along the x,y, and z axes can be written,

$$\mathbf{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Let **a**, **b** and **c** be arbitary vectors with magnitudes *a*, *b* and *c*. Then

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

$$(s\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (s\mathbf{b}) = s(\mathbf{a} \times \mathbf{b}) \quad \text{(s is a scalar)}$$

Let θ be the smaller of the two angles between **a** and **b**. Then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = a_x b_x + a_y b_y + a_z b_z = ab\cos\theta$$

$$|\mathbf{a} \times \mathbf{b}| = ab\sin\theta$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= (a_y b_z - b_y a_z)\,\hat{\mathbf{i}} + (a_z b_x - b_z a_x)\,\hat{\mathbf{j}} + (a_x b_y - b_x a_y)\,\hat{\mathbf{k}}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$

APPENDIX A 6

SI DERIVED UNITS

A 6.1 Some SI Derived Units expressed in SI Base Units

Physical quantity	SI Unit	
	Name	**Symbol**
Area	square metre	m^2
Volume	cubic metre	m^3
Speed, velocity	metre per second	m/s or $m\ s^{-1}$
Angular velocity	radian per second	rad/s or $rad\ s^{-1}$
Acceleration	metre per second square	m/s^2 or $m\ s^{-2}$
Angular acceleration	radian per second square	rad/s^2 or $rad\ s^{-2}$
Wave number	per metre	m^{-1}
Density, mass density	kilogram per cubic metre	kg/m^3 or $kg\ m^{-3}$
Current density	ampere per square metre	A/m^2 or $A\ m^{-2}$
Magnetic field strength, magnetic intensity, magnetic moment density	ampere per metre	A/m or $A\ m^{-1}$
Concentration (of amount of substance)	mole per cubic metre	mol/m^3 or $mol\ m^{-3}$
Specific volume	cubic metre per kilogram	m^3/kg or $m^3\ kg^{-1}$
Luminance, intensity of illumination	candela per square metre	cd/m^2 or $cd\ m^{-2}$
Kinematic viscosity	square metre per second	m^2/s or $m^2\ s^{-1}$
Momentum	kilogram metre per second	$kg\ m\ s^{-1}$
Moment of inertia	kilogram square metre	$kg\ m^2$
Radius of gyration	metre	m
Linear/superficial/volume expansivities	per kelvin	K^{-1}
Flow rate	cubic metre per second	$m^3\ s^{-1}$

A 6.2 SI Derived Units with special names

Physical quantity	SI Unit			
	Name	Symbol	Expression in terms of other units	Expression in terms of SI base Units
Frequency	hertz	Hz	-	s^{-1}
Force	newton	N	-	kg m s^{-2} or kg m/s^2
Pressure, stress	Pascal	Pa	N/m^2 or N m^{-2}	kg m^{-1} s^{-2} or kg /s^2 m
Energy, work, quantity of heat	joule	J	N m	kg m^2 s^{-2} or kg m^2/s^2
Power, radiant flux	watt	W	J/s or J s^{-1}	kg m^2 s^{-3}or kg m^2/s^3
Quantity of electricity, electric charge	coulomb	C	-	A s
Electric potential, potential difference, electromotive force	volt	V	W/A or W A^{-1}	kg $m^2 s^{-3}$ A^{-1} or kg m^2/s^3 A
Capacitance	farad	F	C/V	A^2 s^4 kg^{-1} m^{-2}
Electric resistance	ohm	Ω	V/A	kg m^2 s^{-3} A^{-2}
Conductance	siemens	S	A/V	m^{-2} kg^{-1} s^3 A^2
Magnetic flux	weber	Wb	V s or J/A	kg m^2 s^{-2} A^{-1}
Magnetic field, magnetic flux density, magnetic induction	tesla	T	Wb/m^2	kg s^{-2} A^{-1}
Inductance	henry	H	Wb/A	kg m^2 s^{-2} A^{-2}
Luminous flux, luminous power	lumen	lm	-	cd /sr
Illuminance	lux	lx	lm/m^2	m^{-2} cd sr^{-1}
Activity (of a radio nuclide/radioactive source)	becquerel	Bq	-	s^{-1}
Absorbed dose, absorbed dose index	gray	Gy	J/kg	m^2/s^2 or m^2 s^{-2}

A 6.3 Some SI Derived Units expressed by means of SI Units with special names

Physical quantity	SI Unit		
	Name	Symbol	Expression in terms of SI base units
Magnetic moment	joule per tesla	J T^{-1}	m^2 A
Dipole moment	coulomb metre	C m	s A m
Dynamic viscosity	poiseiulles or pascal second or newton second per square metre	Pl or Pa s or N s m^{-2}	m^{-1} kg s^{-1}
Torque, couple, moment of force	newton metre	N m	m^2 kg s^{-2}
Surface tension	newton per metre	N/m	kg s^{-2}
Power density, irradiance, heat flux density	watt per square metre	W/m^2	kg s^{-3}

Heat capacity, entropy	joule per kelvin	J/K	m^2 kg s^{-2} K^{-1}
Specific heat capacity, specific entropy	joule per kilogram kelvin	J/kg K	m^2 s^{-2} K^{-1}
Specific energy, latent heat	joule per kilogram	J/kg	m^2 s^{-2}
Radiant intensity	watt per steradian	W sr^{-1}	kg m^2 s^{-3} sr^{-1}
Thermal conductivity	watt per metre kelvin	W m^{-1} K^{-1}	m kg s^{-3} K^{-1}
Energy density	joule per cubic metre	J/m^3	kg m^{-1} s^{-2}
Electric field strength	volt per metre	V/m	m kg s^3 A^{-1}
Electric charge density	coulomb per cubic metre	C/m^3	m^{-3} A s
Electric flux density	coulomb per square metre	C/m^2	m^{-2} A s
Permittivity	farad per metre	F/m	m^{-3} kg^{-1} s^4 A^2
Permeability	henry per metre	H/m	m kg s^{-2} A^{-2}
Molar energy	joule per mole	J/mol	m^2 kg $s^{-2}$$mol^{-1}$
Angular momentum, Planck's constant	joule second	J s	kg m^2 s^{-1}
Molar entropy, molar heat capacity	joule per mole kelvin	J/mol K	m^2 kg s^{-2} K^{-1} mol^{-1}
Exposure (x-rays and γ-rays)	coulomb per kilogram	C/kg	kg^{-1} s A
Absorbed dose rate	gray per second	Gy/s	m^2 s^{-3}
Compressibility	per pascal	Pa^{-1}	m kg^{-1} s^2
Elastic moduli	newton per square metre	N/m^2 or N m^{-2}	kg m^{-1} s^{-2}
Pressure gradient	pascal per metre	Pa/m or N m^{-3}	kg m^{-2} s^{-2}
Surface potential	joule per kilogram	J/kg or N m/kg	m^2 s^{-2}
Pressure energy	pascal cubic metre	Pa m^3 or N m	kg m^2 s^{-2}
Impulse	newton second	N s	kg m s^{-1}
Angular impulse	newton metre second	N m s	kg m^2 s^{-1}
Specific resistance	ohm metre	Ωm	kg m^3 s^{-3} A^{-2}
Surface energy	joule per square metre	J/m^2 or N/m	kg s^{-2}

APPENDIX A 7
GENERAL GUIDELINES FOR USING SYMBOLS FOR PHYSICAL QUANTITIES, CHEMICAL ELEMENTS AND NUCLIDES

- Symbols for physical quantities are normally single letters and printed in italic (or sloping) type. However, in case of the two letter symbols, appearing as a factor in a product, some spacing is necessary to separate this symbol from other symbols.
- Abbreviations, i.e., shortened forms of names or expressions, such as p.e. for potential energy, are not used in physical equations. These abbreviations in the text are written in ordinary normal/roman (upright) type.
- Vectors are printed in bold and normal/roman (upright) type. However, in class room situations, vectors may be indicated by an arrow on the top of the symbol.
- Multiplication or product of two physical quantities is written with some spacing between them. Division of one physical quantity by another may be indicated with a horizontal bar or with

solidus, a slash or a short oblique stroke mark (/) or by writing it as a product of the numerator and the inverse first power of the denominator, using brackets at appropriate places to clearly distinguish between the numerator and the denominator.

- Symbols for chemical elements are written in normal/roman (upright) type. The symbol is not followed by a full stop.
  For example, Ca, C, H, He, U, etc.
- The attached numerals specifying a nuclide are placed as a left subscript (atomic number) and superscript (mass number).
  For example, a U-235 nuclide is expressed as $^{235}_{92}U$ (with 235 expressing the mass number and 92 as the atomic number of uranium with chemical symbol U).
- The right superscript position is used, if required, for indicating a state of ionisation (in case of ions).
  For example, Ca^{2+}, PO_4^{3-}

APPENDIX A 8
GENERAL GUIDELINES FOR USING SYMBOLS FOR SI UNITS, SOME OTHER UNITS, AND SI PREFIXES

- Symbols for units of physical quantities are printed/written in Normal/Roman (upright) type.
- Standard and recommended symbols for units are written in lower case roman (upright) type, starting with small letters. The shorter designations for units such as kg, m, s, cd, etc., are symbols and not the abbreviations. The unit names are never capitalised. However, the unit symbols are capitalised only if the symbol for a unit is derived from a proper name of scientist, beginning with a capital, normal/roman letter.
  For example, m for the unit 'metre', d for the unit 'day', atm for the unit 'atmospheric pressure', Hz for the unit 'hertz', Wb for the unit 'weber', J for the unit 'joule', A for the unit 'ampere', V for the unit 'volt', etc. The single exception is L, which is the symbol for the unit 'litre'. This exception is made to avoid confusion of the lower case letter l with the Arabic numeral l.
- Symbols for units do not contain any final full stop at the end of recommended letter and remain unaltered in the plural, using only singular form of the unit.
  For example, for a length of 25 centimetres the unit symbol is written as 25 cm and not 25 cms or 25 cm. or 25 cms., etc.
- Use of solidus (/) is recommended only for indicating a division of one letter unit symbol by another unit symbol. Not more than one solidus is used.
  For example :
  m/s^2 or $m\ s^{-2}$ (with a spacing between m and s^{-2}) but not m/s/s;
  1 Pl =1 N s m^{-2} = 1 N s/m^2 = 1 kg/s m=1 kg m^{-1} s^{-1}, but not 1 kg/m/s;
  J/K mol or J K^{-1} mol^{-1}, but not J/K/mol; etc.
- Prefix symbols are printed in normal/roman (upright) type without spacing between the prefix symbol and the unit symbol. Thus certain approved prefixes written very close to the unit symbol are used to indicate decimal fractions or multiples of a SI unit, when it is inconveniently small or large.
  For example :

megawatt (1MW = 10^6 W);	nanosecond (1 ns = 10^{-9} s);
centimetre (1 cm = 10^{-2} m);	picofarad (1 pF = 10^{-12} F);.
kilometre (1 km = 10^3 m);	microsecond (1μs = 10^{-6} s);
millivolt (1 mV= 10^{-3} V);	gigahertz (1GHz = 10^9 Hz);

kilowatt-hour (1 kW h = 10^3 W h = 3.6 MJ = 3.6×10^6 J);
microampere (1μ A = 10^{-6} A); micron (1μm = 10^{-6} m);
angstrom (1 Å =0.1 nm = 10^{-10} m); etc.

The unit 'micron' which equals 10^{-6} m, i.e. a micrometre, is simply the name given to convenient sub-multiple of the metre. In the same spirit, the unit 'fermi', equal to a femtometre or 10^{-15} m has been used as the convenient length unit in nuclear studies. Similarly, the unit 'barn', equal to 10^{-28} m^2, is a convenient measure of cross-sectional areas in sub-atomic particle collisions. However, the unit 'micron' is preferred over the unit 'micrometre' to avoid confusion of the 'micrometre' with the length measuring instrument called 'micrometer'. These newly formed multiples or sub-multiples (cm, km, μm, μs, ns) of SI units, metre and second, constitute a new composite inseparable symbol for units.

- When a prefix is placed before the symbol of a unit, the combination of prefix and symbol is considered as a new symbol, for the unit, which can be raised to a positive or negative power without using brackets. These can be combined with other unit symbols to form compound unit. Rules for binding-in indices are not those of ordinary algebra.
  For example :
  cm^3 means always $(cm)^3 = (0.01\ m)^3 = (10^{-2}\ m)^3 = 10^{-6}\ m^3$, but never $0.01\ m^3$ or $10^{-2}\ m^3$ or $1c\,m^3$ (prefix c with a spacing with m^3 is meaningless as prefix c is to be attached to a unit symbol and it has no physical significance or independent existence without attachment with a unit symbol).
  Similarly, mA^2 means always $(mA)^2 = (0.001A)^2 = (10^{-3}\ A)^2 = 10^{-6}\ A^2$, but never $0.001\ A^2$ or $10^{-3}\ A^2$ or m A^2;
  $1\ cm^{-1} = (10^{-2}m)^{-1} = 10^2\ m^{-1}$, but not $1c\ m^{-1}$ or $10^{-2}\ m^{-1}$;
  $1\mu s^{-1}$ means always $(10^{-6}s)^{-1} = 10^6\ s^{-1}$, but not $1 \times 10^{-6}\ s^{-1}$;
  $1\ km^2$ means always $(km)^2 = (10^3\ m)^2 = 10^6\ m^2$, but not $10^3\ m^2$;
  $1mm^2$ means always $(mm)^2 = (10^{-3}\ m)^2 = 10^{-6}\ m^2$, but not $10^{-3}\ m^2$.
- A prefix is never used alone. It is always attached to a unit symbol and written or fixed before (pre-fix) the unit symbol.
  For example :
  $10^3/m^3$ means $1000/m^3$ or $1000\ m^{-3}$, but not k/m^3 or k m^{-3}.
  $10^6/m^3$ means $10{,}00{,}000/m^3$ or $10{,}00{,}000\ m^{-3}$, but not M/m^3 or M m^{-3}
- Prefix symbol is written very close to the unit symbol without spacing between them, while unit symbols are written separately with spacing when units are multiplied together.
  For example :
  m s^{-1} (symbols m and s^{-1}, in lower case, small letter m and s, are separate and independent unit symbols for metre and second respectively, with spacing between them) means 'metre per second', but not 'milli per second'.
  Similarly, ms^{-1} [symbol m and s are written very close to each other, with prefix symbol m (for prefix milli) and unit symbol s, in lower case, small letter (for unit 'second') without any spacing between them and making ms as a new composite unit] means 'per millisecond', but never 'metre per second'.
  mS^{-1}[symbol m and S are written very close to each other, with prefix symbol m (for prefix milli) and unit symbol S, in capital roman letter S (for unit 'siemens') without any spacing between them, and making mS as a new composite unit] means 'per millisiemens', but never 'per millisecond'.
  C m [symbol C and m are written separately, representing unit symbols C (for unit 'coulomb') and m (for unit 'metre'), with spacing between them] means 'coulomb metre', but never 'centimetre', etc.
- The use of double prefixes is avoided when single prefixes are available.
  For example :

10^{-9} m = 1nm (nanometre), but not 1mμm (millimicrometre),
10^{-6} m= 1μm (micron), but not 1mmm(millimillimetre),
10^{-12} F= 1 pF (picofarad), but not 1μμF (micromicrofarad),
10^{9} W=1 GW (giga watt), but not 1 kMW (kilomegawatt), etc.

- The use of a combination of unit and the symbols for units is avoided when the physical quantity is expressed by combining two or more units.
  For example :
  joule per mole kelvin is written as J/mol K or J mol^{-1} K^{-1}, but not joule/mole K or J/ mol kelvin or J/mole K, etc.
  joule per tesla is written as J/T or J T^{-1}, but not joule /T or J per tesla or J/tesla, etc.
  newton metre second is written as N m s, but not Newton m second or N m second or N metre s or newton metre s, etc.
  joule per kilogram kelvin is written as J/kg K or J kg^{-1} K^{-1}, but not J/kilog K or joule/kg K or J/ kg kelvin or J/kilogram K, etc.
- To simplify calculations, the prefix symbol is attached to the unit symbol in the numerator and not to the denominator.
  For example :
  10^{6} N/m^2 is written more conveniently as MN/m^2, in preference to N/mm^2.
  A preference has been expressed for multiples or sub-multiples involving the factor 1000, $10^{\pm 3n}$ where n is the integer.
- Proper care is needed when same symbols are used for physical quantities and units of physical quantities.
  For example :
  The physical quantity weight (*W*) expressed as a product of mass (*m*) and acceleration due to gravity (*g*) may be written in terms of symbols *W*, *m* and *g* printed in italic (or sloping) type as $W = m\ g$, preferably with a spacing between *m* and *g*. It should not be confused with the unit symbols for the units watt (W), metre (m) and gram (g). However, in the equation $W=m\ g$, the symbol *W* expresses the weight with a unit symbol J, *m* as the mass with a unit symbol kg and *g* as the acceleration due to gravity with a unit symbol m/s^2. Similarly, in equation $F = m\ a$, the symbol *F* expresses the force with a unit symbol N, *m* as the mass with a unit symbol kg, and *a* as the acceleration with a unit symbol m/s^2. These symbols for physical quantities should not be confused with the unit symbols for the units 'farad' (F), 'metre'(m) and 'are' (a).
  Proper distinction must be made while using the symbols h (prefix hecto, and unit hour), c (prefix centi, and unit carat), d (prefix deci and unit day), T (prefix tera, and unit tesla), a (prefix atto, and unit are), da (prefix deca, and unit deciare), etc.
- SI base unit 'kilogram' for mass is formed by attaching SI prefix (a multiple equal to 10^3) 'kilo' to a cgs (centimetre, gram, second) unit 'gram' and this may seem to result in an anomaly. Thus, while a thousandth part of unit of length (metre) is called a millimetre (mm), a thousandth part of the unit of mass (kg) is not called a millikilogram, but just a gram. This appears to give the impression that the unit of mass is a gram (g) which is not true. Such a situation has arisen because we are unable to replace the name 'kilogram' by any other suitable unit. Therefore, as an exception, name of the multiples and sub-multiples of the unit of mass are formed by attaching prefixes to the word 'gram' and not to the word 'kilogram'.
  For example :
  10^{3} kg =1 megagram (1Mg), but not 1 kilo kilogram (1 kkg);
  10^{-6} kg = 1 milligram (1 mg), but not 1 microkilogram (1μkg);
  10^{-3} kg = 1 gram (1g), but not 1 millikilogram (1 mkg), etc.

It may be emphasised again that you should use the internationally approved and recommended symbols only. Continual practice of following general rules and guidelines in unit symbol writing would make you learn mastering the correct use of SI units, prefixes and related symbols for physical quantities in a proper perspective.

APPENDIX A 9
DIMENSIONAL FORMULAE OF PHYSICAL QUANTITIES

S.No	Physical quantity	Relationship with other physical quantities	Dimensions	Dimensional formula
1.	Area	Length × breadth	$[L^2]$	$[M^0 L^2 T^0]$
2.	Volume	Length × breadth × height	$[L^3]$	$[M^0 L^3 T^0]$
3.	Mass density	Mass/volume	$[M]/[L^3]$ or $[M L^{-3}]$	$[M L^{-3} T^0]$
4.	Frequency	1/time period	$1/[T]$	$[M^0 L^0 T^{-1}]$
5.	Velocity, speed	Displacement/time	$[L]/[T]$	$[M^0 L T^{-1}]$
6.	Acceleration	Velocity /time	$[LT^{-1}]/[T]$	$[M^0 LT^{-2}]$
7.	Force	Mass × acceleration	$[M][LT^{-2}]$	$[M LT^{-2}]$
8.	Impulse	Force × time	$[M LT^{-2}][T]$	$[M LT^{-1}]$
9.	Work, Energy	Force × distance	$[MLT^{-2}] [L]$	$[M L^2 T^{-2}]$
10.	Power	Work/time	$[ML^2 T^{-2}]/ [T]$	$[M L^2 T^{-3}]$
11.	Momentum	Mass × velocity	$[M] [LT^{-1}]$	$[M LT^{-1}]$
12.	Pressure, stress	Force/area	$[M LT^{-2}]/[L^2]$	$[ML^{-1}T^{-2}]$
13.	Strain	$\frac{\text{Change in dimension}}{\text{Oringinal dimension}}$	$[L] / [L]$ or $[L^3] / [L^3]$	$[M^0 L^0 T^0]$
14.	Modulus of elasticity	Stress/strain	$\frac{[M L^{-1} T^{-2}]}{[M^0 L^0 T^0]}$	$[M L^{-1} T^{-2}]$
15	Surface tension	Force/length	$[MLT^{-2}]/[L]$	$[ML^0 T^{-2}]$
16.	Surface energy	Energy/area	$[ML^2 T^{-2}]/[L^2]$	$[ML^0 T^{-2}]$
17.	Velocity gradient	Velocity/distance	$[LT^{-1}]/[L]$	$[M^0 L^0 T^{-1}]$
18.	Pressure gradient	Pressure/distance	$[ML^{-1}T^{-2}]/[L]$	$[ML^{-2} T^{-2}]$
19.	Pressure energy	Pressure × volume	$[ML^{-1}T^{-2}] [L^3]$	$[ML^2 T^{-2}]$
20.	Coefficient of viscosity	Force/area × velocity gradient	$\frac{[MLT^{-2}]}{[L^2][LT^{-1} / L]}$	$[ML^{-1} T^{-1}]$
21.	Angle, Angular displacement	Arc/radius	$[L]/[L]$	$[M^0 L^0 T^0]$
22.	Trigonometric ratio (sinθ, cosθ, tanθ, etc.)	Length/length	$[L]/[L]$	$[M^0 L^0 T^0]$
23.	Angular velocity	Angle/time	$[L^0]/[T]$	$[M^0 L^0 T^{-1}]$

24.	Angular acceleration	Angular velocity/time	$[T^{-1}]/[T]$	$[M^0L^0T^{-2}]$
25.	Radius of gyration	Distance	$[L]$	$[M^0LT^0]$
26.	Moment of inertia	Mass × (radius of gyration)^2	$[M]\,[L^2]$	$[ML^2\,T^0]$
27.	Angular momentum	Moment of inertia × angular velocity	$[ML^2]\,[T^{-1}]$	$[ML^2\,T^{-1}]$
28.	Moment of force, moment of couple	Force x distance	$[MLT^{-2}]\,[L]$	$[ML^2\,T^{-2}]$
29.	Torque	Angular momentum/time, Or Force × distance	$[ML^2\,T^{-1}]\,/\,[T]$ or $[MLT^{-2}]\,[L]$	$[ML^2\,T^{-2}]$
30.	Angular frequency	2π × Frequency	$[T^{-1}]$	$[M^0L^0T^{-1}]$
31.	Wavelength	Distance	$[L]$	$[M^0LT^0]$
32.	Hubble constant	Recession speed/distance	$[LT^{-1}]/[L]$	$[M^0L^0T^{-1}]$
33.	Intensity of wave	(Energy/time)/area	$[ML^2\,T^{-2}/T]/[L^2]$	$[ML^0T^{-3}]$
34.	Radiation pressure	$\frac{\text{Intensity of wave}}{\text{Speed of light}}$	$[MT^{-3}]/[LT^{-1}]$	$[ML^{-1}\,T^{-2}]$
35.	Energy density	Energy/volume	$[ML^2T^{-2}]/[L^3]$	$[ML^{-1}\,T^{-2}]$
36.	Critical velocity	$\frac{\text{Reynold's number} \times \text{coefficient of viscocity}}{\text{Mass density} \times \text{radius}}$	$\frac{[M^0L^0T^0][ML^{-1}\,T^{-1}]}{[ML^{-3}][L]}$	$[M^0LT^{-1}]$
37.	Escape velocity	(2 × acceleration due to gravity × earth's radius)$^{1/2}$	$[LT^{-2}]^{1/2}$ x $[L]^{1/2}$	$[M^0LT^{-1}]$
38.	Heat energy, internal energy	Work (= Force × distance)	$[MLT^{-2}]\,[L]$	$[ML^2\,T^{-2}]$
39.	Kinetic energy	(1/2) mass × (velocity)^2	$[M]\,[LT^{-1}]^2$	$[ML^2T^{-2}]$
40	Potential energy	Mass × acceleration due to gravity × height	$[M]\,[LT^{-2}]\,[L]$	$[ML^2\,T^{-2}]$
41.	Rotational kinetic energy	½ × moment of inertia × (angular velocity)^2	$[M^0L^0T^0]\,[ML^2]$x$[T^{-1}]^2$	$[M\,L^2\,T^{-2}]$
42.	Efficiency	$\frac{\text{Output work or energy}}{\text{Input work or energy}}$	$\frac{[ML^2\,T^{-2}]}{[ML^2\,T^{-2}]}$	$[M^0L^0T^0]$
43.	Angular impulse	Torque × time	$[ML^2\,T^{-2}]\,[T]$	$[M\,L^2\,T^{-1}]$
44.	Gravitational constant	$\frac{\text{Force} \times (\text{distance})^2}{\text{mass} \times \text{mass}}$	$\frac{[MLT^{-2}][L^2]}{[M]\,[M]}$	$[M^{-1}L^3T^{-2}]$
45.	Planck constant	Energy/frequency	$[ML^2\,T^{-2}]\,/[T^{-1}]$	$[ML^2T^{-1}]$

46.	Heat capacity, entropy	Heat energy / temperature	$[ML^2T^{-2}]/[K]$	$[ML^2T^{-2}K^{-1}]$
47.	Specific heat capacity	$\frac{\text{Heat Energy}}{\text{Mass} \times \text{temperature}}$	$[ML^2T^{-2}]/[M][K]$	$[M^0L^2T^{-2}K^{-1}]$
48.	Latent heat	Heat energy/mass	$[ML^2T^{-2}]/[M]$	$[M^0L^2T^{-2}]$
49.	Thermal expansion coefficient or Thermal expansivity	$\frac{\text{Change in dimension}}{\text{Original dimension} \times \text{temperature}}$	$[L]/[L][K]$	$[M^0L^0K^{-1}]$
50.	Thermal conductivity	$\frac{\text{Heat energy} \times \text{thickness}}{\text{Area} \times \text{temperature} \times \text{time}}$	$\frac{[ML^2T^{-2}][L]}{[L^2][K][T]}$	$[MLT^{-3}K^{-1}]$
51.	Bulk modulus or (compressibility)$^{-1}$	$\frac{\text{Volume} \times (\text{change in pressure})}{(\text{change in volume})}$	$\frac{[L^3][ML^{-1}T^{-2}]}{[L^3]}$	$[ML^{-1}T^{-2}]$
52.	Centripetal acceleration	(Velocity)^2 /radius	$[LT^{-1}]^2/[L]$	$[M^0LT^{-2}]$
53.	Stefan constant	$\frac{(\text{Energy / area} \times \text{time})}{(\text{Temperature})^4}$	$\frac{[ML^2T^{-2}]}{[L^2][T][K]^4}$	$[ML^0T^{-3}K^{-4}]$
54.	Wien constant	Wavelength × temperature	$[L][K]$	$[M^0LT^0K]$
55.	Boltzmann constant	Energy/temperature	$[ML^2T^{-2}]/[K]$	$[ML^2T^{-2}K^{-1}]$
56.	Universal gas constant	$\frac{\text{Pressure} \times \text{volume}}{\text{mole} \times \text{temperature}}$	$\frac{[ML^{-1}T^{-2}][L^3]}{[mol][K]}$	$[ML^2T^{-2}K^{-1}mol^{-1}]$
57.	Charge	Current × time	$[A][T]$	$[M^0L^0TA]$
58.	Current density	Current /area	$[A]/[L^2]$	$[M^0L^{-2}T^0A]$
59.	Voltage, electric potential, electromotive force	Work/charge	$[ML^2T^{-2}]/[AT]$	$[ML^2T^{-3}A^{-1}]$
60.	Resistance	$\frac{\text{Potential difference}}{\text{Current}}$	$\frac{[ML^2T^{-3}A^{-1}]}{[A]}$	$[ML^2T^{-3}A^{-2}]$
61.	Capacitance	Charge/potential difference	$\frac{[AT]}{[ML^2T^{-3}A^{-1}]}$	$[M^{-1}L^{-2}T^4A^2]$
62.	Electrical resistivity or (electrical conductivity)$^{-1}$	$\frac{\text{Resistance} \times \text{area}}{\text{length}}$	$[ML^2T^{-3}A^{-2}][L^2]/[L]$	$[ML^3T^{-3}A^{-2}]$
63.	Electric field	Electrical force/charge	$[MLT^{-2}]/[AT]$	$[MLT^{-3}A^{-1}]$
64.	Electric flux	Electric field × area	$[MLT^{-3}A^{-1}][L^2]$	$[ML^3T^{-3}A^{-1}]$

65.	Electric dipole moment	Torque/electric field	$\frac{[ML^2\ T^{-2}]}{[MLT^{-3}\ A^{-1}]}$	$[M^0\ LTA]$
66.	Electric field strength or electric intensity	$\frac{\text{Potential difference}}{\text{distance}}$	$\frac{[ML^2\ T^{-3}\ A^{-1}]}{[L]}$	$[MLT^{-3}\ A^{-1}]$
67.	Magnetic field, magnetic flux density, magnetic induction	$\frac{\text{Force}}{\text{Current} \times \text{length}}$	$[MLT^{-2}]/[A]\ [L]$	$[ML^0\ T^{-2}\ A^{-1}]$
68.	Magnetic flux	Magnetic field × area	$[MT^{-2}\ A^{-2}]\ [L^2]$	$[ML^2\ T^{-2}\ A^{-1}]$
69.	Inductance	$\frac{\text{Magnetic flux}}{\text{Current}}$	$\frac{[ML^2\ T^{-2}\ A^{-1}]}{[A]}$	$[ML^2\ T^{-2}\ A^{-2}]$
70.	Magnetic dipole moment	Torque/magnetic field or current × area	$[ML^2\ T^{-2}]\ /\ [MT^{-2}\ A^{-1}]$ or $[A]\ [L^2]$	$[M^0L^2T^0A]$
71.	Magnetic field strength, magnetic intensity or magnetic moment density	$\frac{\text{Magnetic moment}}{\text{Volume}}$	$\frac{[L^2A]}{[L^3]}$	$[M^0L^{-1}T^0A]$
72	Permittivity constant (of free space)	$\frac{\text{Charge} \times \text{charge}}{4\pi \times \text{electric force} \times (\text{distance})^2}$	$\frac{[AT][AT]}{[MLT^{-2}][L]^2}$	$[M^{-1}L^{-3}T^4\ A^2]$
73.	Permeability constant (of free space)	$\frac{2\pi \times \text{force} \times \text{distance}}{\text{current} \times \text{current} \times \text{length}}$	$\frac{[M^0\ L^0\ T^0][MLT^{-2}][L]}{[A][A][L]}$	$[MLT^{-2}\ A^{-2}]$
74.	Refractive index	$\frac{\text{Speed of light in vacuum}}{\text{Speed of light in medium}}$	$[LT^{-1}]/LT^{-1}]$	$[M^0L^0T^0]$
75.	Faraday constant	Avogadro constant × elementary charge	[AT]/[mol]	$[M^0L^0TA\ mol^{-1}]$
76.	Wave number	2π/wavelength	$[M^0L^0T^0]\ /\ [L]$	$[M^0L^{-1}\ T^0]$
77.	Radiant flux, Radiant power	Energy emitted/time	$[ML^2T^{-2}]/[T]$	$[ML^2T^{-3}]$
78.	Luminosity of radiant flux or radiant intensity	$\frac{\text{Radiant power or radiant flus of source}}{\text{Solid angle}}$	$[ML^2T^{-3}]\ /\ [M^0L^0T^0]$	$[ML^2\ T^{-3}]$
79.	Luminous power or luminous flux of source	$\frac{\text{Luminous energy emitted}}{\text{time}}$	$[ML^2\ T^{-2}]/[T]$	$[ML^2\ T^{-3}]$

80.	Luminous intensity or illuminating power of source	$\frac{\text{Luminous flux}}{\text{Soild angle}}$	$\frac{[ML^2 T^{-3}]}{[M^0L^0T^0]}$	$[ML^2 T^{-3}]$
81.	Intensity of illumination or luminance	$\frac{\text{Luminous intensity}}{(\text{distance})^2}$	$[ML^2 T^{-3}]/[L^2]$	$[ML^0T^{-3}]$
82.	Relative luminosity	$\frac{\text{Luminous flux of a source of given wavelength}}{\text{luminous flux of peak sensitivity wavelength (555 nm) source of same power}}$	$\frac{[ML^2T^{-1}]}{[ML^2T^{-3}]}$	$[M^0L^0T^0]$
83.	Luminous efficiency	$\frac{\text{Total luminous flux}}{\text{Total radiant flux}}$	$[ML^2 T^{-3}] / [ML^2T^{-3}]$	$[M^0L^0T^0]$
84.	Illuminance or illumination	$\frac{\text{Luminous flux incident}}{\text{area}}$	$[ML^2T^{-3}]/[L^2]$	$[ML^0T^{-3}]$
85.	Mass defect	(sum of masses of nucleons)-(mass of the nucleus)	[M]	$[ML^0T^0]$
86.	Binding energy of nucleus	Mass defect × (speed of light in vacuum)^2	$[M] [L T^{-1}]^2$	$[ML^2 T^{-2}]$
87.	Decay constant	0.693/half life	$[T^{-1}]$	$[M^0L^0T^{-1}]$
88.	Resonant frequency	$(\text{Inductance} \times \text{capacitance})^{-\frac{1}{2}}$	$[ML^2 T^{-2} A^{-2}]^{-\frac{1}{2}}$ x $[M^{-1} L^{-2} T^4 A^2]^{-\frac{1}{2}}$	$[M^0L^0 A^0 T^{-1}]$
89.	Quality factor or Q-factor of coil	$\frac{\text{Resonant frequency} \times \text{inductance}}{\text{Resistance}}$	$\frac{[T^{-1}][ML^2 T^{-2} A^{-2}]}{[ML^2 T^{-3} A^{-2}]}$	$[M^0L^0T^0]$
90.	Power of lens	(Focal length)$^{-1}$	$[L^{-1}]$	$[M^0L^{-1} T^0]$
91.	Magnification	$\frac{\text{Image distance}}{\text{Object distance}}$	[L] /[L]	$[M^0L^0T^0]$
92.	Fluid flow rate	$\frac{(\pi / 8)(\text{pressure}) \times (\text{radius})^4}{(\text{viscosity coefficient}) \times (\text{length})}$	$\frac{[ML^{-1} T^{-2}]\ [L^4]}{[ML^{-1} T^{-1}]\ [L]}$	$[M^0L^3T^{-1}]$
93	Capacitive reactance	(Angular frequency × capacitance)$^{-1}$	$[T^{-1}]^{-1} [M^{-1} L^{-2} T^4 A^2]^{-1}$	$[ML^2 T^{-3} A^{-2}]$
94.	Inductive reactance	(Angular frequency × inductance)	$[T^{-1}][ML^2 T^{-2} A^{-2}]$	$[ML^2 T^{-3} A^{-2}]$

ANSWERS

Chapter 2

2.1 (a) 10^{-6} ; (b) 1.5×10^{4} ; (c) 5 ; (d) 11.3, 1.13×10^{4}.

2.2 (a) 10^{7} ; (b) 10^{-16} ; (c) 3.9×10^{4} ; (d) 6.67×10^{-8}.

2.5 500

2.6 (c)

2.7 0.035 mm

2.9 94.1

2.10 (a) 1 ; (b) 3 ; (c) 4 ; (d) 4 ; (e) 4 ; (f) 4.

2.11 8.72 m^2; 0.0855 m^3

2.12 (a) 2.3 kg ; (b) 0.02 g

2.13 13%; 3.8

2.14 (b) and (c) are wrong on dimensional grounds. Hint: The argument of a trigonometric function must always be dimensionless.

2.15 The correct formula is $m = m_0 (1 - v^2/c^2)^{-1/2}$

2.16 $\cong 3 \times 10^{-7}$ m^3

2.17 $\cong$ 10^4; intermolecular separation in a gas is much larger than the size of a molecule.

2.18 Near objects make greater angle than distant (far off) objects at the eye of the observer. When you are moving, the angular change is less for distant objects than nearer objects. So, these distant objects seem to move along with you, but the nearer objects in opposite direction.

2.19 $\cong 3 \times 10^{16}$ m; as a unit of length 1 parsec is defined to be equal to 3.084×10^{16} m.

2.20 1.32 parsec; 2.64″ (second of arc)

2.23 1.4×10^{3} kg m^{-3}; the mass density of the Sun is in the range of densities of liquids / solids and *not* gases. This high density arises due to inward gravitational attraction on outer layers due to inner layers of the Sun.

2.24 1.429×10^{5} km

2.25 Hint: tan θ must be dimensionless. The correct formula is tan $\theta = \upsilon/\upsilon'$ where υ' is the speed of rainfall.

2.26 Accuracy of 1 part in 10^{11} to 10^{12}

2.27 $\cong$ 0.7×10^3 kg m^{-3}. In the solid phase atoms are tightly packed, so the atomic mass density is close to the mass density of the solid.

2.28 $\cong 0.3 \times 10^{18}$ m^{-3} – Nuclear density is typically 10^{15} times atomic density of matter.

2.29 3.84×10^8 m

2.30 55.8 km

2.31 2.8×10^{22} km

2.32 3,581 km

2.33 Hint: the quantity $e^4/\ (16\ \pi^2\ \epsilon_0^2 m_p\ m_e^2\ c^3\ G)$ has the dimension of time.

Chapter 3

3.1 (a), (b)

3.2 (a) A....B, (b) A....B, (c) B....A, (d) Same, (e) B....A....once.

3.4 37 s

3.5 1000 km/h

3.6 3.06 m s^{-2}; 11.4 s

3.7 1250 m (Hint: view the motion of B relative to A)

3.8 1 m s^{-2} (Hint: view the motion of B and C relative to A)

3.9 T = 9 min, speed = 40 km/h. Hint: $\upsilon T / (\upsilon - 20) = 18$; $\upsilon T / (\upsilon + 20) = 6$

3.10 (a) Vertically downwards; (b) zero velocity, acceleration of 9.8 m s^{-2} downwards; (c) $x > 0$ (upward and downward motion); $\upsilon < 0$ (upward), $\upsilon > 0$ (downward), $a > 0$ throughout; (d) 44.1 m, 6 s.

3.11 (a) True;, (b) False; (c) True (if the particle rebounds instantly with the same speed, it implies infinite acceleration which is unphysical); (d) False (true only when the chosen positive direction is along the direction of motion)

3.14 (a) 5 km h^{-1}, 5 km h^{-1}; (b) 0, 6 km h^{-1}; (c) $\frac{15}{8}$ km h^{-1}, $\frac{45}{8}$ km h^{-1}

3.15 Because, for an arbitrarily small interval of time, the magnitude of displacement is equal to the length of the path.

3.16 All the four graphs are impossible. (a) a particle cannot have two different positions at the same time; (b) a particle cannot have velocity in opposite directions at the same time; (c) speed is always non-negative; (d) total path length of a particle can never decrease with time. (Note, the arrows on the graphs are meaningless).

3.17 No, wrong. x-t plot does not show the trajectory of a particle. Context: A body is dropped from a tower ($x = 0$) at $t = 0$.

3.18 105 m s^{-1}

3.19 (a) A ball at rest on a smooth floor is kicked, it rebounds from a wall with reduced speed and moves to the opposite wall which stops it; (b) A ball thrown up with some initial velocity rebounding from the floor with reduced speed after each hit; (c) A uniformly moving cricket ball turned back by hitting it with a bat for a very short time-interval.

3.20 $x < 0$, $v < 0$, $a > 0$; $x > 0$, $v > 0$, $a < 0$; $x < 0$, $v > 0$, $a > 0$.

3.21 Greatest in 3, least in 2; $v > 0$ in 1 and 2, $v < 0$ in 3.

3.22 Acceleration magnitude greatest in 2; speed greatest in 3; $v > 0$ in 1, 2 and 3; $a > 0$ in 1 and 3, $a < 0$ in 2; $a = 0$ at A, B, C, D.

3.23 A straight line inclined with the time-axis for uniformly accelerated motion; parallel to the time- axis for uniform motion.

3.24 10 s, 10 s

3.25 (a) 13 km h^{-1} ; (b) 5 km h^{-1}; (c) 20 s in either direction, viewed by any one of the parents, the speed of the child is 9 km h^{-1} in either direction; answer to (c) is unaltered.

3.26 $x_2 - x_1 = 15\,t$ (linear part); $x_2 - x_1 = 200 + 30\,t - 5\,t^2$ (curved part).

3.27 (a) 60 m, 6 m s^{-1} ; (b) 36 m, 9 m s^{-1}

3.28 (c), (d), (f)

Chapter 4

4.1 Volume, mass, speed, density, number of moles, angular frequency are scalars; the rest are vectors.

4.2 Work, current

4.3 Impulse

4.4 Only (c) and (d) are permissible

4.5 (a) T, (b) F, (c) F, (d) T, (e) T

4.6 Hint: The sum (difference) of any two sides of a triangle is never less (greater) than the third side. Equality holds for collinear vectors.

4.7 All statements except (a) are correct

4.8 400 m for each; B

4.9 (a) O; (b) O; (c) 21.4 km h^{-1}

4.10 Displacement of magnitude 1 km and direction 60° with the initial direction; total path length = 1.5 km (third turn); null displacement vector; path length = 3 km (sixth turn); 866 m, 30°, 4 km (eighth turn)

4.11 (a) 49.3 km h^{-1} ; (b) 21.4 km h^{-1}. No, the average speed equals average velocity magnitude only for a straight path.

4.12 About 18° with the vertical, towards the south.

4.13 15 min, 750 m

4.14 East (approximately)

4.15 150.5 m

4.16 50 m

4.17 9.9 m s^{-2}, along the radius at every point towards the centre.

4.18 6.4 g

4.19 (a) False (true only for uniform circular motion)

(b) True, (c) True.

4.20 (a) $\mathbf{v}(t) = (3.0\ \hat{\mathbf{i}} - 4.0\,t\ \hat{\mathbf{j}})\ \ \hat{\mathbf{a}}(t) = -4.0\,\hat{\mathbf{j}}$

(b) 8.54 m s^{-1}, 70° with x-axis.

4.21 (a) 2 s, 24 m, 21.26 m s^{-1}

4.22 $\sqrt{2}$, 45° with the x-axis; $\sqrt{2}$, – 45° with the x - axis, $\left(5/\sqrt{2},\ -1/\sqrt{2}\right)$.

4.23 (b) and (e)

4.24 Only (e) is true

4.25 182 m s^{-1}

4.27 No. Rotations in *general* cannot be associated with vectors

4.28 A vector can be associated with a plane area

4.29 No

4.30 At an angle of $\sin^{-1}(1/3) = 19.5°$ with the vertical; 16 km.

4.31 0.86 m s^{-2}, 54.5° with the direction of velocity

Chapter 5

5.1 (a) to (d) No net force according to the First Law
(e) No force, since it is far away from all material agencies producing electromagnetic and gravitational forces.

5.2 The only force in each case is the force of gravity, (neglecting effects of air) equal to 0.5 N vertically downward. The answers do not change, even if the motion of the pebble is not along the vertical. The pebble is not at rest at the highest point. It has a constant horizontal component of velocity throughout its motion.

5.3 (a) 1 N vertically downwards (b) same as in (a)
(c) same as in (a); force at an instant depends on the situation at that instant, not on history.
(d) 0.1 N in the direction of motion of the train.

5.4 (i) T

5.5 $a = -2.5$ m s^{-2}. Using $v = u + at$, $0 = 15 - 2.5\,t$ i.e., $t = 6.0$ s

5.6 $a = 1.5/25 = 0.06$ m s^{-2}

$F = 3 \times 0.06 = 0.18$ N in the direction of motion.

5.7 Resultant force = 10 N at an angle of $\tan^{-1}(3/4) = 37°$ with the direction of 8 N force. Acceleration = 2 m s^{-2} in the direction of the resultant force.

5.8 $a = -2.5$ m s^{-2}, Retarding force = 465 × 2.5 = 1.2 × 10^3 N

5.9 $F - 20{,}000 \times 10 = 20000 \times 5.0$, i.e., $F = 3.0 \times 10^5$ N

5.10 $a = -20$ m s^{-2} $0 \le t \le 30$ s

$t = -5$ s : $x = u\,t = -\,10 \times 5 = -50$ m

$t = 25$ s : $x = u\,t + (½)\ a\,t^2 = (10 \times 25 - 10 \times 625)$m $= -\,6$ km

$t = 100$ s : First consider motion up to 30 s

$x_1 = 10 \times 30 - 10 \times 900 = -\,8700$ m

At $t = 30$ s, $v = 10 - 20 \times 30 = -590$ m s^{-1}

For motion from 30 s to 100 s : $x_2 = -\,590 \times 70 = -\,41300$ m

$x = x_1 + x_2 = -\,50$ km

5.11 (a) Velocity of car (at $t = 10$ s) $= 0 + 2 \times 10 = 20$ m s^{-1}
By the First Law, the horizontal component of velocity is 20 m s^{-1} throughout.
Vertical component of velocity (at $t = 11$s) $= 0 + 10 \times 1 = 10$ m s^{-1}

Velocity of stone (at $t = 11$s) $= \sqrt{20^2 + 10^2} = \sqrt{500} = 22.4$ m s^{-1} at an angle of $\tan^{-1}$ (½) with the horizontal.
(b)10 m s^{-2} vertically downwards.

5.12 (a) At the extreme position, the speed of the bob is zero. If the string is cut, it will fall vertically downwards.
(b) At the mean position, the bob has a horizontal velocity. If the string is cut, it will fall along a parabolic path.

5.13 The reading on the scale is a measure of the force on the floor by the man. By the Third Law, this is equal and opposite to the normal force N on the man by the floor.

(a) $N = 70 \times 10 = 700$ N ; Reading is 70 kg
(b) $70 \times 10 - N = 70 \times 5$; Reading is 35 kg
(c) $N - 70 \times 10 = 70 \times 5$; Reading is 105 kg
(d) $70 \times 10 - N = 70 \times 10$; Reading would be zero; the scale would read zero.

5.14 (a) In all the three intervals, acceleration and, therefore, force are zero.
(b) 3 kg m s^{-1} at $t = 0$; (c) –3 kg m s^{-1} at $t = 4$ s.

5.15 If the 20 kg mass is pulled,
$600 - T = 20\,a$, $T = 10\,a$
$a = 20$ m s^{-2}, $T = 200$ N
If the 10 kg mass is pulled, a = 20 m s^{-2}, $T = 400$ N

5.16 $T - 8 \times 10 = 8\,a, 12 \times 10 - T = 12a$
i.e. $a = 2$ m s^{-2}, $T = 96$ N

5.17 By momentum conservation principle, total final momentum is zero. Two momentum vectors cannot sum to a null momentum unless they are equal and opposite.

5.18 Impulse on each ball = 0.05 ×12 = 0.6 kg m s^{-1} in magnitude. The two impulses are opposite in direction.

5.19 Use momentum conservation : $100\,v = 0.02 \times 80$
$v = 0.016$ m s^{-1} = 1.6 cm s^{-1}

5.20 Impulse is directed along the bisector of the initial and final directions. Its magnitude is 0.15 × 2 × 15 × cos 22.5° = 4.2 kg m s^{-1}

5.21 $$v \quad 2 \quad 1.5 \quad \frac{40}{60} \quad 2 \quad \text{m s}^{-1}$$

$$T \quad \frac{mv^2}{R} \quad \frac{0.25 \quad 4 \quad ^2}{1.5} \quad 6.6\,\text{N}$$

$$200 \quad \frac{mv_{max}^2}{R}, \text{which gives } \upsilon_{max} \quad 35 \text{ m s}^{-1}$$

5.22 Alternative (b) is correct, according to the First Law

5.23 (a) The horse-cart system has no external force in empty space. The mutual forces between the horse and the cart cancel (Third Law). On the ground, the contact force between the system and the ground (friction) causes their motion from rest.

(b) Due to inertia of the body not directly in contact with the seat.

(c) A lawn mower is pulled or pushed by applying force at an angle. When you push, the normal force (N) must be more than its weight, for equilibrium in the vertical direction. This results in greater friction f ($f \propto N$) and, therefore, a greater applied force to move. Just the opposite happens while pulling.

(d) To reduce the rate of change of momentum and hence to reduce the force necessary to stop the ball.

5.24 A body with a constant speed of 1 cm s^{-1} receives impulse of magnitude 0.04 kg × 0.02 m s^{-1} = 8×10^{-4} kg m s^{-1} after every 2 s from the walls at $x = 0$ and $x = 2$ cm.

5.25 Net force = 65 kg × 1 m s^{-2} = 65 N

$a_{max} = \mu_s g = 2 \text{ m s}^{-2}$

5.26 Alternative (a) is correct. Note $mg + T_2 = m\boldsymbol{v}_2^2/R$; $T_1 - mg = m\boldsymbol{v}_1^2/R$

The moral is : do not confuse the actual material forces on a body (tension, gravitational force, etc) with the effects they produce : centripetal acceleration $\boldsymbol{v}_2^2/R$ or $\boldsymbol{v}_1^2/R$ in this example.

5.27 (a) 'Free body' : crew and passengers

Force on the system by the floor = F upwards; weight of system = mg downwards;

$\therefore F - mg = ma$

$F - 300 \times 10 = 300 \times 15$

$F = 7.5 \times 10^3$ N upward

By the Third Law, force on the floor by the crew and passengers = 7.5×10^3 N downwards.

(b) 'Free body' : helicopter plus the crew and passengers

Force by air on the system = R upwards; weight of system = mg downwards

$\therefore R - mg = ma$

$R - 1300 \times 10 = 1300 \times 15$

$R = 3.25 \times 10^4$ N upwards

By the Third Law, force (action) on the air by the helicopter = 3.25×10^4 N downwards.

(c) 3.25×10^4 N upwards

5.28 Mass of water hitting the wall per second

$= 10^3 \text{ kg m}^{-3} \times 10^{-2} \text{ m}^2 \times 15 \text{ m s}^{-1} = 150 \text{ kg s}^{-1}$

Force by the wall = momentum loss of water per second = 150 kg s^{-1} × 15 m s^{-1} = 2.25×10^3 N

5.29 (a) 3 $m g$ (down) (b) 3 $m g$ (down) (c) 4 $m g$ (up)

5.30 If N is the normal force on the wings,

$$N\cos\theta = mg, \qquad N\sin\theta = \frac{mv^2}{R}$$

$$\text{which give } R = \frac{v^2}{g\ \tan\ \theta} = \frac{200\times200}{10\times\tan 15°} = 15\text{km}$$

5.31 The centripetal force is provided by the lateral thrust by the rail on the flanges of the wheels. By the Third Law, the train exerts an equal and opposite thrust on the rail causing its wear and tear.

$$\text{Angle of banking} = \tan^{-1}\left(\frac{v^2}{R\ g}\right) = \tan^{-1}\left(\frac{15\times15}{30\times10}\right) \approx 37°$$

5.32 Consider the forces on the man in equilibrium : his weight, force due to the rope and normal force due to the floor.

(a) 750 N (b) 250 N; mode (b) should be adopted.

5.33 (a) $T - 400 = 240$, $T = 640$ N

(b) $400 - T = 160$, $T = 240$ N

(c) $T = 400$ N

(*d*) $T = 0$

The rope will break in case (a).

5.34 We assume perfect contact between bodies A and B and the rigid partition. In that case, the self-adjusting normal force on B by the partition (reaction) equals 200 N. There is no impending motion and no friction. The action-reaction forces between A and B are also 200 N. When the partition is removed, kinetic friction comes into play.

Acceleration of A + B = [200 – (150 × 0.15)] / 15 =11.8 m s^{-2}

Friction on A = 0.15 × 50 = 7.5 N

$200 - 7.5 - F_{AB} = 5 \times 11.8$

$F_{AB} = 1.3 \times 10^2$ N; opposite to motion .

$F_{BA} = 1.3 \times 10^2$ N; in the direction of motion.

5.35 (a) Maximum frictional force possible for opposing impending relative motion between the block and the trolley = 150 × 0.18 = 27 N, which is more than the frictional force of 15 × 0.5 = 7.5 N needed to accelerate the box with the trolley. When the trolley moves with uniform velocity, there is no force of friction acting on the block.

(b) For the accelerated (non-inertial) observer, frictional force is opposed by the pseudo-force of the same magnitude, keeping the box at rest relative to the observer. When the trolley moves with uniform velocity there is no pseudo-force for the moving (inertial) observer and no friction.

5.36 Acceleration of the box due to friction = μg = 0.15 × 10 = 1.5 m s^{-2}. But the acceleration of the truck is greater. The acceleration of the box relative to the truck is 0.5 m s^{-2} towards the rear end. The time taken for the box to fall off the truck = $\sqrt{\frac{2\ \ 5}{0.5}}\ \ \sqrt{20}$ s .

During this time, the truck covers a distance = ½ × 2 × 20 = 20 m.

5.37 For the coin to revolve with the disc, the force of friction should be enough to provide the necessary centripetal force, i.e $\frac{mv^2}{r} \le \mu m g$. Now $v = r\omega$, where $\omega = \frac{2\pi}{T}$ is the angular frequency of the disc. For a given μ and ω, the condition is $r \le \mu g / \omega^2$. The condition is satisfied by the nearer coin (4 cm from the centre).

5.38 At the uppermost point, $N + mg = \frac{mv^2}{R}$, where N is the normal force (downwards) on the motorcyclist by the ceiling of the chamber. The minimum possible speed at the uppermost point corresponds to $N = 0$.

i.e. $v_{\text{min}} \quad \sqrt{Rg} \quad \sqrt{25 \quad 10} \quad 16 \text{ m s}^{-1}$

5.39 The horizontal force N by the wall on the man provides the needed centripetal force : $N = m R \omega^2$. The frictional force f (vertically upwards) opposes the weight mg. The man remains stuck to the wall after the floor is removed if $mg = f < \mu N$ i.e. $mg < \mu m R \omega^2$. The minimum angular speed of rotation of the cylinder is $\omega_{min} = \sqrt{g/\mu R} \;=\; \quad 5 \text{ s}^{-1}$

5.40 Consider the free-body diagram of the bead when the radius vector joining the centre of the wire makes an angle θ with the vertical downward direction. We have $mg = N\cos\theta$ and $mR\sin\theta\ \omega^2 = N\sin\theta$. These equations give $\cos\theta = g/R\omega^2$. Since $\cos\theta \le 1$, the bead remains at its lowermost point for $\omega \le \sqrt{\frac{g}{R}}$.

For $\boldsymbol{\omega} = \sqrt{\frac{2g}{R}}$, $\cos\theta = \frac{1}{2}$ i.e. $\theta = 60^0$.

Chapter 6

6.1 (a) +ve (b) –ve (c) –ve (d) + ve (e) – ve

6.2 (a) 882 J ; (b) –247 J; (c) 635 J ; (d) 635 J;
Work done by the net force on a body equals change in its kinetic energy.

6.3 (a) $x > a$; 0 (c) $x < a$, $x > b$; $-V_1$
(b) $-\infty < x < \infty$; V_1 (d) $-b/2 < x < -a/2$, $a/2 < x < b/2$; $-V_1$

6.5 (a) rocket; (b) For a conservative force work done over a path is minus of change in potential energy. Over a complete orbit, there is no change in potential energy; (c) K.E. increases, but P.E. decreases, and the sum decreases due to dissipation against friction; (d) in the second case.

6.6 (a) decrease; (b) kinetic energy; (c) external force; (d) total linear momentum, and also total energy (if the system of two bodies is isolated).

6.7 (a) F ; (b) F ; (c) F ; (d) F (true usually but not always, why?)

6.8 (a) No
(b) Yes
(c) Linear momentum is conserved during an inelastic collision, kinetic energy is, of course, not conserved even after the collision is over.
(d) elastic.

6.9 (b) t

6.10 (c)

6.11 12 J

6.12 The electron is faster, $v_e / v_p = 13.5$

6.13 0.082 J in each half ; – 0.163 J

6.14 Yes, momentum of the molecule + wall system is conserved. The wall has a recoil momentum such that the momentum of the wall + momentum of the outgoing molecule equals momentum of the incoming molecule, assuming the wall to be stationary initially. However, the recoil momentum produces negligible velocity because of the large mass of the wall. Since kinetic energy is also conserved, the collision is elastic.

6.15 43.6 kW

6.16 (b)

6.17 It transfers its entire momentum to the ball on the table, and does not rise at all.

6.18 5.3 m s^{-1}

6.19 27 km h^{-1} (no change in speed)

6.20 50 J

6.21 (a) $m \quad Avt$ (b) $K \quad Av^3 t/2$ (c) $P = 4.5$ kW

6.22 (a) 49,000 J (b) 6.45 10^{-3} kg

6.23 (a) 200 m^2(b) comparable to the roof of a large house of dimension 14m × 14m.

6.24 21.2 cm, 28.5 J

6.25 No, the stone on the steep plane reaches the bottom earlier; yes, they reach with the same speed v, [since $mgh = (1/2)\, m v^2$]

$v_B = v_C = 14.1$ m s^{-1} , $t_B = 2\sqrt{2}$ s , $t_C = 2\sqrt{2}$ s

6.26 0.125

6.27 8.82 J for both cases.

6.28 The child gives an impulse to the trolley at the start and then runs with a constant relative velocity of 4 m s^{-1} with respect to the trolley's new velocity. Apply momentum conservation for an observer outside. 10.36 m s^{-1}, 25.9 m.

6.29 All except (V) are impossible.

Chapter 7

7.1 The geometrical centre of each. No, the CM may lie outside the body, as in case of a ring, a hollow sphere, a hollow cylinder, a hollow cube etc.

7.2 Located on the line joining H and C1 nuclei at a distance of 1.24 Å from the H end.

7.3 The speed of the CM of the (trolley + child) system remains unchanged (equal to v) because no external force acts on the system. The forces involved in running on the trolley are internal to this system.

7.6 $l_z = xp_y - yp_x$, $l_x = yp_z - zp_y$, $l_y = zp_x - xp_z$

7.8 72 cm

7.9 3675 N on each front wheel, 5145 N on each back wheel.

7.10 (a) 7/5 MR^2 (b) 3/2 MR^2

7.11 Sphere

7.12 Kinetic Energy = 3125 J; Angular Momentum = 62.5 J s

7.13 (a) 100 rev/min (use angular momentum conservation).

(b) The new kinetic energy is 2.5 times the initial kinetic energy of rotation. The child uses his internal energy to increase his rotational kinetic energy.

7.14 $25\ s^{-2}$; $10\ m\ s^{-2}$

7.15 36 kW

7.16 at R/6 from the center of original disc opposite to the center of cut portion.

7.17 66.0 g

7.18 (a) Yes; (b) Yes, (c) the plane with smaller inclination ($\because a \propto \sin\theta$)

7.19 4J

7.20 6.75×10^{12} rad s^{-1}

7.21 (a) 3.8 m (b) 3.0 s

7.22 Tension = 98 N, N_B = 245 N, N_C = 147 N.

7.23 (a) 59 rev/min, (b) No, the K.E. is increased and it comes from work done by man in the process.

7.24 0.625 rad s^{-1}

7.27 (a) By angular momentum conservation, the common angular speed

$\omega = (I_1\,\omega_1 + I_2\,\omega_2)\ /\ (I_1 + I_2)$

(b) The loss is due to energy dissipation in frictional contact which brings the two discs to a common angular speed ω. However, since frictional torques are internal to the system, angular momentum is unaltered.

7.28 Velocity of A = $\omega_o R$ in the same direction as the arrow; velocity of B = $\omega_o R$ in the opposite direction to the arrow; velocity of C = $\omega_o R/2$ in the same direction as the arrow. The disc will not roll on a frictionless plane.

7.29 (a) Frictional force at B opposes velocity of B. Therefore, frictional force is in the same direction as the arrow. The sense of frictional torque is such as to oppose angular motion. $\boldsymbol{\omega}_o$ and $\boldsymbol{\tau}$ are both normal to the paper, the first into the paper, and the second coming out of the paper.

(b) Frictional force decreases the velocity of the point of contact B. Perfect rolling ensues when this velocity is zero. Once this is so, the force of friction is zero.

7.30 Frictional force causes the CM to accelerate from its initial zero velocity. Frictional torque causes retardation in the initial angular speed ω_o. The equations of motion are : $\mu_k\, m g = m a$ and $\mu_k\, m g R = -I\alpha$, which yield $v = \mu_k\, g\, t$, $\omega = \omega_o - \mu_k\, m g R t / I$. Rolling begins when $v = R\,\omega$. For a ring, $I = m R^2$, and rolling begins at $t = \omega_o\ R/2\,\mu_k g$. For a disc, $I = \frac{1}{2}\, m R^2$ and rolling starts at break line $t = R\,\omega_o/3\,\mu_k\, g$. Thus, the disc begins to roll earlier than the ring, for the same R and ω_o. The actual times can be obtained for R = 10 cm, $\omega_o = 10\,\pi$ rad s^{-1}, $\mu_k = 0.2$

7.31 (a) 16.4 N

(b) Zero

(c) 37° approx.

Chapter 8

8.1 (a) No.

(b) Yes, if the size of the space ship is large enough for him to detect the variation in g.

(c) Tidal effect depends inversely on the cube of the distance unlike force, which depends inversely on the square of the distance.

8.2 (a) decreases; (b) decreases; (c) mass of the body; (d) more.

8.3 Smaller by a factor of 0.63.

8.5 3.54×10^8 years.

8.6 (a) Kinetic energy, (b) less,

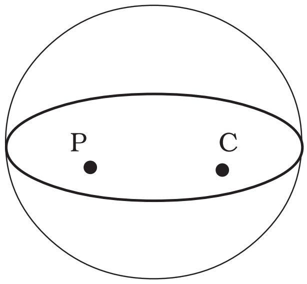
8.7 (a) No, (b) No, (c) No, (d) Yes

[The escape velocity is independent of mass of the body and the direction of projection. It depends upon the gravitational potential at the point from where the body is launched. Since this potential depends (slightly) on the latitude and height of the point, the escape velocity (speed) depends (slightly) on these factors.]

8.8 All quantities vary over an orbit except angular momentum and total energy.

8.9 (b), (c) and (d)

8.10 and **8.11** For these two problems, complete the hemisphere to sphere. At both P, and C, potential is constant and hence intensity = 0. Therefore, for the hemisphere, (c) and (e) are correct.



8.12 2.6×10^8 m

8.13 2.0×10^{30} kg

8.14 1.43×10^{12} m

8.15 28 N

8.16 125 N

8.17 8.0×10^6 m from the earth's centre

8.18 31.7 km/s

8.19 5.9×10^9 J

8.20 2.6×10^{6} m/s

8.21 0, 2.7×10^{-8} J/kg; an object placed at the mid point is in an unstable equilibrium

8.22 -9.4×10^{6} J/kg

8.23 $GM / R^2 = 2.3 \times 10^{12}$ m s^{-2} , $\omega^2 R = 1.1 \times 10^{6}$ m s^{-2}; here ω is the angular speed of rotation. Thus in the rotating frame of the star, the inward force is much greater than the outward centrifugal force at its equator. The object will remain stuck (and not fly off due to centrifugal force). Note, if angular speed of rotation increases say by a factor of 2000, the object will fly off.

8.24 3×10^{11} J

8.25 495 km

Chapter Nine

Mechanical Properties of Solids

9.1 INTRODUCTION

In Chapter 7, we studied the rotation of the bodies and then realised that the motion of a body depends on how mass is distributed within the body. We restricted ourselves to simpler situations of rigid bodies. A rigid body generally means a hard solid object having a definite shape and size. But in reality, bodies can be stretched, compressed and bent. Even the appreciably rigid steel bar can be deformed when a sufficiently large external force is applied on it. This means that solid bodies are not perfectly rigid.

A solid has definite shape and size. In order to change (or deform) the shape or size of a body, a force is required. If you stretch a helical spring by gently pulling its ends, the length of the spring increases slightly. When you leave the ends of the spring, it regains its original size and shape. The property of a body, by virtue of which it tends to regain its original size and shape when the applied force is removed, is known as **elasticity** and the deformation caused is known as **elastic** deformation. However, if you apply force to a lump of putty or mud, they have no gross tendency to regain their previous shape, and they get permanently deformed. Such substances are called **plastic** and this property is called **plasticity.** Putty and mud are close to ideal plastics.

The elastic behaviour of materials plays an important role in engineering design. For example, while designing a building, knowledge of elastic properties of materials like steel, concrete etc. is essential. The same is true in the design of bridges, automobiles, ropeways etc. One could also ask — Can we design an aeroplane which is very light but sufficiently strong? Can we design an artificial limb which is lighter but stronger? Why does a railway track have a particular shape like **I**? Why is glass brittle while brass is not? Answers to such questions begin with the study of how relatively simple kinds of loads or forces act to deform different solids bodies. In this chapter, we shall study the

elastic behaviour and mechanical properties of solids which would answer many such questions.

9.2 ELASTIC BEHAVIOUR OF SOLIDS

We know that in a solid, each atom or molecule is surrounded by neighbouring atoms or molecules. These are bonded together by interatomic or intermolecular forces and stay in a stable equilibrium position. When a solid is deformed, the atoms or molecules are displaced from their equilibrium positions causing a change in the interatomic (or intermolecular) distances. When the deforming force is removed, the interatomic forces tend to drive them back to their original positions. Thus the body regains its original shape and size. The restoring mechanism can be visualised by taking a model of spring-ball system shown in the Fig. 9.1. Here the balls represent atoms and springs represent interatomic forces.

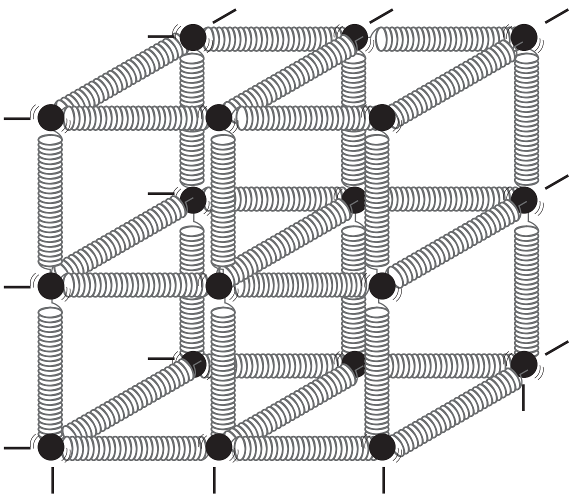


Fig. 9.1 *Spring-ball model for the illustration of elastic behaviour of solids.*

If you try to displace any ball from its equilibrium position, the spring system tries to restore the ball back to its original position. Thus elastic behaviour of solids can be explained in terms of microscopic nature of the solid. Robert Hooke, an English physicist (1635 - 1703 A.D) performed experiments on springs and found that the elongation (change in the length) produced in a body is proportional to the applied force or load. In 1676, he presented his law of elasticity, now called Hooke's law. We shall study about it in Section 9.4. This law, like Boyle's law, is one of the earliest quantitative relationships in science. It is very important to know the behaviour of the materials under various kinds of load from the context of engineering design.

9.3 STRESS AND STRAIN

When a force is applied on body, it is deformed to a small or large extent depending upon the nature of the material of the body and the magnitude of the deforming force. The deformation may not be noticeable visually in many materials but it is there. When a body is subjected to a deforming force, a restoring force is developed in the body. This restoring force is equal in magnitude but opposite in direction to the applied force. The restoring force per unit area is known as **stress.** If F is the force applied and A is the area of cross section of the body,

Magnitude of the stress = F/A (9.1)

The SI unit of stress is N m^{-2} or pascal (Pa) and its dimensional formula is [$ML^{-1}T^{-2}$].

There are three ways in which a solid may change its dimensions when an external force acts on it. These are shown in Fig. 9.2. In Fig.9.2(a), a cylinder is stretched by two equal forces applied normal to its cross-sectional area. The restoring force per unit area in this case is called **tensile stress**. If the cylinder is compressed under the action of applied forces, the restoring force per unit area is known as **compressive stress**. Tensile or compressive stress can also be termed as longitudinal stress.

In both the cases, there is a change in the length of the cylinder. The change in the length ΔL to the original length L of the body (cylinder in this case) is known as **longitudinal strain**.

$$\text{Longitudinal strain} \quad \frac{L}{L} \qquad (9.2)$$

However, if two equal and opposite deforming forces are applied parallel to the cross-sectional area of the cylinder, as shown in Fig. 9.2(b), there is relative displacement between the opposite faces of the cylinder. The restoring force per unit area developed due to the applied tangential force is known as **tangential** or **shearing stress**.

Robert Hooke
(1635 – 1703 A.D.)

Robert Hooke was born on July 18, 1635 in Freshwater, Isle of Wight. He was one of the most brilliant and versatile seventeenth century English scientists. He attended Oxford University but never graduated. Yet he was an extremely talented inventor, instrument-maker and building designer. He assisted Robert Boyle in the construction of Boylean air pump. In 1662, he was appointed as Curator of Experiments to the newly founded Royal Society. In 1665, he became Professor of Geometry in Gresham College where he carried out his astronomical observations. He built a Gregorian reflecting telescope; discovered the fifth star in the trapezium and an asterism in the constellation Orion; suggested that Jupiter rotates on its axis; plotted detailed sketches of Mars which were later used in the 19^{th} century to determine the planet's rate of rotation; stated the inverse square law to describe planetary motion, which Newton modified later etc. He was elected Fellow of Royal Society and also served as the Society's Secretary from 1667 to 1682. In his series of observations presented in Micrographia, he suggested wave theory of light and first used the word 'cell' in a biological context as a result of his studies of cork.

Robert Hooke is best known to physicists for his discovery of law of elasticity: **Ut tensio, sic vis** (This is a Latin expression and it means as the distortion, so the force). This law laid the basis for studies of stress and strain and for understanding the elastic materials.

As a result of applied tangential force, there is a relative displacement Δx between opposite faces of the cylinder as shown in the Fig. 9.2(b). The strain so produced is known as **shearing strain** and it is defined as the ratio of relative displacement of the faces Δx to the length of the cylinder L.

$$\text{Shearing strain} \quad \frac{x}{L} = \tan\theta \qquad (9.3)$$

where θ is the angular displacement of the cylinder from the vertical (original position of the cylinder). Usually θ is very small, tan θ is nearly equal to angle θ, (if $\theta = 10°$, for example, there is only 1% difference between θ and tan θ).

It can also be visualised, when a book is pressed with the hand and pushed horizontally, as shown in Fig. 9.2 (c).

$$\text{Thus, shearing strain} = \tan\theta \approx \theta \qquad (9.4)$$

In Fig. 9.2 (d), a solid sphere placed in the fluid under high pressure is compressed uniformly on all sides. The force applied by the fluid acts in perpendicular direction at each point of the surface and the body is said to be under hydraulic compression. This leads to decrease in its volume without any change of its geometrical shape.

The body develops internal restoring forces that are equal and opposite to the forces applied by the fluid (the body restores its original shape and size when taken out from the fluid). The internal restoring force per unit area in this case

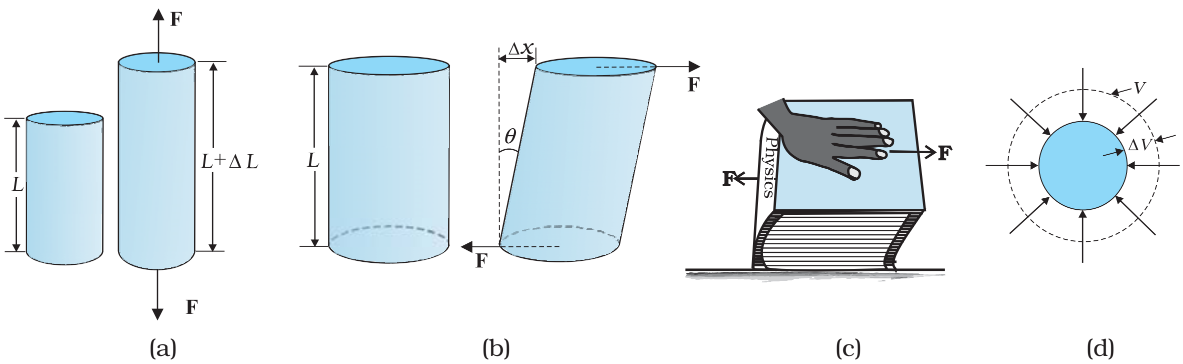


Fig. 9.2 *(a) Cylinder subjected to tensile stress stretches it by an amount ΔL. (b) A cylinder subjected to shearing (tangential) stress deforms by an angle θ .(c) A book subjected to a shearing stress (d) A solid sphere subjected to a uniform hydraulic stress shrinks in volume by an amount ΔV.*

is known as **hydraulic stress** and in magnitude is equal to the hydraulic pressure (applied force per unit area).

The strain produced by a hydraulic pressure is called **volume strain** and is defined as the ratio of change in volume (ΔV) to the original volume (V).

$$\text{Volume strain} \quad \frac{V}{V} \qquad (9.5)$$

Since the strain is a ratio of change in dimension to the original dimension, it has no units or dimensional formula.

9.4 HOOKE'S LAW

Stress and strain take different forms in the situations depicted in the Fig. (9.2). For small deformations the stress and strain are proportional to each other. This is known as Hooke's law.

Thus,

stress $\propto$ strain

$$\text{stress} = k \times \text{strain} \qquad (9.6)$$

where k is the proportionality constant and is known as modulus of elasticity.

Hooke's law is an empirical law and is found to be valid for most materials. However, there are some materials which do not exhibits this linear relationship.

9.5 STRESS-STRAIN CURVE

The relation between the stress and the strain for a given material under tensile stress can be found experimentally. In a standard test of tensile properties, a test cylinder or a wire is stretched by an applied force. The fractional change in length (the strain) and the applied force needed to cause the strain are recorded. The applied force is gradually increased in steps and the change in length is noted. A graph is plotted between the stress (which is equal in magnitude to the applied force per unit area) and the strain produced. A typical graph for a metal is shown in Fig. 9.3. Analogous graphs for compression and shear stress may also be obtained. The stress-strain curves vary from material to material. These curves help us to understand how a given material deforms with increasing loads. From the graph, we can see that in the region between O to A, the curve is linear. In this region, Hooke's law is obeyed. The body regains its original dimensions when the applied force is removed. In this region, the solid behaves as an elastic body.

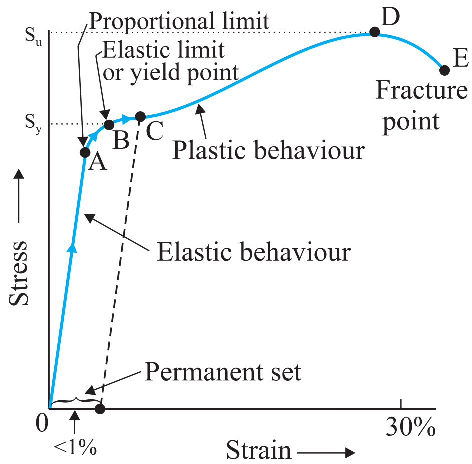


Fig. 9.3 *A typical stress-strain curve for a metal.*

In the region from A to B, stress and strain are not proportional. Nevertheless, the body still returns to its original dimension when the load is removed. The point B in the curve is known as **yield point** (also known as **elastic limit**) and the corresponding stress is known as **yield strength** (S_y) of the material.

If the load is increased further, the stress developed exceeds the yield strength and strain increases rapidly even for a small change in the stress. The portion of the curve between B and D shows this. When the load is removed, say at some point C between B and D, the body does not regain its original dimension. In this case, even when the stress is zero, the strain is not zero. The material is said to have a **permanent set**. The deformation is said to be **plastic deformation**. The point D on the graph is the ultimate **tensile strength** (S_u) of the material. Beyond this point, additional strain is produced even by a reduced applied force and fracture occurs at point E. If the ultimate strength and fracture points D and E are close, the material is said to be brittle. If they are far apart, the material is said to be ductile.

As stated earlier, the stress-strain behaviour varies from material to material. For example, rubber can be pulled to several times its original length and still returns to its original shape. Fig. 9.4 shows stress-strain curve for the elastic tissue of aorta, present in the heart. Note that although elastic region is very large, the material

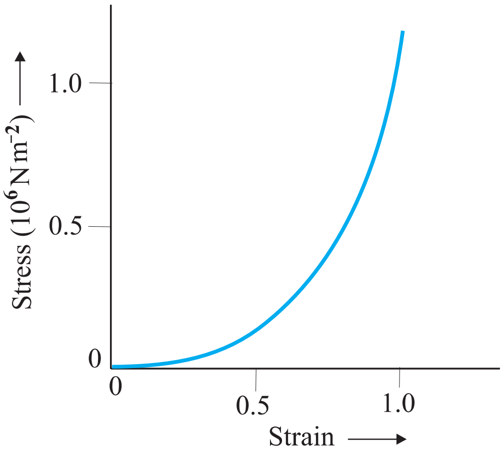


Fig. 9.4 *Stress-strain curve for the elastic tissue of Aorta, the large tube (vessel) carrying blood from the heart.*

does not obey Hooke's law over most of the region. Secondly, there is no well defined plastic region. Substances like tissue of aorta, rubber etc. which can be stretched to cause large strains are called **elastomers**.

9.6 ELASTIC MODULI

The proportional region within the elastic limit of the stress-strain curve (region OA in Fig. 9.3) is of great importance for structural and manufacturing engineering designs. The ratio of stress and strain, called **modulus of elasticity**, is found to be a characteristic of the material.

9.6.1 Young's Modulus

Experimental observation show that for a given material, the magnitude of the strain produced is same whether the stress is tensile or compressive. The ratio of tensile (or compressive) stress (σ) to the longitudinal strain (ε) is defined as **Young's modulus** and is denoted by the symbol Y.

$$Y = — \qquad (9.7)$$

From Eqs. (9.1) and (9.2), we have

$$Y = (F/A)/(\Delta L/L)$$
$$= (F \times L)/(A \times \Delta L) \qquad (9.8)$$

Since strain is a dimensionless quantity, the unit of Young's modulus is the same as that of stress *i.e.*, N m^{-2} or Pascal (Pa). Table 9.1 gives the values of Young's moduli and yield strengths of some materials.

From the data given in Table 9.1, it is noticed that for metals Young's moduli are large. Therefore, these materials require a large force to produce small change in length. To increase the length of a thin steel wire of 0.1 cm^2 cross-sectional area by 0.1%, a force of 2000 N is required. The force required to produce the same strain in aluminium, brass and copper wires having the same cross-sectional area are 690 N, 900 N and 1100 N respectively. It means that steel is more elastic than copper, brass and aluminium. It is for this reason that steel is

Table 9.1 **Young's moduli and yield strengths of some materials.**

Substance	Density ρ (kg m^{-3})	Young's modulus Y (10^9 N m^{-2})	Ultimate strength, S_u (10^6 N m^{-2})	Yield strength S_y (10^6 N m^{-2})
Aluminium	2710	70	110	95
Copper	8890	110	400	200
Iron (wrought)	7800-7900	190	330	170
Steel	7860	200	400	250
Glass#	2190	65	50	—
Concrete	2320	30	40	—
Wood#	525	13	50	—
Bone#	1900	9	170	—
Polystyrene	1050	3	48	—

 # substance tested under compression

preferred in heavy-duty machines and in structural designs. Wood, bone, concrete and glass have rather small Young's moduli.

Example 9.1 A structural steel rod has a radius of 10 mm and a length of 1.0 m. A 100 kN force stretches it along its length. Calculate (a) stress, (b) elongation, and (c) strain on the rod. Young's modulus, of structural steel is 2.0×10^{11} N m^{-2}.

Answer We assume that the rod is held by a clamp at one end, and the force F is applied at the other end, parallel to the length of the rod. Then the stress on the rod is given by

$$Stress = \frac{F}{A} = \frac{F}{\pi r^2}$$

$$\frac{100 \quad 10^{3}\,\mathrm{N}}{3.14 \quad 10^{2}\,\mathrm{m}^{2}}$$

$$= 3.18 \times 10^{8}\ \mathrm{N\ m^{-2}}$$

The elongation,

$$\Delta L = \frac{(F/A)L}{Y}$$

$$= \frac{3.18 \quad 10^{8}\,\mathrm{N\,m^{-2}} \quad 1\mathrm{m}}{2 \quad 10^{11}\,\mathrm{N\,m^{-2}}}$$

$$= 1.59 \times 10^{-3}\ \mathrm{m}$$

$$= 1.59\ \mathrm{mm}$$

The strain is given by

$$\text{Strain} = \Delta L/L$$

$$= (1.59 \times 10^{-3}\ \mathrm{m})/(1\mathrm{m})$$

$$= 1.59 \times 10^{-3}$$

$$= 0.16\ \%$$

Example 9.2 A copper wire of length 2.2 m and a steel wire of length 1.6 m, both of diameter 3.0 mm, are connected end to end. When stretched by a load, the net elongation is found to be 0.70 mm. Obtain the load applied.

Answer The copper and steel wires are under a tensile stress because they have the same tension (equal to the load W) and the same area of cross-section A. From Eq. (9.7) we have stress = strain × Young's modulus. Therefore

$$W/A = Y_c \times (\Delta L_c/L_c) = Y_s \times (\Delta L_s/L_s)$$

where the subscripts c and s refer to copper and stainless steel respectively. Or,

$$\Delta L_c/\Delta L_s = (Y_s/Y_c) \times (L_c/L_s)$$

Given $L_c = 2.2$ m, $L_s = 1.6$ m,

From Table 9.1 $Y_c = 1.1 \times 10^{11}$ N.m^{-2}, and

$Y_s = 2.0 \times 10^{11}$ N.m^{-2}.

$$\Delta L_c/\Delta L_s = (2.0 \times 10^{11}/1.1 \times 10^{11}) \times (2.2/1.6) = 2.5.$$

The total elongation is given to be

$$\Delta L_c + \Delta L_s = 7.0 \times 10^{-4}\ \mathrm{m}$$

Solving the above equations,

$$\Delta L_c = 5.0 \times 10^{-4}\ \mathrm{m}, \quad \text{and} \quad \Delta L_s = 2.0 \times 10^{-4}\ \mathrm{m}.$$

Therefore

$$W = (A \times Y_c \times \Delta L_c)/L_c$$

$$= \pi (1.5 \times 10^{-3})^2 \times [(5.0 \times 10^{-4} \times 1.1 \times 10^{11})/2.2]$$

$$= 1.8 \times 10^{2}\ \mathrm{N}$$

Example 9.3 In a human pyramid in a circus, the entire weight of the balanced group is supported by the legs of a performer who is lying on his back (as shown in Fig. 9.5). The combined mass of all the persons performing the act, and the tables, plaques etc. involved is 280 kg. The mass of the performer lying on his back at the bottom of the pyramid is 60 kg. Each thighbone (femur) of this performer has a length of 50 cm and an effective radius of 2.0 cm. Determine the amount by which each thighbone gets compressed under the extra load.

Fig. 9.5 *Human pyramid in a circus.*

Answer Total mass of all the performers, tables, plaques etc. = 280 kg

Mass of the performer = 60 kg

Mass supported by the legs of the performer at the bottom of the pyramid

= 280 – 60 = 220 kg

Weight of this supported mass

= 220 kg wt. = 220 × 9.8 N = 2156 N.

Weight supported by each thighbone of the performer = ½ (2156) N = 1078 N.

From Table 9.1, the Young's modulus for bone is given by

$Y = 9.4 \times 10^9$ N m^{-2}.

Length of each thighbone $L = 0.5$ m

the radius of thighbone = 2.0 cm

Thus the cross-sectional area of the thighbone

$A = \pi \times (2 \times 10^{-2})^2$ m^2 = 1.26×10^{-3} m^2.

Using Eq. (9.8), the compression in each thighbone (ΔL) can be computed as

$\Delta L = [(F \times L)/(Y \times A)]$

$= [(1078 \times 0.5)/(9.4 \times 10^9 \times 1.26 \times 10^{-3})]$

$= 4.55 \times 10^{-5}$ m or 4.55×10^{-3} cm.

This is a very small change! The fractional decrease in the thighbone is $\Delta L/L = 0.000091$ or 0.0091%. t

9.6.2 Determination of Young's Modulus of the Material of a Wire

A typical experimental arrangement to determine the Young's modulus of a material of wire under tension is shown in Fig. 9.6. It consists of two long straight wires of same length and equal radius suspended side by side from a fixed rigid support. The wire A (called the *reference wire*) carries a millimetre main scale M and a pan to place a weight. The wire B (called the *experimental wire*) of uniform area of cross-section also carries a pan in which known weights can be placed. A vernier scale V is attached to a pointer at the bottom of the experimental wire B, and the main scale M is fixed to the reference wire A. The weights placed in the pan exert a downward force and stretch the experimental wire under a tensile stress. The elongation of the wire (increase in length) is measured by the vernier arrangement. The reference wire is used to compensate for any change in length that may occur due to change in room temperature, since any change in length of the reference wire due to temperature change will be accompanied by an equal change in experimental wire. (We shall study these temperature effects in detail in Chapter 11.)

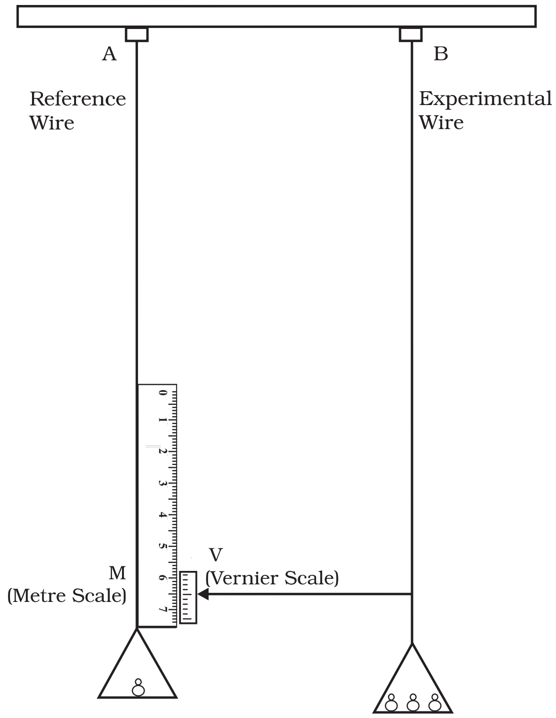


Fig. 9.6 *An arrangement for the determination of Young's modulus of the material of a wire.*

Both the reference and experimental wires are given an initial small load to keep the wires straight and the vernier reading is noted. Now the experimental wire is gradually loaded with more weights to bring it under a tensile stress and the vernier reading is noted again. The difference between two vernier readings gives the elongation produced in the wire. Let r and L be the initial radius and length of the experimental wire, respectively. Then the area of cross-section of the wire would be πr^2. Let M be the mass that produced an elongation ΔL in the wire. Thus the applied force is equal to Mg, where g is the acceleration due to gravity. From Eq. (9.8), the Young's modulus of the material of the experimental wire is given by

$$Y = \frac{\sigma}{\varepsilon} = \frac{Mg}{\pi r^2} \cdot \frac{L}{\Delta L}$$

$$= Mg \times L/(\pi r^2 \times \Delta L) \quad (9.9)$$

9.6.3 Shear Modulus

The ratio of shearing stress to the corresponding shearing strain is called the *shear modulus* of the material and is represented by G. It is also called the *modulus of rigidity*.

G = shearing stress (σ_s)/shearing strain

$G = (F/A)/(\Delta x/L)$

$= (F \times L)/(A \times \Delta x)$ (9.10)

Similarly, from Eq. (9.4)

$G = (F/A)/\theta$

$= F/(A \times \theta)$ (9.11)

The shearing stress σ_s can also be expressed as

$\sigma_s = G \times \theta$ (9.12)

SI unit of shear modulus is N m^{-2} or Pa. The shear moduli of a few common materials are given in Table 9.2. It can be seen that shear modulus (or modulus of rigidity) is generally less than Young's modulus (from Table 9.1). For most materials $G \approx Y/3$.

Table 9.2 Shear moduli *(G)* of some common materials

Material	G (10^9 Nm^{-2} or GPa)
Aluminium	25
Brass	36
Copper	42
Glass	23
Iron	70
Lead	5.6
Nickel	77
Steel	84
Tungsten	150
Wood	10

Example 9.4 A square lead slab of side 50 cm and thickness 10 cm is subject to a shearing force (on its narrow face) of 9.0×10^4 N. The lower edge is riveted to the floor. How much will the upper edge be displaced?

Answer The lead slab is fixed and the force is applied parallel to the narrow face as shown in Fig. 9.7. The area of the face parallel to which this force is applied is

A = 50 cm × 10 cm

= 0.5 m × 0.1 m

= 0.05 m^2

Therefore, the stress applied is

= $(9.4 \times 10^4$ N/0.05 $m^2)$

= 1.80×10^6 N.m^{-2}

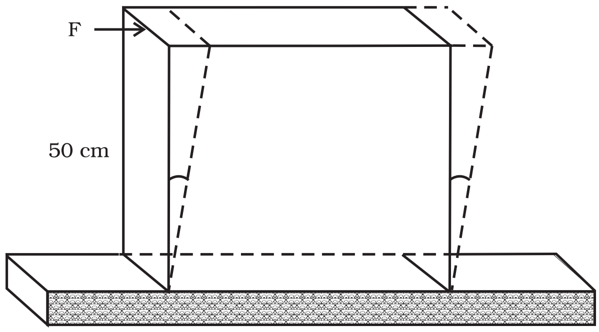


Fig. 9.7

We know that shearing strain = $(\Delta x/L)$= Stress /G.

Therefore the displacement Δx = (Stress × L)/G

= $(1.8\times 10^6$ N $m^{-2} \times 0.5$m$)/(5.6 \times 10^9$ N $m^{-2})$

= 1.6×10^{-4} m = 0.16 mm

9.6.4 Bulk Modulus

In Section (9.3), we have seen that when a body is submerged in a fluid, it undergoes a hydraulic stress (equal in magnitude to the hydraulic pressure). This leads to the decrease in the volume of the body thus producing a strain called volume strain [Eq. (9.5)]. The ratio of hydraulic stress to the corresponding hydraulic strain is called *bulk modulus*. It is denoted by symbol B.

$B = - p/(\Delta V/V)$ (9.13)

The negative sign indicates the fact that with an increase in pressure, a decrease in volume occurs. That is, if p is positive, ΔV is negative. Thus for a system in equilibrium, the value of bulk modulus B is always positive. SI unit of bulk modulus is the same as that of pressure *i.e.*, N m^{-2} or Pa. The bulk moduli of a few common materials are given in Table 9.3.

The reciprocal of the bulk modulus is called *compressibility* and is denoted by k. It is defined as the fractional change in volume per unit increase in pressure.

$k = (1/B) = -(1/\Delta p) \times (\Delta V/V)$ (9.14)

It can be seen from the data given in Table 9.3 that the bulk moduli for solids are much larger than for liquids, which are again much larger than the bulk modulus for gases (air).

Table 9.3 **Bulk moduli (B) of some common Materials**

Material	B (10^9 N m^{-2} or GPa)
Solids	
Aluminium	72
Brass	61
Copper	140
Glass	37
Iron	100
Nickel	260
Steel	160
Liquids	
Water	2.2
Ethanol	0.9
Carbon disulphide	1.56
Glycerine	4.76
Mercury	25
Gases	
Air (at STP)	1.0×10^{-4}

Thus solids are least compressible whereas gases are most compressible. Gases are about a million times more compressible than solids! Gases have large compressibilities, which vary with pressure and temperature. The incompressibility of the solids is primarily due to the tight coupling between the neighbouring atoms. The molecules in liquids are also bound with their neighbours but not as strong as in solids. Molecules in gases are very poorly coupled to their neighbours.

Table 9.4 shows the various types of stress, strain, elastic moduli, and the applicable state of matter at a glance.

Example 9.5 The average depth of Indian Ocean is about 3000 m. Calculate the fractional compression, $\Delta V/V$, of water at the bottom of the ocean, given that the bulk modulus of water is 2.2×10^9 N m^{-2}. (Take g = 10 m s^{-2})

Answer The pressure exerted by a 3000 m column of water on the bottom layer

$$p = h\rho\, g = 3000 \text{ m} \times 1000 \text{ kg m}^{-3} \times 10 \text{ m s}^{-2}$$
$$= 3 \times 10^7 \text{ kg m}^{-1} \text{ s}^{-2}$$
$$= 3 \times 10^7 \text{ N m}^{-2}$$

Fractional compression $\Delta V/V$, is

$$\Delta V/V = \text{stress}/B = (3\times 10^7 \text{ N m}^{-2})/(2.2\times 10^9 \text{ N m}^{-2})$$
$$= 1.36 \times 10^{-2} \text{ or } 1.36\ \%$$

Table 9.4 **Stress, strain and various elastic moduli**

Type of stress	Stress	Strain	Change in shape	Change in volume	Elastic modulus	Name of modulus	State of Mater
Tensile or compressive	Two equal and opposite forces perpendicular to opposite faces ($\sigma = F/A$)	Elongation or compression parallel to force direction ($\Delta L/L$) (longitudinal strain)	Yes	No	$Y = (F\times L)/(A\times\Delta L)$	Young's modulus	Solid
Shearing	Two equal and opposite forces parallel to oppoiste surfaces [forces in each case such that total force and total torque on the body vanishes ($\sigma_s = F/A$)	Pure shear, θ	Yes	No	$G = (F\times\theta)/A$	Shear modulus	Solid
Hydraulic	Forces perpendicular everywhere to the surface, force per unit area (pressure) same everywhere.	Volume change (compression or elongation ($\Delta V/V$)	No	Yes	$B = -p/(\Delta V/V)$	Bulk modulus	Solid, liquid and gas

9.7 APPLICATIONS OF ELASTIC BEHAVIOUR OF MATERIALS

The elastic behaviour of materials plays an important role in everyday life. All engineering designs require precise knowledge of the elastic behaviour of materials. For example while designing a building, the structural design of the columns, beams and supports require knowledge of strength of materials used. Have you ever thought why the beams used in construction of bridges, as supports etc. have a cross-section of the type **I**? Why does a heap of sand or a hill have a pyramidal shape? Answers to these questions can be obtained from the study of structural engineering which is based on concepts developed here.

Cranes used for lifting and moving heavy loads from one place to another have a thick metal rope to which the load is attached. The rope is pulled up using pulleys and motors. Suppose we want to make a crane, which has a lifting capacity of 10 tonnes or metric tons (1 metric ton = 1000 kg). How thick should the steel rope be? We obviously want that the load does not deform the rope permanently. Therefore, the extension should not exceed the elastic limit. From Table 9.1, we find that mild steel has a yield strength (S_y) of about 300×10^6 N m^{-2}. Thus, the area of cross-section (A) of the rope should at least be

$$A \geq W/S_y = Mg/S_y \qquad (9.15)$$
$$= (10^4 \text{ kg} \times 10 \text{ m s}^{-2})/(300 \times 10^6 \text{ N m}^{-2})$$
$$= 3.3 \times 10^{-4} \text{ m}^2$$

corresponding to a radius of about 1 cm for a rope of circular cross-section. Generally a large margin of safety (of about a factor of ten in the load) is provided. Thus a thicker rope of radius about 3 cm is recommended. A single wire of this radius would practically be a rigid rod. So the ropes are always made of a number of thin wires braided together, like in pigtails, for ease in manufacture, flexibility and strength.

A bridge has to be designed such that it can withstand the load of the flowing traffic, the force of winds and its own weight. Similarly, in the design of buildings use of beams and columns is very common. In both the cases, the overcoming of the problem of bending of beam under a load is of prime importance. The beam should not bend too much or break. Let us consider the case of a beam loaded at the centre and supported near its ends as shown in Fig. 9.8. A bar of length l, breadth b, and depth d when loaded at the centre by a load W sags by an amount given by

$$\delta = W\, l^3/(4bd^3Y) \qquad (9.16)$$

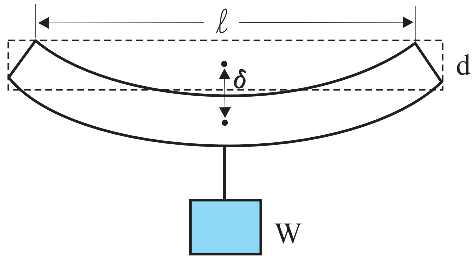


Fig. 9.8 *A beam supported at the ends and loaded at the centre.*

This relation can be derived using what you have already learnt and a little calculus. From Eq. (9.16), we see that to reduce the bending for a given load, one should use a material with a large Young's modulus Y. For a given material, increasing the depth d rather than the breadth b is more effective in reducing the bending, since δ is proportional to d^{-3} and only to b^{-1}(of course the length l of the span should be as small as possible). But on increasing the depth, unless the load is exactly at the right place (difficult to arrange in a bridge with moving traffic), the deep bar may bend as shown in Fig. 9.9(b). This is called buckling. To avoid this, a common compromise is the cross-sectional shape shown in Fig. 9.9(c). This section provides a large load-bearing surface and enough depth to prevent bending. This shape reduces the weight of the beam without sacrificing the strength and hence reduces the cost.

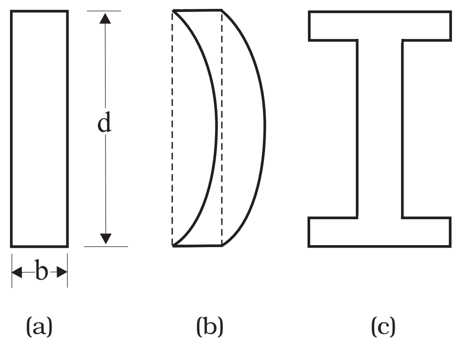


Fig. 9.9 *Different cross-sectional shapes of a beam. (a) Rectangular section of a bar; (b) A thin bar and how it can buckle; (c) Commonly used section for a load bearing bar.*

Use of pillars or columns is also very common in buildings and bridges. A pillar with rounded ends as shown in Fig. 9.10(a) supports less load than that with a distributed shape at the ends [Fig. 9.10(b)]. The precise design of a bridge or a building has to take into account the conditions under which it will function, the cost and long period, reliability of usable materials etc.

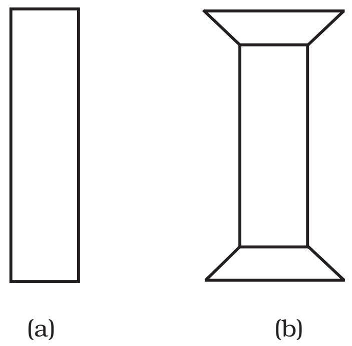


Fig. 9.10 *Pillars or columns: (a) a pillar with rounded ends, (b) Pillar with distributed ends.*

The answer to the question why the maximum height of a mountain on earth is ~10 km can also be provided by considering the elastic properties of rocks. A mountain base is not under uniform compression and this provides some shearing stress to the rocks under which they can flow. The stress due to all the material on the top should be less than the critical shearing stress at which the rocks flow.

At the bottom of a mountain of height h, the force per unit area due to the weight of the mountain is $h\rho g$ where ρ is the density of the material of the mountain and g is the acceleration due to gravity. The material at the bottom experiences this force in the vertical direction, and the sides of the mountain are free. Therefore this is not a case of pressure or bulk compression. There is a shear component, approximately $h\rho g$ itself. Now the elastic limit for a typical rock is 30×10^7 N m^{-2}. Equating this to $h\rho g$, with $\rho = 3 \times 10^3$ kg m^{-3} gives

$h\rho g = 30 \times 10^7$ N m^{-2}. Or

$h = 30 \times 10^7$ N m^{-2}/(3×10^3 kg m$^{-3} \times 10$ m s^{-2})

$= 10$ km

which is more than the height of Mt. Everest!

SUMMARY

1. Stress is the restoring force per unit area and strain is the fractional change in dimension. In general there are three types of stresses (a) tensile stress — longitudinal stress (associated with stretching) or compressive stress (associated with compression), (b) shearing stress, and (c) hydraulic stress.
2. For small deformations, stress is directly proportional to the strain for many materials. This is known as Hooke's law. The constant of proportionality is called modulus of elasticity. Three elastic moduli *viz.*, Young's modulus, shear modulus and bulk modulus are used to describe the elastic behaviour of objects as they respond to deforming forces that act on them.
   A class of solids called elastomers does not obey Hooke's law.
3. When an object is under tension or compression, the Hooke's law takes the form
   $F/A = Y\Delta L/L$
   where $\Delta L/L$ is the tensile or compressive strain of the object, F is the magnitude of the applied force causing the strain, A is the cross-sectional area over which F is applied (perpendicular to A) and Y is the Young's modulus for the object. The stress is F/A.
4. A pair of forces when applied parallel to the upper and lower faces, the solid deforms so that the upper face moves sideways with respect to the lower. The horizontal displacement ΔL of the upper face is perpendicular to the vertical height L. This type of deformation is called shear and the corresponding stress is the shearing stress. This type of stress is possible only in solids.
   In this kind of deformation the Hooke's law takes the form
   $F/A = G \times \Delta L/L$
   where ΔL is the displacement of one end of object in the direction of the applied force F, and G is the shear modulus.

5. When an object undergoes hydraulic compression due to a stress exerted by a surrounding fluid, the Hooke's law takes the form

$$p = B\,(\Delta V/V),$$

where p is the pressure (hydraulic stress) on the object due to the fluid, $\Delta V/V$ (the volume strain) is the absolute fractional change in the object's volume due to that pressure and B is the bulk modulus of the object.

POINTS TO PONDER

1. In the case of a wire, suspended from celing and stretched under the action of a weight (F) suspended from its other end, the force exerted by the ceiling on it is equal and opposite to the weight. However, the tension at any cross-section A of the wire is just F and not $2F$. Hence, tensile stress which is equal to the tension per unit area is equal to F/A.
2. Hooke's law is valid only in the linear part of stress-strain curve.
3. The Young's modulus and shear modulus are relevant only for solids since only solids have lengths and shapes.
4. Bulk modulus is relevant for solids, liquid and gases. It refers to the change in volume when every part of the body is under the uniform stress so that the shape of the body remains unchanged.
5. Metals have larger values of Young's modulus than alloys and elastomers. A material with large value of Young's modulus requires a large force to produce small changes in its length.
6. In daily life, we feel that a material which stretches more is more elastic, but it a is misnomer. In fact material which stretches to a lesser extent for a given load is considered to be more elastic.
7. In general, a deforming force in one direction can produce strains in other directions also. The proportionality between stress and strain in such situations cannot be described by just one elastic constant. For example, for a wire under longitudinal strain, the lateral dimensions (radius of cross section) will undergo a small change, which is described by another elastic constant of the material (called *Poisson ratio*).
8. Stress is not a vector quantity since, unlike a force, the stress cannot be assigned a specific direction. Force acting on the portion of a body on a specified side of a section has a definite direction.

EXERCISES

9.1 A steel wire of length 4.7 m and cross-sectional area 3.0×10^{-5} m^2 stretches by the same amount as a copper wire of length 3.5 m and cross-sectional area of 4.0×10^{-5} m^2 under a given load. What is the ratio of the Young's modulus of steel to that of copper?

9.2 Figure 9.11 shows the strain-stress curve for a given material. What are (a) Young's modulus and (b) approximate yield strength for this material?

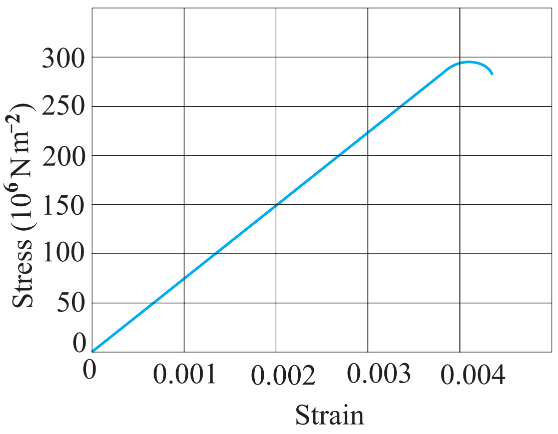


Fig. 9.11

9.3 The stress-strain graphs for materials *A* and *B* are shown in Fig. 9.12.

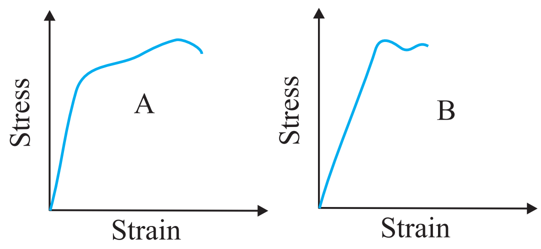


Fig. 9.12

The graphs are drawn to the same scale.

(a) Which of the materials has the greater Young's modulus?

(b) Which of the two is the stronger material?

9.4 Read the following two statements below carefully and state, with reasons, if it is true or false.

(a) The Young's modulus of rubber is greater than that of steel;

(b) The stretching of a coil is determined by its shear modulus.

9.5 Two wires of diameter 0.25 cm, one made of steel and the other made of brass are loaded as shown in Fig. 9.13. The unloaded length of steel wire is 1.5 m and that of brass wire is 1.0 m. Compute the elongations of the steel and the brass wires.

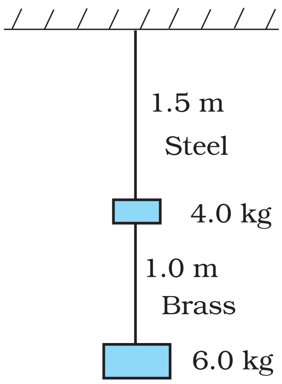


Fig. 9.13

9.6 The edge of an aluminium cube is 10 cm long. One face of the cube is firmly fixed to a vertical wall. A mass of 100 kg is then attached to the opposite face of the cube. The shear modulus of aluminium is 25 GPa. What is the vertical deflection of this face?

9.7 Four identical hollow cylindrical columns of mild steel support a big structure of mass 50,000 kg. The inner and outer radii of each column are 30 and 60 cm respectively. Assuming the load distribution to be uniform, calculate the compressional strain of each column.

9.8 A piece of copper having a rectangular cross-section of 15.2 mm × 19.1 mm is pulled in tension with 44,500 N force, producing only elastic deformation. Calculate the resulting strain?

9.9 A steel cable with a radius of 1.5 cm supports a chairlift at a ski area. If the maximum stress is not to exceed 10^8 N m^{-2}, what is the maximum load the cable can support ?

9.10 A rigid bar of mass 15 kg is supported symmetrically by three wires each 2.0 m long. Those at each end are of copper and the middle one is of iron. Determine the ratios of their diameters if each is to have the same tension.

9.11 A 14.5 kg mass, fastened to the end of a steel wire of unstretched length 1.0 m, is whirled in a vertical circle with an angular velocity of 2 rev/s at the bottom of the circle. The cross-sectional area of the wire is 0.065 cm^2. Calculate the elongation of the wire when the mass is at the lowest point of its path.

9.12 Compute the bulk modulus of water from the following data: Initial volume = 100.0 litre, Pressure increase = 100.0 atm (1 atm = 1.013 × 10^5 Pa), Final volume = 100.5 litre. Compare the bulk modulus of water with that of air (at constant temperature). Explain in simple terms why the ratio is so large.

9.13 What is the density of water at a depth where pressure is 80.0 atm, given that its density at the surface is 1.03 × 103 kg m^{-3}?

9.14 Compute the fractional change in volume of a glass slab, when subjected to a hydraulic pressure of 10 atm.

9.15 Determine the volume contraction of a solid copper cube, 10 cm on an edge, when subjected to a hydraulic pressure of 7.0 × 10^6 Pa.

9.16 How much should the pressure on a litre of water be changed to compress it by 0.10%?

Additional Exercises

9.17 Anvils made of single crystals of diamond, with the shape as shown in Fig. 9.14, are used to investigate behaviour of materials under very high pressures. Flat faces at the narrow end of the anvil have a diameter of 0.50 mm, and the wide ends are subjected to a compressional force of 50,000 N. What is the pressure at the tip of the anvil?

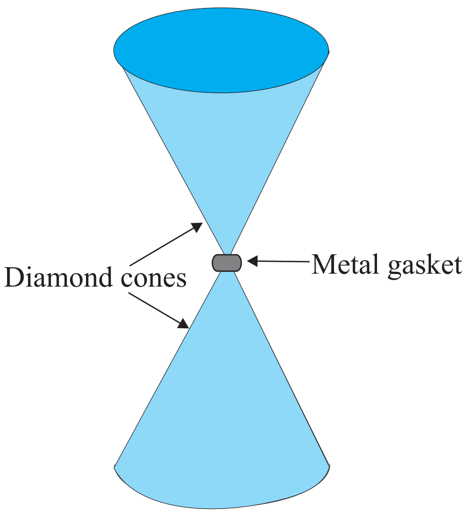


Fig. 9.14

9.18 A rod of length 1.05 m having negligible mass is supported at its ends by two wires of steel (wire A) and aluminium (wire B) of equal lengths as shown in Fig. 9.15. The cross-sectional areas of wires A and B are 1.0 mm^2 and 2.0 mm^2, respectively. At what point along the rod should a mass *m* be suspended in order to produce (a) equal stresses and (b) equal strains in both steel and aluminium wires.

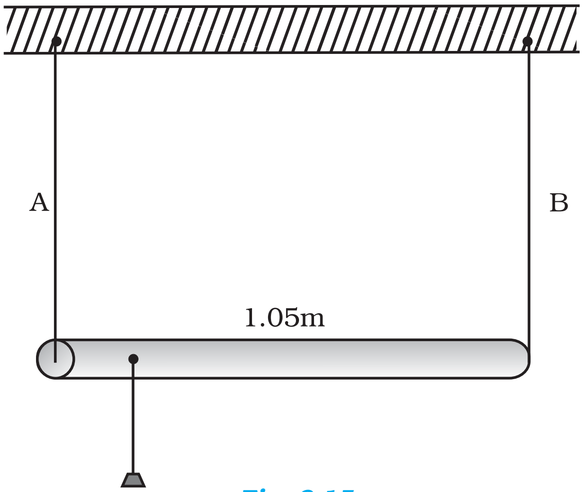


Fig. 9.15

9.19 A mild steel wire of length 1.0 m and cross-sectional area 0.50×10^{-2} cm^2 is stretched, well within its elastic limit, horizontally between two pillars. A mass of 100 g is suspended from the mid-point of the wire. Calculate the depression at the mid-point.

9.20 Two strips of metal are riveted together at their ends by four rivets, each of diameter 6.0 mm. What is the maximum tension that can be exerted by the riveted strip if the shearing stress on the rivet is not to exceed 6.9×10^7 Pa? Assume that each rivet is to carry one quarter of the load.

9.21 The Marina trench is located in the Pacific Ocean, and at one place it is nearly eleven km beneath the surface of water. The water pressure at the bottom of the trench is about 1.1×10^8 Pa. A steel ball of initial volume 0.32 m^3 is dropped into the ocean and falls to the bottom of the trench. What is the change in the volume of the ball when it reaches to the bottom?

CHAPTER TEN

MECHANICAL PROPERTIES OF FLUIDS

10.1 INTRODUCTION

In this chapter, we shall study some common physical properties of liquids and gases. Liquids and gases can flow and are therefore, called fluids. It is this property that distinguishes liquids and gases from solids in a basic way.

Fluids are everywhere around us. Earth has an envelop of air and two-thirds of its surface is covered with water. Water is not only necessary for our existence; every mammalian body constitute mostly of water. All the processes occurring in living beings including plants are mediated by fluids. Thus understanding the behaviour and properties of fluids is important.

How are fluids different from solids? What is common in liquids and gases? Unlike a solid, a fluid has no definite shape of its own. Solids and liquids have a fixed volume, whereas a gas fills the entire volume of its container. We have learnt in the previous chapter that the volume of solids can be changed by stress. The volume of solid, liquid or gas depends on the stress or pressure acting on it. When we talk about fixed volume of solid or liquid, we mean its volume under atmospheric pressure. The difference between gases and solids or liquids is that for solids or liquids the change in volume due to change of external pressure is rather small. In other words solids and liquids have much lower compressibility as compared to gases.

Shear stress can change the shape of a solid keeping its volume fixed. The key property of fluids is that they offer very little resistance to shear stress; their shape changes by application of very small shear stress. The shearing stress of fluids is about million times smaller than that of solids.

10.2 PRESSURE

A sharp needle when pressed against our skin pierces it. Our skin, however, remains intact when a blunt object with a wider contact area (say the back of a spoon) is pressed against it with the same force. If an elephant were to step on a man's chest, his ribs would crack. A circus performer across whose

chest a large, light but strong wooden plank is placed first, is saved from this accident. Such everyday experiences convince us that both the force and its coverage area are important. Smaller the area on which the force acts, greater is the impact. This concept is known as pressure.

When an object is submerged in a fluid at rest, the fluid exerts a force on its surface. This force is always normal to the object's surface. This is so because if there were a component of force parallel to the surface, the object will also exert a force on the fluid parallel to it; as a consequence of Newton's third law. This force will cause the fluid to flow parallel to the surface. Since the fluid is at rest, this cannot happen. Hence, the force exerted by the fluid at rest has to be perpendicular to the surface in contact with it. This is shown in Fig.10.1(a).

The normal force exerted by the fluid at a point may be measured. An idealised form of one such pressure-measuring device is shown in Fig. 10.1(b). It consists of an evacuated chamber with a spring that is calibrated to measure the force acting on the piston. This device is placed at a point inside the fluid. The inward force exerted by the fluid on the piston is balanced by the outward spring force and is thereby measured.

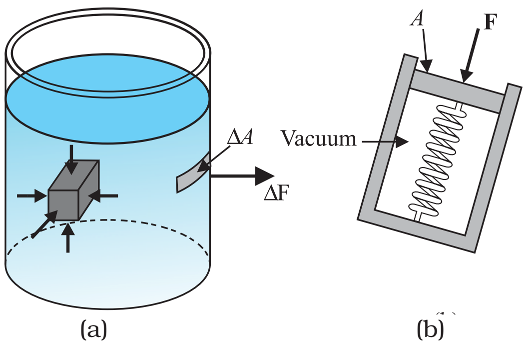


Fig. 10.1 *(a) The force exerted by the liquid in the beaker on the submerged object or on the walls is normal (perpendicular) to the surface at all points.*
(b) An idealised device for measuring pressure.

If F is the magnitude of this normal force on the piston of area A then the **average pressure** P_{av} is defined as the normal force acting per unit area.

$$P_{av} = \frac{F}{A} \tag{10.1}$$

In principle, the piston area can be made arbitrarily small. The pressure is then defined in a limiting sense as

$$P = \lim_{\Delta A \to 0} \frac{F}{A} \tag{10.2}$$

Pressure is a scalar quantity. We remind the reader that it is the component of the force normal to the area under consideration and not the (vector) force that appears in the numerator in Eqs. (10.1) and (10.2). Its dimensions are $[ML^{-1}T^{-2}]$. The SI unit of pressure is N m^{-2}. It has been named as pascal (Pa) in honour of the French scientist Blaise Pascal (1623-1662) who carried out pioneering studies on fluid pressure. A common unit of pressure is the atmosphere (atm), i.e. the pressure exerted by the atmosphere at sea level (1 atm = 1.013×10^5 Pa).

Another quantity, that is indispensable in describing fluids, is the density ρ. For a fluid of mass m occupying volume V,

$$\rho = \frac{m}{V} \tag{10.3}$$

The dimensions of density are $[ML^{-3}]$. Its SI unit is kg m^{-3}. It is a positive scalar quantity. A liquid is largely incompressible and its density is therefore, nearly constant at all pressures. Gases, on the other hand exhibit a large variation in densities with pressure.

The density of water at 4°C (277 K) is 1.0×10^3 kg m^{-3}. The relative density of a substance is the ratio of its density to the density of water at 4°C. It is a dimensionless positive scalar quantity. For example the relative density of aluminium is 2.7. Its density is 2.7×10^3 kg m^{-3}. The densities of some common fluids are displayed in Table 10.1.

Table 10.1 Densities of some common fluids at STP*

Fluid	ρ (kg m^{-3})
Water	1.00×10^3
Sea water	1.03×10^3
Mercury	13.6×10^3
Ethyl alcohol	0.806×10^3
Whole blood	1.06×10^3
Air	1.29
Oxygen	1.43
Hydrogen	9.0×10^{-2}
Interstellar space	$\approx 10^{-20}$

* *STP means standard temperature (0°C) and 1 atm pressure.*

Example 10.1 The two thigh bones (femurs), each of cross-sectional area 10 cm² support the upper part of a human body of mass 40 kg. Estimate the average pressure sustained by the femurs.

Answer Total cross-sectional area of the femurs is $A = 2 \times 10\ \text{cm}^2 = 20 \times 10^{-4}\ \text{m}^2$. The force acting on them is $F = 40$ kg wt $= 400$ N (taking $g = 10\ \text{m s}^{-2}$). This force is acting vertically down and hence, normally on the femurs. Thus, the average pressure is

$$P_{av} = \frac{F}{A} = 2 \times 10^5\ \text{N m}^{-2}$$

10.2.1 Pascal's Law

The French scientist Blaise Pascal observed that the pressure in a fluid at rest is the same at all points if they are at the same height. This fact may be demonstrated in a simple way.

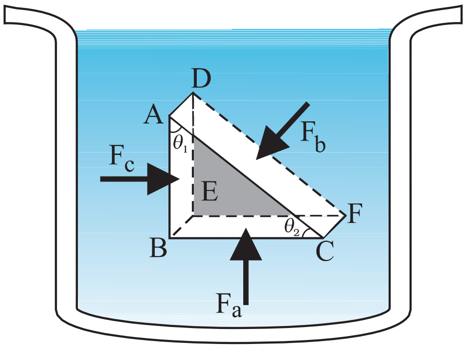


Fig. 10.2 *Proof of Pascal's law. ABC-DEF is an element of the interior of a fluid at rest. This element is in the form of a right-angled prism. The element is small so that the effect of gravity can be ignored, but it has been enlarged for the sake of clarity.*

Fig. 10.2 shows an element in the interior of a fluid at rest. This element ABC-DEF is in the form of a right-angled prism. In principle, this prismatic element is very small so that every part of it can be considered at the same depth from the liquid surface and therefore, the effect of the gravity is the same at all these points. But for clarity we have enlarged this element. The forces on this element are those exerted by the rest of the fluid and they must be normal to the surfaces of the element as discussed above. Thus, the fluid exerts pressures P_a, P_b and P_c on this element of area corresponding to the normal forces F_a, F_b and F_c as shown in Fig. 10.2 on the faces BEFC, ADFC and ADEB denoted by A_a, A_b and A_c respectively. Then

$F_b \sin\theta = F_c$, $\quad F_b \cos\theta = F_a$ (by equilibrium)
$A_b \sin\theta = A_c$, $\quad A_b \cos\theta = A_a$ (by geometry)

Thus,

$$\frac{F_b}{A_b} = \frac{F_c}{A_c} = \frac{F_a}{A_a}; \qquad P_b = P_c = P_a \qquad (10.4)$$

Hence, pressure exerted is same in all directions in a fluid at rest. It again reminds us that like other types of stress, pressure is not a vector quantity. No direction can be assigned to it. The force against any area within (or bounding) a fluid at rest and under pressure is normal to the area, regardless of the orientation of the area.

Now consider a fluid element in the form of a horizontal bar of uniform cross-section. The bar is in equilibrium. The horizontal forces exerted at its two ends must be balanced or the pressure at the two ends should be equal. This proves that for a liquid in equilibrium the pressure is same at all points in a horizontal plane. Suppose the pressure were not equal in different parts of the fluid, then there would be a flow as the fluid will have some net force acting on it. Hence in the absence of flow the pressure in the fluid must be same everywhere. Wind is flow of air due to pressure differences.

10.2.2 Variation of Pressure with Depth

Consider a fluid at rest in a container. In Fig. 10.3 point 1 is at height h above a point 2. The pressures at points 1 and 2 are P_1 and P_2 respectively. Consider a cylindrical element of fluid having area of base A and height h. As the fluid is at rest the resultant horizontal forces should be zero and the resultant vertical forces should balance the weight of the element. The forces acting in the vertical direction are due to the fluid pressure at the top (P_1A) acting downward, at the bottom (P_2A) acting upward. If mg is weight of the fluid in the cylinder we have

$$(P_2 - P_1)\ A = mg \qquad (10.5)$$

Now, if ρ is the mass density of the fluid, we have the mass of fluid to be $m = \rho V = \rho h A$ so that

$$P_2 - P_1 = \rho g h \qquad (10.6)$$

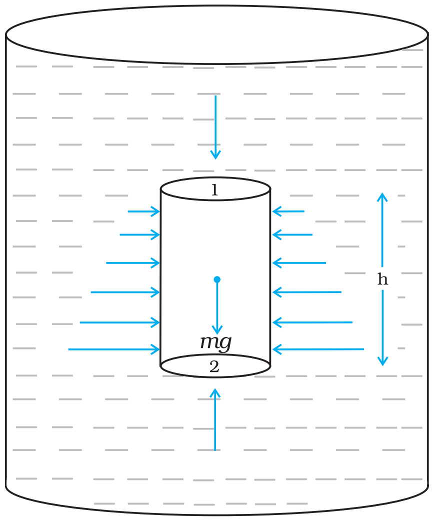


Fig.10.3 *Fluid under gravity. The effect of gravity is illustrated through pressure on a vertical cylindrical column.*

Pressure difference depends on the vertical distance h between the points (1 and 2), mass density of the fluid ρ and acceleration due to gravity g. If the point 1 under discussion is shifted to the top of the fluid (say water), which is open to the atmosphere, P_1 may be replaced by atmospheric pressure (P_a) and we replace P_2 by P. Then Eq. (10.6) gives

$$P = P_a + \rho g h \qquad (10.7)$$

Thus, the pressure P, at depth below the surface of a liquid open to the atmosphere is greater than atmospheric pressure by an amount ρgh. The excess of pressure, $P - P_a$, at depth h is called a **gauge pressure** at that point.

The area of the cylinder is not appearing in the expression of absolute pressure in Eq. (10.7). Thus, the height of the fluid column is important and not cross sectional or base area or the shape of the container. The liquid pressure is the same at all points at the same horizontal level (same depth). The result is appreciated through the example of **hydrostatic paradox**. Consider three vessels A, B and C [Fig.10.4] of different shapes. They are connected at the bottom by a horizontal pipe. On filling with water the level in the three vessels is the same though they hold different amounts of water. This is so, because water at the bottom has the same pressure below each section of the vessel.

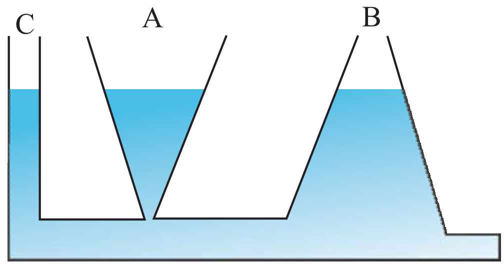


Fig 10.4 *Illustration of hydrostatic paradox. The three vessels A, B and C contain different amounts of liquids, all upto the same height.*

Example 10.2 What is the pressure on a swimmer 10 m below the surface of a lake?

Answer Here

h = 10 m *and* ρ = 1000 kg m^{-3}. Take g = 10 m s^{-2}

From Eq. (10.7)

$P = P_a + \rho g h$

$= 1.01 \times 10^5$ Pa + 1000 kg m^{-3} $\times$ 10 m s^{-2} $\times$ 10 m

$= 2.01 \times 10^5$ Pa

$\approx$ 2 atm

This is a 100% increase in pressure from surface level. At a depth of 1 km the increase in pressure is 100 atm! Submarines are designed to withstand such enormous pressures. ⊥

10.2.3 Atmospheric Pressure and Gauge Pressure

The pressure of the atmosphere at any point is equal to the weight of a column of air of unit cross sectional area extending from that point to the top of the atmosphere. At sea level it is 1.013×10^5 Pa (1 atm). Italian scientist Evangelista Torricelli (1608-1647) devised for the first time, a method for measuring atmospheric pressure. A long glass tube closed at one end and filled with mercury is inverted into a trough of mercury as shown in Fig.10.5 (a). This device is known as mercury barometer. The space above the mercury column in the tube contains only mercury vapour whose pressure P is so small that it may be neglected. The pressure inside the column at point A must equal the pressure at point B, which is at the same level. Pressure at B = atmospheric pressure = P_a

$$P_a = \rho g h \qquad (10.8)$$

where ρ is the density of mercury and h is the height of the mercury column in the tube.

In the experiment it is found that the mercury column in the barometer has a height of about 76 cm at sea level equivalent to one atmosphere (1 atm). This can also be obtained using the value of ρ in Eq. (10.8). A common way of stating pressure is in terms of cm or mm of mercury (Hg). A pressure equivalent of 1 mm is called a torr (after Torricelli).

1 torr = 133 Pa.

The mm of Hg and torr are used in medicine and physiology. In meteorology, a common unit is the bar and millibar.

1 bar = 10^5 Pa

An open-tube manometer is a useful instrument for measuring pressure differences. It consists of a U-tube containing a suitable liquid i.e. a low density liquid (such as oil) for measuring small pressure differences and a high density liquid (such as mercury) for large pressure differences. One end of the tube is open to the atmosphere and other end is connected to the system whose pressure we want to measure [see Fig. 10.5 (b)]. The pressure *P* at A is equal to pressure at point B. What we normally measure is the gauge pressure, which is $P - P_a$, given by Eq. (10.8) and is proportional to manometer height *h*.

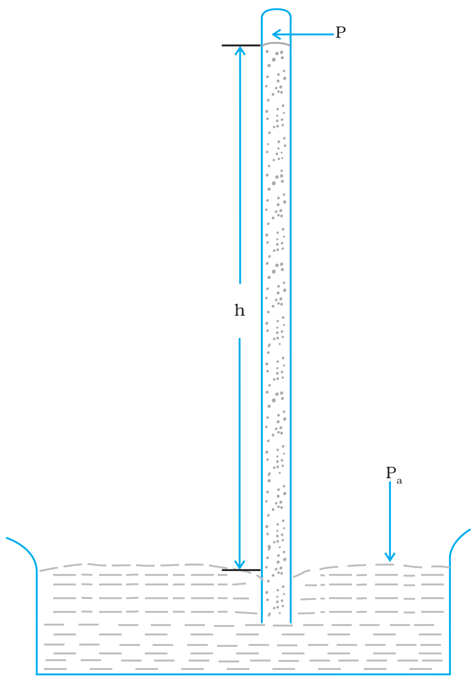
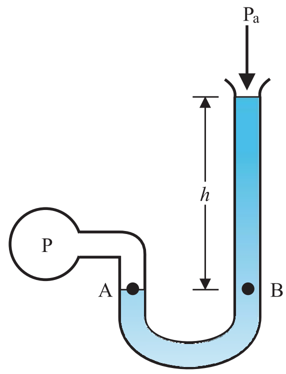


Fig 10.5 *(a) The mercury barometer.*



(b) the open tube manometer

Fig 10.5 *Two pressure measuring devices.*

Pressure is same at the same level on both sides of the U-tube containing a fluid. For liquids the density varies very little over wide ranges in pressure and temperature and we can treat it safely as a constant for our present purposes. Gases on the other hand, exhibits large variations of densities with changes in pressure and temperature. Unlike gases, liquids are therefore, largely treated as incompressible.

Example 10.3 The density of the atmosphere at sea level is 1.29 kg/m³. Assume that it does not change with altitude. Then how high would the atmosphere extend?

Answer We use Eq. (10.7)

$\rho g h = 1.29 \text{ kg m}^{-3} \times 9.8 \text{ m s}^{2} \times h \text{ m} = 1.01 \times 10^5 \text{ Pa}$

$\therefore h = 7989 \text{ m} \approx 8 \text{ km}$

In reality the density of air decreases with height. So does the value of *g*. The atmospheric cover extends with decreasing pressure over 100 km. We should also note that the sea level atmospheric pressure is not always 760 mm of Hg. A drop in the Hg level by 10 mm or more is a sign of an approaching storm.

Example 10.4 At a depth of 1000 m in an ocean (a) what is the absolute pressure? (b) What is the gauge pressure? (c) Find the force acting on the window of area 20 cm × 20 cm of a submarine at this

depth, the interior of which is maintained at sea-level atmospheric pressure. (The density of sea water is 1.03×10^3 kg m^{-3}, g = 10m s^{-2}.)

Answer Here h = 1000 m and $\rho = 1.03 \times 10^3$ kg m^{-3}.

(a) From Eq. (10.6), absolute pressure

$P = P_a + \rho gh$
$= 1.01 \times 10^5$ Pa
$+ 1.03 \times 10^3$ kg m^{-3} $\times$ 10 m s^{-2} $\times$ 1000 m
$= 104.01 \times 10^5$ Pa
$\approx$ 104 atm

(b) Gauge pressure is $P - P_a = \rho gh = P_g$
$P_g = 1.03 \times 10^3$ kg m^{-3} $\times$ 10 ms^2 $\times$ 1000 m
$= 103 \times 10^5$ Pa
$\approx$ 103 atm

(c) The pressure outside the submarine is $P = P_a + \rho gh$ and the pressure inside it is P_a. Hence, the net pressure acting on the window is gauge pressure, $P_g = \rho gh$. Since the area of the window is A = 0.04 m^2, the force acting on it is
$F = P_g A = 103 \times 10^5$ Pa $\times$ 0.04 m^2 = 4.12×10^5 N

t

10.2.4 Hydraulic Machines

Let us now consider what happens when we change the pressure on a fluid contained in a vessel. Consider a horizontal cylinder with a piston and three vertical tubes at different points. The pressure in the horizontal cylinder is indicated by the height of liquid column in the vertical tubes.It is necessarily the same in all. If we push the piston, the fluid level rises in all the tubes, again reaching the same level in each one of them.

This indicates that when the pressure on the cylinder was increased, it was distributed uniformly throughout. We can say **whenever external pressure is applied on any part of a fluid contained in a vessel, it is transmitted undiminished and equally in all directions. This is the Pascal's law for transmission of fluid pressure and has many applications in daily life.**

A number of devices such as **hydraulic lift** and **hydraulic brakes** are based on the Pascal's law. In these devices fluids are used for transmitting pressure. In a hydraulic lift as shown in Fig. 10.6 two pistons are separated by the space filled with a liquid. A piston of small cross section A_1 is used to exert a force F_1 directly on the liquid. The pressure $P = \dfrac{F_1}{A_1}$ is transmitted throughout the liquid to the larger cylinder attached with a larger piston of area A_2, which results in an upward force of $P \times A_2$. Therefore, the piston is capable of supporting a large force (large weight of, say a car, or a truck, placed on the platform) $F_2 = PA_2 = \dfrac{F_1 A_2}{A_1}$. By changing the force at A_1, the platform can be

Archemedes' Principle

Fluid appears to provide partial support to the objects placed in it. When a body is wholly or partially immersed in a fluid at rest, the fluid exerts pressure on the surface of the body in contact with the fluid. The pressure is greater on lower surfaces of the body than on the upper surfaces as pressure in a fluid increases with depth. The resultant of all the forces is an upward force called buoyant force. Suppose that a cylindrical body is immersed in the fluid. The upward force on the bottom of the body is more than the downward force on its top. The fluid exerts a resultant upward force or buoyant force on the body equal to $(P_2\text{-}P_1)$ A. We have seen in equation 10.4 that $(P_2\text{-}P_1)A = \rho ghA$. Now hA is the volume of the solid and ρhA is the weight of an equivaliant volume of the fluid. $(P_2\text{-}P_1)A = mg$. Thus the upward force exerted is equal to the weight of the displaced fluid.

The result holds true irrespective of the shape of the object and here cylindrical object is considered only for convenience. This is Archimedes' principle. For totally immersed objects the volume of the fluid displaced by the object is equal to its own volume. If the density of the immersed object is more than that of the fluid, the object will sink as the weight of the body is more than the upward thrust. If the density of the object is less than that of the fluid, it floats in the fluid partially submerged. To calculate the volume submerged. Suppose the total volume of the object is V_s and a part V_p of it is submerged in the fluid. Then the upward force which is the weight of the displaced fluid is $\rho_f g V_p$, which must equal the weight of the body; $\rho_s g V_s = \rho_f g V_p$ or $\rho_s/\rho_f = V_p/V_s$ The apparent weight of the floating body is zero.

This principle can be summarised as; 'the loss of weight of a body submerged (partially or fully) in a fluid is equal to the weight of the fluid displaced'.

moved up or down. Thus, the applied force has been increased by a factor of $\frac{A_2}{A_1}$ and this factor is the mechanical advantage of the device. The example below clarifies it.

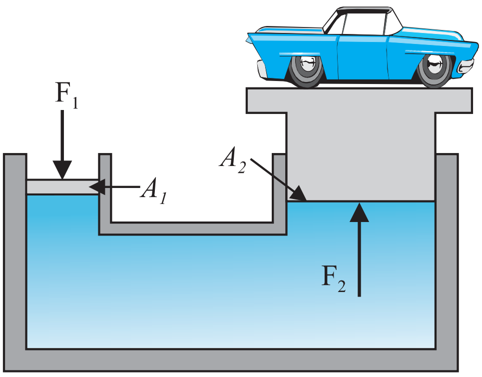


Fig 10.6 *Schematic diagram illustrating the principle behind the hydraulic lift, a device used to lift heavy loads.*

> ***Example 10.5*** Two syringes of different cross sections (without needles) filled with water are connected with a tightly fitted rubber tube filled with water. Diameters of the smaller piston and larger piston are 1.0 cm and 3.0 cm respectively. (a) Find the force exerted on the larger piston when a force of 10 N is applied to the smaller piston. (b) If the smaller piston is pushed in through 6.0 cm, how much does the larger piston move out?

Answer (a) Since pressure is transmitted undiminished throughout the fluid,

$$F_2 \quad \frac{A_2}{A_1} F_1 \quad \frac{3/2 \quad 10^{-2}\,\text{m}^{\;2}}{1/2 \quad 10^{-2}\,\text{m}^{\;2}} \quad 10\,\text{N}$$

$= 90$ N

(b) Water is considered to be perfectly incompressible. Volume covered by the movement of smaller piston inwards is equal to volume moved outwards due to the larger piston.

$$L_1 A_1 = L_2 A_2$$

$\simeq 0.67 \times 10^{-2}$ m = 0.67 cm

Note, atmospheric pressure is common to both pistons and has been ignored.

> ***Example 10.6*** In a car lift compressed air exerts a force F_1 on a small piston having a radius of 5.0 cm. This pressure is transmitted to a second piston of radius 15 cm (Fig 10.7). If the mass of the car to be lifted is 1350 kg, calculate F_1. What is the pressure necessary to accomplish this task? (g = 9.8 ms^{-2}).

Answer Since pressure is transmitted undiminished throughout the fluid,

$$F_1 \quad \frac{A_1}{A_2} F_2 \quad \frac{5 \quad 10^{-2}\,\text{m}^{\;2}}{15 \quad 10^{-2}\,\text{m}^{\;2}} \quad 1350\,\text{N} \quad 9.8\,\text{m s}^{-2}$$

= 1470 N

$\approx 1.5 \times 10^3$ N

The air pressure that will produce this force is

$$P \quad \frac{F_1}{A_1} \quad \frac{1.5 \quad 10^3\,\text{N}}{5 \quad 10^{-2}\;^{2}\,\text{m}} \quad 1.9 \quad 10^5\,\text{Pa}$$

This is almost double the atmospheric pressure.

Hydraulic brakes in automobiles also work on the same principle. When we apply a little

Archimedes (287 – 212 B.C.)

Archimedes was a Greek philosopher, mathematician, scientist and engineer. He invented the catapult and devised a system of pulleys and levers to handle heavy loads. The king of his native city Syracuse, Hiero II asked him to determine if his gold crown was alloyed with some cheaper metal such as silver without damaging the crown. The partial loss of weight he experienced while lying in his bathtub suggested a solution to him. According to legend, he ran naked through the streets of Syracuse exclaiming "Eureka, eureka!", which means "I have found it, I have found it!"

force on the pedal with our foot the master piston moves inside the master cylinder, and the pressure caused is transmitted through the brake oil to act on a piston of larger area. A large force acts on the piston and is pushed down expanding the brake shoes against brake lining. In this way a small force on the pedal produces a large retarding force on the wheel. An important advantage of the system is that the pressure set up by pressing pedal is transmitted equally to all cylinders attached to the four wheels so that the braking effort is equal on all wheels.

10.3 STREAMLINE FLOW

So far we have studied fluids at rest. The study of the fluids in motion is known as fluid dynamics. When a water-tap is turned on slowly, the water flow is smooth initially, but loses its smoothness when the speed of the outflow is increased. In studying the motion of fluids we focus our attention on what is happening to various fluid particles at a particular point in space at a particular time. The flow of the fluid is said to be **steady** if at any given point, the velocity of each passing fluid particle remains constant in time. This does not mean that the velocity at different points in space is same. The velocity of a particular particle may change as it moves from one point to another. That is, at some other point the particle may have a different velocity, but every other particle which passes the second point behaves exactly as the previous particle that has just passed that point. Each particle follows a smooth path, and the paths of the particles do not cross each other.

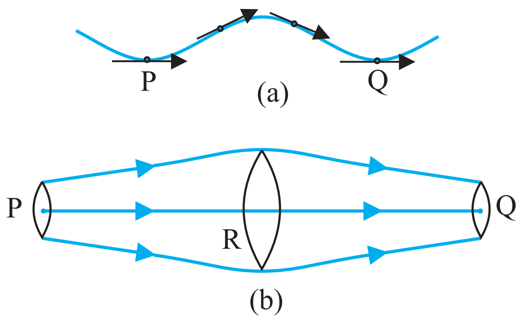


Fig. 10.7 *The meaning of streamlines. (a) A typical trajectory of a fluid particle. (b) A region of streamline flow.*

The path taken by a fluid particle under a steady flow is a **streamline**. It is defined as a curve whose tangent at any point is in the direction of the fluid velocity at that point. Consider the path of a particle as shown in Fig.10.7 (a), the curve describes how a fluid particle moves with time. The curve PQ is like a permanent map of fluid flow, indicating how the fluid streams. No two streamlines can cross, for if they do, an oncoming fluid particle can go either one way or the other and the flow would not be steady. Hence, in steady flow, the map of flow is stationary in time. How do we draw closely spaced streamlines ? If we intend to show streamline of every flowing particle, we would end up with a continuum of lines. Consider planes perpendicular to the direction of fluid flow e.g., at three points P, R and Q in Fig.10.7 (b). The plane pieces are so chosen that their boundaries be determined by the same set of streamlines. This means that number of fluid particles crossing the surfaces as indicated at P, R and Q is the same. If area of cross-sections at these points are A_P,A_R and A_Q and speeds of fluid particles are v_P, v_R and v_Q, then mass of fluid Δm_P crossing at A_P in a small interval of time Δt is $\rho_P A_P v_P \Delta t$. Similarly mass of fluid Δm_R flowing or crossing at A_R in a small interval of time Δt is $\rho_R A_R v_R \Delta t$ and mass of fluid Δm_Q is $\rho_Q A_Q v_Q \Delta t$ crossing at A_Q. The mass of liquid flowing out equals the mass flowing in, holds in all cases. Therefore,

$$\rho_P A_P v_P \Delta t = \rho_R A_R v_R \Delta t = \rho_Q A_Q v_Q \Delta t \qquad (10.9)$$

For flow of incompressible fluids

$$\rho_P = \rho_R = \rho_Q$$

Equation (10.9) reduces to

$$A_P v_P = A_R v_R = A_Q v_Q \qquad (10.10)$$

which is called the **equation of continuity** and it is a statement of conservation of mass in flow of incompressible fluids. In general

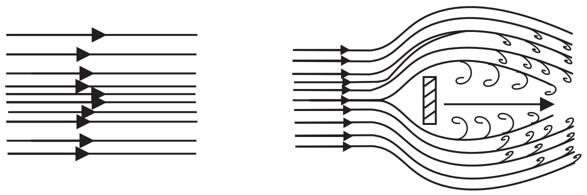
$$Av = \text{constant} \qquad (10.11)$$

Av gives the volume flux or flow rate and remains constant throughout the pipe of flow. Thus, at narrower portions where the streamlines are closely spaced, velocity increases and its vice versa. From (Fig 10.7b) it is clear that $A_R > A_Q$ or $v_R < v_Q$, the fluid is accelerated while passing from R to Q. This is associated with a change in pressure in fluid flow in horizontal pipes.

Steady flow is achieved at low flow speeds. Beyond a limiting value, called critical speed, this flow loses steadiness and becomes **turbulent**. One sees this when a fast flowing

stream encounters rocks, small foamy whirlpool-like regions called 'white water rapids are formed.

Figure 10.8 displays streamlines for some typical flows. For example, Fig. 10.8(a) describes a laminar flow where the velocities at different points in the fluid may have different magnitudes but their directions are parallel. Figure 10.8 (b) gives a sketch of turbulent flow.



(a) (b)

Fig. 10.8 *(a) Some streamlines for fluid flow. (b) A jet of air striking a flat plate placed perpendicular to it. This is an example of turbulent flow.*

10.4 BERNOULLI'S PRINCIPLE

Fluid flow is a complex phenomenon. But we can obtain some useful properties for steady or streamline flows using the conservation of energy.

Consider a fluid moving in a pipe of varying cross-sectional area. Let the pipe be at varying heights as shown in Fig. 10.9. We now suppose that an incompressible fluid is flowing through the pipe in a steady flow. Its velocity must change as a consequence of equation of continuity. A force is required to produce this acceleration, which is caused by the fluid surrounding it, the pressure must be different in different regions. Bernoulli's equation is a general expression that relates the pressure difference between two points in a pipe to both velocity changes (kinetic energy change) and elevation (height) changes (potential energy change). The Swiss Physicist Daniel Bernoulli developed this relationship in 1738.

Consider the flow at two regions 1 (i.e. BC) and 2 (i.e. DE). Consider the fluid initially lying between B and D. In an infinitesimal time interval Δt, this fluid would have moved. Suppose v_1 is the speed at B and v_2 at D, then fluid initially at B has moved a distance $v_1\Delta t$ to C ($v_1\Delta t$ is small enough to assume constant cross-section along BC). In the same interval Δt the fluid initially at D moves to E, a distance equal to $v_2\Delta t$. Pressures P_1 and P_2 act as shown on the plane faces of areas A_1 and A_2 binding the two regions. The work done on the fluid at left end (BC) is $W_1 = P_1A_1(v_1\Delta t) = P_1\Delta V$. Since the same volume ΔV passes through both the regions (from the equation of continuity) the work done by the fluid at the other end (DE) is $W_2 = P_2A_2(v_2\Delta t) = P_2\Delta V$ or, the work done on the fluid is $-P_2\Delta V$. So the total work done on the fluid is

$$W_1 - W_2 = (P_1 - P_2)\,\Delta V$$

Part of this work goes into changing the kinetic energy of the fluid, and part goes into changing the gravitational potential energy. If the density of the fluid is ρ and $\Delta m = \rho A_1 v_1 \Delta t = \rho\Delta V$ is the mass passing through the pipe in time Δt, then change in gravitational potential energy is

$$\Delta U = \rho g \Delta V\,(h_2 - h_1)$$

The change in its kinetic energy is

$$\Delta K = \left(\frac{1}{2}\right)\rho\,\Delta V\,(v_2^2 - v_1^2)$$

We can employ the work – energy theorem (Chapter 6) to this volume of the fluid and this yields

$$(P_1 - P_2)\,\Delta V = \left(\frac{1}{2}\right)\rho\,\Delta V\,(v_2^2 - v_1^2) + \rho g \Delta V\,(h_2 - h_1)$$

We now divide each term by ΔV to obtain

$$(P_1 - P_2) = \left(\frac{1}{2}\right)\rho\,(v_2^2 - v_1^2) + \rho g\,(h_2 - h_1)$$

Daniel Bernoulli (1700-1782)

Daniel Bernoulli was a Swiss scientist and mathematician who along with Leonard Euler had the distinction of winning the French Academy prize for mathematics ten times. He also studied medicine and served as a professor of anatomy and botany for a while at Basle, Switzerland. His most well known work was in hydrodynamics, a subject he developed from a single principle: the conservation of energy. His work included calculus, probability, the theory of vibrating strings, and applied mathematics. He has been called the founder of mathematical physics.

We can rearrange the above terms to obtain

$$P_1 + \left(\frac{1}{2}\right)\rho v_1^2 + \rho g h_1 = P_2 + \left(\frac{1}{2}\right)\rho v_2^2 + \rho g h_2 \qquad (10.12)$$

This is **Bernoulli's equation**. Since 1 and 2 refer to any two locations along the pipeline, we may write the expression in general as

$$P + \left(\frac{1}{2}\right)\rho v^2 + \rho g h = \text{constant} \qquad (10.13)$$

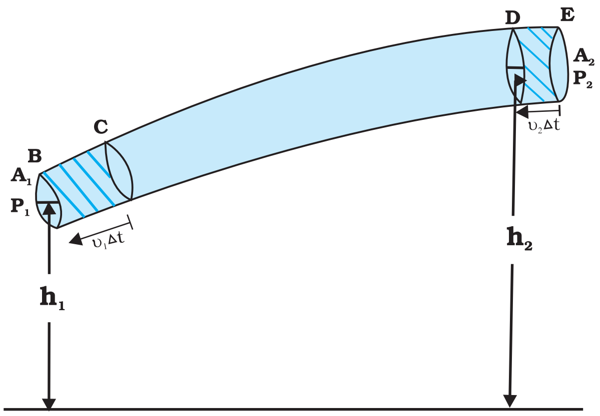


Fig. 10.9 *The flow of an ideal fluid in a pipe of varying cross section. The fluid in a section of length $v_1\Delta t$ moves to the section of length $v_2\Delta t$ in time Δt.*

In words, the Bernoulli's relation may be stated as follows: As we move along a streamline the sum of the pressure (P), the kinetic energy per unit volume $\frac{\rho v^2}{2}$ and the potential energy per unit volume ($\rho g h$) remains a constant.

Note that in applying the energy conservation principle, there is an assumption that no energy is lost due to friction. But in fact, when fluids flow, some energy does get lost due to internal friction. This arises due to the fact that in a fluid flow, the different layers of the fluid flow with different velocities. These layers exert frictional forces on each other resulting in a loss of energy. This property of the fluid is called viscosity and is discussed in more detail in a later section. The lost kinetic energy of the fluid gets converted into heat energy. Thus, Bernoulli's equation ideally applies to fluids with zero viscosity or non-viscous fluids. Another restriction on application of Bernoulli theorem is that the fluids must be incompressible, as the elastic energy of the fluid is also not taken into consideration. In practice, it has a large number of useful applications and can help explain a wide variety of phenomena for low viscosity incompressible fluids. Bernoulli's equation also does not hold for non-steady or turbulent flows, because in that situation velocity and pressure are constantly fluctuating in time.

When a fluid is at rest i.e. its velocity is zero everywhere, Bernoulli's equation becomes

$$P_1 + \rho g h_1 = P_2 + \rho g h_2$$

$$(P_1 - P_2) = \rho g\ (h_2 - h_1)$$

which is same as Eq. (10.6).

10.4.1 Speed of Efflux: Torricelli's Law

The word efflux means fluid outflow. Torricelli discovered that the speed of efflux from an open tank is given by a formula identical to that of a freely falling body. Consider a tank containing a liquid of density ρ with a small hole in its side at a height y_1 from the bottom (see Fig. 10.10). The air above the liquid, whose surface is at height y_2, is at pressure P. From the equation of continuity [Eq. (10.10)] we have

$$v_1 A_1 = v_2 A_2$$

$$v_2 = \frac{A_1}{A_2} v_1$$

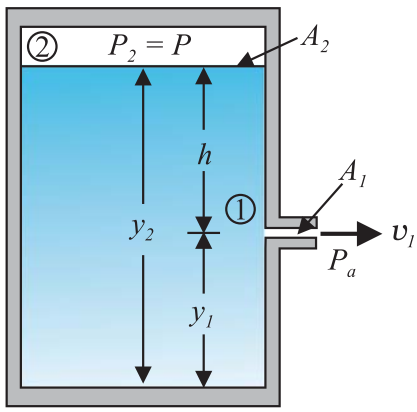


Fig. 10.10 *Torricelli's law. The speed of efflux, v_1, from the side of the container is given by the application of Bernoulli's equation. If the container is open at the top to the atmosphere then* v_1 $\sqrt{2gh}$.

If the cross sectional area of the tank A_2 is much larger than that of the hole ($A_2 >> A_1$), then we may take the fluid to be approximately at rest at the top, i.e. $v_2 = 0$. Now applying the Bernoulli equation at points 1 and 2 and noting that at the hole $P_1 = P_a$, the atmospheric pressure, we have from Eq. (10.12)

$$P_a \quad \frac{1}{2} \quad v_1^2 \qquad g\, y_1 \quad P \qquad g\, y_2$$

Taking $y_2 - y_1 = h$ we have

$$v_1 \quad \sqrt{2g\; h \quad \frac{2\; P \quad P_a}{}} \qquad (10.14)$$

When $P >> P_a$ and $2\, g\, h$ may be ignored, the speed of efflux is determined by the container pressure. Such a situation occurs in rocket propulsion. On the other hand if the tank is open to the atmosphere, then $P = P_a$ and

$$v_1 = \sqrt{2gh} \qquad (10.15)$$

This is the speed of a freely falling body. Equation (10.15) is known as **Torricelli's law**.

10.4.2 Venturi-meter

The Venturi-meter is a device to measure the flow speed of incompressible fluid. It consists of a tube with a broad diameter and a small constriction at the middle as shown in Fig. (10.11). A manometer in the form of a U-tube is also attached to it, with one arm at the broad neck point of the tube and the other at constriction as shown in Fig. (10.11). The manometer contains a liquid of density ρ_m. The speed v_1 of the liquid flowing through the tube at the broad neck area A is to be measured from equation of continuity Eq. (10.10) the speed at the constriction becomes $v_2 = \frac{A}{a} v_1$.

Then using Bernoulli's equation, we get

$$P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_1^2 (A/a)^2$$

So that

$$P_1 - P_2 = \frac{1}{2} \rho v_1^2 [\left(\frac{A}{a}\right)^2 - 1] \qquad (10.16)$$

This pressure difference causes the fluid in the U tube connected at the narrow neck to rise in comparison to the other arm. The difference in height h measure the pressure difference.

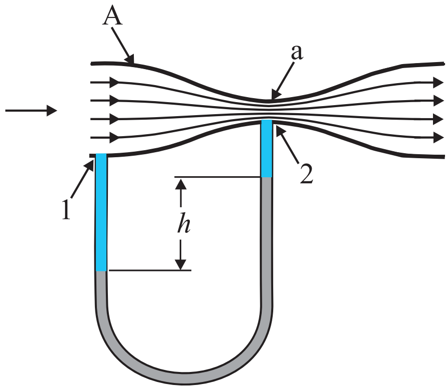


Fig. 10.11 A schematic diagram of Venturi-meter.

$$P_1 - P_2 = \rho_m g h = \frac{1}{2} \rho v_1^2 \left[\left(\frac{A}{a}\right)^2 - 1\right]$$

So that the speed of fluid at wide neck is

$$v_1 = \sqrt{\left(\frac{2\rho_m g h}{\rho}\right)} \left(\left(\frac{A}{a}\right)^2 - 1\right)^{-1/2} \qquad (10.17)$$

The principle behind this meter has many applications. The carburetor of automobile has a Venturi channel (nozzle) through which air flows with a large speed. The pressure is then lowered at the narrow neck and the petrol (gasoline) is sucked up in the chamber to provide the correct mixture of air to fuel necessary for combustion. Filter pumps or aspirators, Bunsen burner, atomisers and sprayers [See Fig. 10.12] used for perfumes or to spray insecticides work on the same principle.

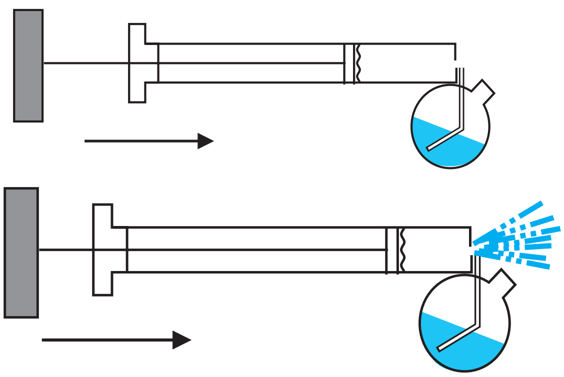


Fig. 10.12 The spray gun. Piston forces air at high speeds causing a lowering of pressure at the neck of the container.

Example 10.7 **Blood velocity:** The flow of blood in a large artery of an anesthetised dog is diverted through a Venturi meter. The wider part of the meter has a cross-sectional area equal to that of the artery. $A = 8\ mm^2$. The narrower part has an area $a = 4\ mm^2$. The pressure drop in the artery is 24 Pa. What is the speed of the blood in the artery?

Answer We take the density of blood from Table 10.1 to be $1.06 \times 10^3\ kg\ m^{-3}$. The ratio of the areas is $\left(\frac{A}{a}\right) = 2$. Using Eq. (10.17) we obtain

$$v_1 = \sqrt{\frac{2 \times 24 Pa}{1060\ kg\ m^{-3} \times (2^2 - 1)}} = 0.125\ m\ s^{-1}$$

10.4.3 Blood Flow and Heart Attack

Bernoulli's principle helps in explaining blood flow in artery. The artery may get constricted due to the accumulation of plaque on its inner walls. In order to drive the blood through this constriction a greater demand is placed on the activity of the heart. The speed of the flow of the blood in this region is raised which lowers the pressure inside and the artery may collapse due to the external pressure. The heart exerts further pressure to open this artery and forces the blood through. As the blood rushes through the opening, the internal pressure once again drops due to same reasons leading to a repeat collapse. This may result in heart attack.

10.4.4 Dynamic Lift

Dynamic lift is the force that acts on a body, such as airplane wing, a hydrofoil or a spinning ball, by virtue of its motion through a fluid. In many games such as cricket, tennis, baseball, or golf, we notice that a spinning ball deviates from its parabolic trajectory as it moves through air. This deviation can be partly explained on the basis of Bernoulli's principle.

(i) **Ball moving without spin**: Fig. 10.13(a) shows the streamlines around a non-spinning ball moving relative to a fluid. From the symmetry of streamlines it is clear that the velocity of fluid (air) above and below the ball at corresponding points is the same resulting in zero pressure difference. The air therefore, exerts no upward or downward force on the ball.

(ii) **Ball moving with spin**: A ball which is spinning drags air along with it. If the surface is rough more air will be dragged. Fig 10.13(b) shows the streamlines of air for a ball which is moving and spinning at the same time. The ball is moving forward and relative to it the air is moving backwards. Therefore, the velocity of air above the ball relative to it is larger and below it is smaller. The stream lines thus get crowded above and rarified below.

This difference in the velocities of air results in the pressure difference between the lower and upper faces and their is a net upward force on the ball. This dynamic lift due to spining is called **Magnus effect**.

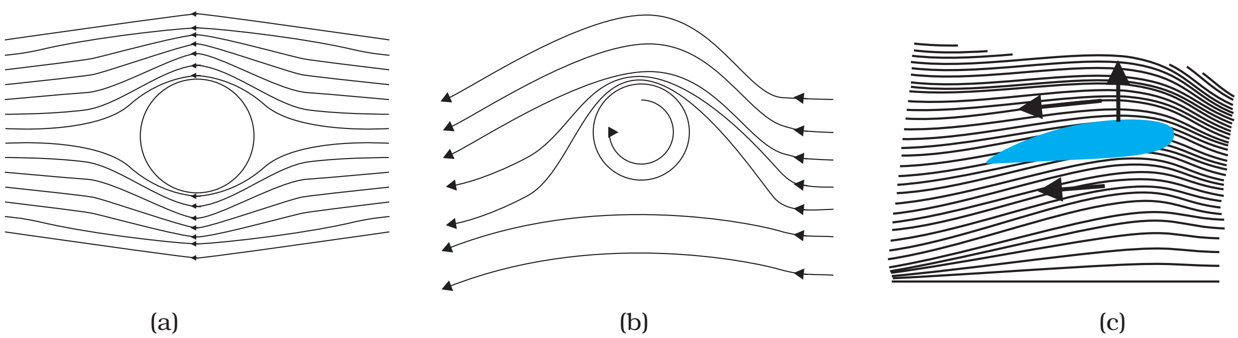


Fig 10.13 *(a) Fluid streaming past a static sphere. (b) Streamlines for a fluid around a sphere spinning clockwise. (c) Air flowing past an aerofoil.*

Aerofoil or lift on aircraft wing: Figure 10.13 (c) shows an aerofoil, which is a solid piece shaped to provide an upward dynamic lift when it moves horizontally through air. The cross-section of the wings of an aeroplane looks somewhat like the aerofoil shown in Fig. 10.13 (c) with streamlines around it. When the aerofoil moves against the wind, the orientation of the wing relative to flow direction causes the streamlines to crowd together above the wing more than those below it. The flow speed on top is higher than that below it. There is an upward force resulting in a dynamic lift of the wings and this balances the weight of the plane. The following example illustrates this.

Example 10.8 A fully loaded Boeing aircraft has a mass of 3.3×10^5 kg. Its total wing area is 500 m^2. It is in level flight with a speed of 960 km/h. (a) Estimate the pressure difference between the lower and upper surfaces of the wings (b) Estimate the fractional increase in the speed of the air on the upper surface of the wing relative to the lower surface. [The density of air is ρ = 1.2 kg m^{-3}]

Answer (a) The weight of the Boeing aircraft is balanced by the upward force due to the pressure difference

$\Delta P \times A = 3.3 \times 10^5 \text{ kg} \times 9.8$

$\Delta P = (3.3 \times 10^5 \text{ kg} \times 9.8 \text{ m s}^{-2}) / 500 \text{ m}^2$

$= 6.5 \times 10^3 \text{ Nm}^{-2}$

(b) We ignore the small height difference between the top and bottom sides in Eq. (10.12). The pressure difference between them is then

where v_2 is the speed of air over the upper surface and v_1 is the speed under the bottom surface.

$$v_2 - v_1 \quad \frac{2\ P}{v_2 \quad v_1}$$

Taking the average speed

$v_{av} = (v_2 + v_1)/2 = 960 \text{ km/h} = 267 \text{ m s}^{-1}$,

we have

$$v_2 - v_1 \ / v_{av} \quad \frac{P}{v_{av}^2} \approx 0.08$$

The speed above the wing needs to be only 8 % higher than that below.

10.5 VISCOSITY

Most of the fluids are not ideal ones and offer some resistance to motion. This resistance to fluid motion is like an internal friction analogous to friction when a solid moves on a surface. It is called viscosity. This force exists when there is relative motion between layers of the liquid. Suppose we consider a fluid like oil enclosed between two glass plates as shown in Fig. 10.15 (a). The bottom plate is fixed while the top plate is moved with a constant velocity **v** relative to the fixed plate. If oil is replaced by honey, a greater force is required to move the plate with the same velocity. Hence we say that honey is more viscous than oil. The fluid in contact with a surface has the same velocity as that of the surfaces. Hence, the layer of the liquid in contact with top surface moves with a velocity **v** and the layer of the liquid in contact with the fixed surface is stationary. The velocities of layers increase uniformly from bottom (zero velocity) to the top layer (velocity **v**). For any layer of liquid, its upper layer pulls it forward while lower layer pulls it backward. This results in force between the layers. This type of flow is known as laminar. The layers of liquid slide over one another as the pages of a book do when it is placed flat on a table and a horizontal force is applied to the top cover. When a fluid is flowing in a pipe or a tube, then velocity of the liquid layer along the axis of the tube is maximum and decreases gradually as we move towards the walls where it becomes zero, Fig. 10.14 (b). The velocity on a cylindrical surface in a tube is constant.

On account of this motion, a portion of liquid, which at some instant has the shape ABCD, take the shape of AEFD after short interval of time (Δt). During this time interval the liquid has undergone a shear strain of $\Delta x/l$. Since, the strain in a flowing fluid increases with time continuously. Unlike a solid, here the stress is found experimentally to depend on 'rate of change of strain' or 'strain rate' i.e. $\Delta x/(l\,\Delta t)$ or v/l instead of strain itself. The coefficient of viscosity (pronounced 'eta') for a fluid is defined as the ratio of shearing stress to the strain rate.

$$\frac{F/A}{v/l} \quad \frac{Fl}{vA} \qquad (10.18)$$

The SI unit of viscosity is poiseiulle (Pl). Its other units are N s m^{-2} or Pa s. The dimensions

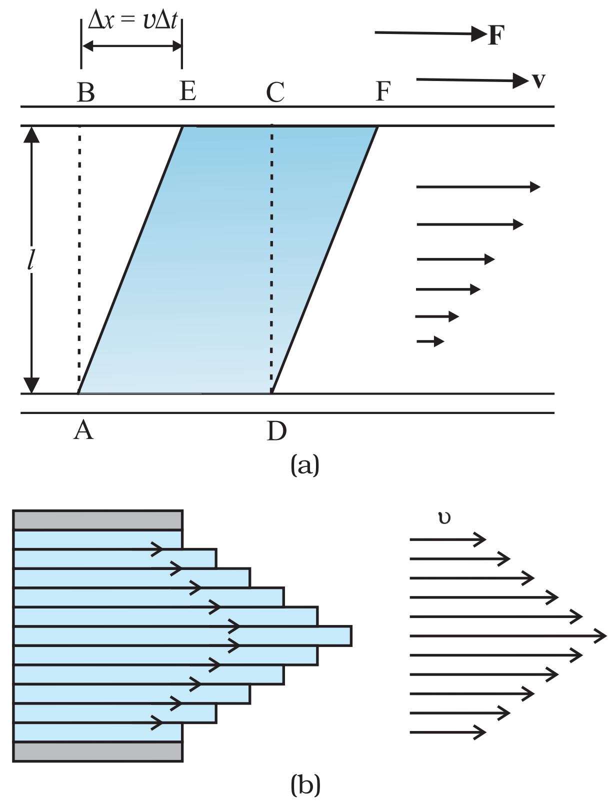


Fig 10.14 *(a) A layer of liquid sandwiched between two parallel glass plates in which the lower plate is fixed and the upper one is moving to the right with velocity* **v** *(b) velocity distribution for viscous flow in a pipe.*

of viscosity are [$ML^{-1}T^{-1}$]. Generally thin liquids like water, alcohol etc. are less viscous than thick liquids like coal tar, blood, glycerin etc. The coefficients of viscosity for some common fluids are listed in Table 10.2. We point out two facts about blood and water that you may find interesting. As Table 10.2 indicates, blood is 'thicker' (more viscous) than water. Further the relative viscosity (η/η_{water}) of blood remains constant between 0°C and 37°C.

The viscosity of liquids decreases with temperature while it increases in the case of gases.

Example 10.9 A metal block of area 0.10 m^2 is connected to a 0.010 kg mass via a string that passes over an ideal pulley (considered massless and frictionless), as in Fig. 10.15. A liquid with a film thickness of 0.30 mm is placed between the block and the table. When released the block moves to the right with a constant speed of 0.085 m s^{-1}. Find the coefficient of viscosity of the liquid.

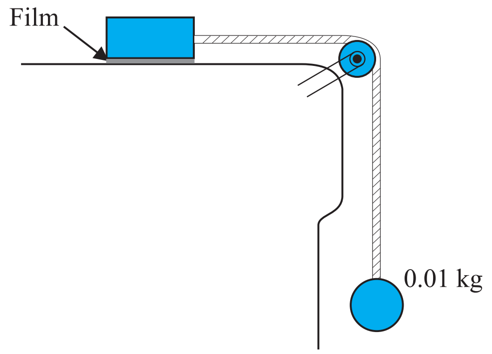


Fig. 10.15 *Measurement of the coefficient of viscosity of a liquid.*

Answer The metal block moves to the right because of the tension in the string. The tension T is equal in magnitude to the weight of the suspended mass m. Thus the shear force F is $F = T = mg = 0.010 \text{ kg} \times 9.8 \text{ m s}^{-2} = 9.8 \times 10^{-2}$ N

$$\text{Shear stress on the fluid} = F/A = \frac{9.8 \quad 10^{-2}}{0.10}$$

$$\text{Strain rate} = \frac{v}{l} \quad \frac{0.085}{0.030}$$

$$\eta = \frac{stress}{strain\ rate}$$

$$\frac{9.8 \quad 10^{-2} N \quad 0.30 \quad 10^{-3} m}{0.085 m\, s^{-1} \quad 0.10 m^2}$$

$= 3.45 \times 10^{-3}$ Pa s

Table10.2 **The viscosities of some fluids**

Fluid	T(°C)	Viscosity (mPl)
Water	20	1.0
	100	0.3
Blood	37	2.7
Machine Oil	16	113
	38	34
Glycerine	20	830
Honey		200
Air	0	0.017
	40	0.019

10.5.1 Stokes' Law

When a body falls through a fluid it drags the layer of the fluid in contact with it. A relative

motion between the different layers of the fluid is set and as a result the body experiences a retarding force. Falling of a raindrop and swinging of a pendulum bob are some common examples of such motion. It is seen that the viscous force is proportional to the velocity of the object and is opposite to the direction of motion. The other quantities on which the force *F* depends on viscosity η of the fluid and radius *a* of the sphere. Sir George G. Stokes (1819-1903), an English scientist enunciated clearly the viscous drag force *F* as

$$F \quad 6 \quad av \tag{10.19}$$

This is known as Stokes' law.We shall not derive Stokes' law.

This law is an interesting example of retarding force which is proportional to velocity. We can study its consequences on an object falling through a viscous medium. We consider a raindrop in air. It accelerates initially due to gravity. As the velocity increases, the retarding force also increases. Finally when viscous force plus buoyant force becomes equal to force due to gravity, the net force becomes zero and so does the acceleration. The sphere (raindrop) then descends with a constant velocity. Thus in equilibrium, this terminal velocity v_t is given by

$$6\pi\eta a v_t = (4\pi/3)\ a^3\ (\rho - \sigma) g$$

where ρ and σ are mass densities of sphere and the fluid respectively. We obtain

$$v_t = 2a^2\ (\rho - \sigma) g\ /\ (9\eta) \tag{10.20}$$

So the terminal velocity v_t depends on the square of the radius of the sphere and inversely on the viscosity of the medium.

You may like to refer back to Example 6.2 in this context.

Example 10.10 The terminal velocity of a copper ball of radius 2.0 mm falling through a tank of oil at 20°C is 6.5 cm s^{-1}. Compute the viscosity of the oil at 20°C. Density of oil is 1.5×10^3 kg m^{-3}, density of copper is 8.9×10^3 kg m^{-3}.

Answer We have $v_t = 6.5 \times 10^{-2}$ ms^{-1}, $a = 2 \times 10^{-3}$ m, $g = 9.8$ ms^{-2}, $\rho = 8.9 \times 10^3$ kg m^{-3}, $\sigma = 1.5 \times 10^3$ kg m^{-3}. From Eq. (10.20)

$$\frac{2}{9} \quad \frac{2 \quad 10^{-3}\ \text{m}^2 \quad 9.8\,\text{m s}^{-2}}{6.5 \quad 10^{-2}\ \text{m s}^{-1}} \quad 7.4 \quad 10^3\ \text{kg m}^{-3}$$

$$= 9.9 \times 10^{-1}\ \text{kg m}^{-1}\ \text{s}^{-1}$$

10.6 REYNOLDS NUMBER

When the rate of flow of a fluid is large, the flow no longer remain laminar, but becomes turbulent. In a turbulent flow the velocity of the fluids at any point in space varies rapidly and randomly with time. Some circular motions called eddies are also generated. An obstacle placed in the path of a fast moving fluid causes turbulence [Fig. 10.8 (b)]. The smoke rising from a burning stack of wood, oceanic currents are turbulent. Twinkling of stars is the result of atmospheric turbulence. The wakes in the water and in the air left by cars, aeroplanes and boats are also turbulent.

Osborne Reynolds (1842-1912) observed that turbulent flow is less likely for viscous fluid flowing at low rates. He defined a dimensionless number, whose value gives one an approximate idea whether the flow would be turbulent . This number is called the Reynolds R_e.

$$R_e = \rho v d / \eta \tag{10.21}$$

where ρ is the density of the fluid flowing with a speed *v*, *d* stands for the dimension of the pipe, and η is the viscosity of the fluid. R_e is a dimensionless number and therefore, it remains same in any system of units. It is found that flow is streamline or laminar for R_e less than 1000. The flow is turbulent for $R_e > 2000$. The flow becomes unsteady for R_e between 1000 and 2000. The critical value of R_e (known as critical Reynolds number), at which turbulence sets, is found to be the same for the geometrically similar flows. For example when oil and water with their different densities and viscosities, flow in pipes of same shapes and sizes, turbulence sets in at almost the same value of R_e. Using this fact a small scale laboratory model can be set up to study the character of fluid flow. They are useful in designing of ships, submarines, racing cars and aeroplanes.

R_e can also be written as

$$R_e = \rho v^2\ /\ (\eta v/d) = \rho A v^2\ /\ (\eta A v/d) \tag{10.22}$$

= inertial force/force of viscosity.

Thus R_e represents the ratio of inertial force (force due to inertia i.e. mass of moving fluid or due to inertia of obstacle in its path) to viscous force.

Turbulence dissipates kinetic energy usually in the form of heat. Racing cars and planes are engineered to precision in order to minimise turbulence. The design of such vehicles involves

experimentation and trial and error. On the other hand turbulence (like friction) is sometimes desirable. Turbulence promotes mixing and increases the rates of transfer of mass, momentum and energy. The blades of a kitchen mixer induce turbulent flow and provide thick milk shakes as well as beat eggs into a uniform texture.

> ***Example 10.11*** The flow rate of water from a tap of diameter 1.25 cm is 0.48 L/min. The coefficient of viscosity of water is 10^{-3} Pa s. After sometime the flow rate is increased to 3 L/min. Characterise the flow for both the flow rates.

Answer Let the speed of the flow be v and the diameter of the tap be d = 1.25 cm. The volume of the water flowing out per second is

$Q = v \times \pi d^2 / 4$

$v = 4\ Q /\ \ d^2$

We then estimate the Reynolds number to be

$R_e = 4\ \rho\ Q\ /\ \pi\ d\ \eta$

$= 4 \times 10^3 \text{kg m}^{-3} \times Q/(3.14 \times 1.25 \times 10^{-2} \text{m} \times 10^{-3} \text{Pa s})$

$= 1.019 \times 10^8 \text{ m}^{-3} \text{ s } Q$

Since initially

$Q = 0.48 \text{ L / min} = 8 \text{ cm}^3 \text{ / s} = 8 \times 10^{-6} \text{ m}^3 \text{ s}^{-1}$,

we obtain,

$R_e = 815$

Since this is below 1000, the flow is steady.

After some time when

$Q = 3 \text{ L / min} = 50 \text{ cm}^3 \text{ / s} = 5 \times 10^{-5} \text{ m}^3 \text{ s}^{-1}$,

we obtain,

$R_e = 5095$

The flow will be turbulent. You may carry out an experiment in your washbasin to determine the transition from laminar to turbulent flow.

10.7 SURFACE TENSION

You must have noticed that, oil and water do not mix; water wets you and me but not ducks; mercury does not wet glass but water sticks to it, oil rises up a cotton wick, inspite of gravity, Sap and water rise up to the top of the leaves of the tree, hairs of a paint brush do not cling together when dry and even when dipped in water but form a fine tip when taken out of it. All these and many more such experiences are related with the free surfaces of liquids. As liquids have no definite shape but have a definite volume, they acquire a free surface when poured in a container. These surfaces possess some additional energy. This phenomenon is known as surface tension and it is concerned with only liquid as gases do not have free surfaces. Let us now understand this phenomena.

10.7.1 Surface Energy

A liquid stays together because of attraction between molecules. Consider a molecule well inside a liquid. The intermolecular distances are such that it is attracted to all the surrounding molecules [Fig. 10.16(a)]. This attraction results in a negative potential energy for the molecule, which depends on the number and distribution of molecules around the chosen one. But the average potential energy of all the molecules is the same. This is supported by the fact that to take a collection of such molecules (the liquid) and to disperse them far away from each other in order to evaporate or vaporise, the heat of evaporation required is quite large. For water it is of the order of 40 kJ/mol.

Let us consider a molecule near the surface Fig. 10.16(b). Only lower half side of it is surrounded by liquid molecules. There is some negative potential energy due to these, but obviously it is less than that of a molecule in bulk, i.e., the one fully inside. Approximately it is half of the latter. Thus, molecules on a liquid surface have some extra energy in comparison to molecules in the interior. A liquid thus tends to have the least surface area which external conditions permit. Increasing surface area requires energy. Most surface phenomenon can be understood in terms of this fact. What is the energy required for having a molecule at the surface? As mentioned above, roughly it is half the energy required to remove it entirely from the liquid i.e., half the heat of evaporation.

Finally, what is a surface? Since a liquid consists of molecules moving about, there cannot be a perfectly sharp surface. The density of the liquid molecules drops rapidly to zero around $z = 0$ as we move along the direction indicated Fig 10.16 (c) in a distance of the order of a few molecular sizes.

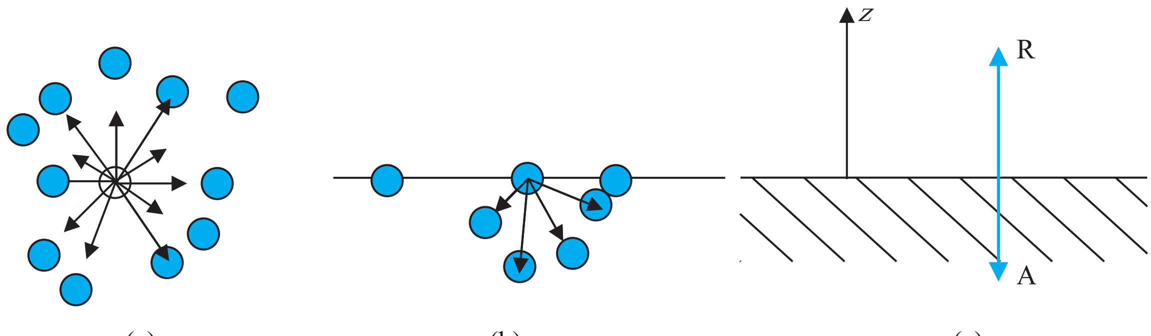


Fig. 10.16 *Schematic picture of molecules in a liquid, at the surface and balance of forces. (a) Molecule inside a liquid. Forces on a molecule due to others are shown. Direction of arrows indicates attraction of repulsion. (b) Same, for a molecule at a surface. (c) Balance of attractive (A) and repulsive (R) forces.*

10.7.2 Surface Energy and Surface Tension

As we have discussed that an extra energy is associated with surface of liquids, the creation of more surface (spreading of surface) keeping other things like volume fixed requires additional energy. To appreciate this, consider a horizontal liquid film ending in bar free to slide over parallel guides Fig (10.17).

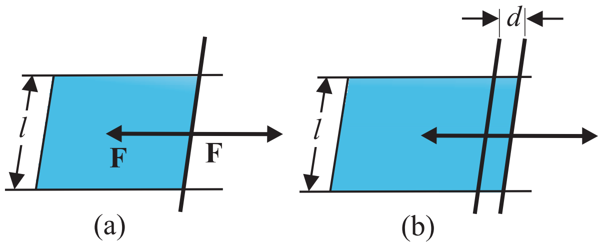


Fig. 10.17 *Stretching a film. (a) A film in equilibrium; (b) The film stretched an extra distance.*

Suppose that we move the bar by a small distance d as shown. Since the area of the surface increases, the system now has more energy, this means that some work has been done against an internal force. Let this internal force be **F**, the work done by the applied force is $\mathbf{F}\cdot\mathbf{d} = Fd$. From conservation of energy, this is stored as additional energy in the film. If the surface energy of the film is S per unit area, the extra area is $2dl$. A film has two sides and the liquid in between, so there are two surfaces and the extra energy is

$$S\,(2dl) = Fd \qquad (10.23)$$

$$\text{Or, } S = Fd/2dl = F/2l \qquad (10.24)$$

This quantity S is the magnitude of surface tension. It is equal to the surface energy per unit area of the liquid interface and is also equal to the force per unit length exerted by the fluid on the movable bar.

So far we have talked about the surface of one liquid. More generally, we need to consider fluid surface in contact with other fluids or solid surfaces. The surface energy in that case depends on the materials on both sides of the surface. For example, if the molecules of the materials attract each other, surface energy is reduced while if they repel each other the surface energy is increased. Thus, more appropriately, the surface energy is the energy of the interface between two materials and depends on both of them.

We make the following observations from above:

(i) Surface tension is a force per unit length (or surface energy per unit area) acting in the plane of the interface between the plane of the liquid and any other substance; it also is the extra energy that the molecules at the interface have as compared to molecules in the interior.

(ii) At any point on the interface besides the boundary, we can draw a line and imagine equal and opposite surface tension forces S per unit length of the line acting perpendicular to the line, in the plane of the interface. The line is in equilibrium. To be more specific, imagine a line of atoms or molecules at the surface. The atoms to the left pull the line towards them; those to the right pull it towards them! This line of atoms is in equilibrium under tension. If the line really marks the end of the interface, as in

Figure 10.16 (a) and (b) there is only the force S per unit length acting inwards.

Table 10.3 gives the surface tension of various liquids. The value of surface tension depends on temperature. Like viscosity, the surface tension of a liquid usually falls with temperature.

Table 10.3 **Surface tension of some liquids at the temperatures indicated with the heats of the vaporisation**

Liquid	Temp (°C)	Surface Tension (N/m)	Heat of vaporisation (kJ/mol)
Helium	–270	0.000239	0.115
Oxygen	–183	0.0132	7.1
Ethanol	20	0.0227	40.6
Water	20	0.0727	44.16
Mercury	20	0.4355	63.2

A fluid will stick to a solid surface if the surface energy between fluid and the solid is smaller than the sum of surface energies between solid-air, and fluid-air. Now there is cohesion between the solid surface and the liquid. It can be directly measured experimentaly as schematically shown in Fig. 10.18. A flat vertical glass plate, below which a vessel of some liquid is kept, forms one arm of the balance. The plate is balanced by weights on the other side, with its horizontal edge just over water. The vessel is raised slightly till the liquid just touches the glass plate and pulls it down a little because of surface tension. Weights are added till the plate just clears water.

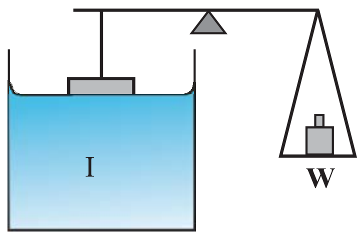


Fig. 10.18 *Measuring Surface Tension.*

Suppose the additional weight required is W. Then from Eq. 10.24 and the discussion given there, the surface tension of the liquid-air interface is

$$S_{la} = (W/2l) = (mg/2l) \qquad (10.25)$$

where m is the extra mass and l is the length of the plate edge. The subscript (la) emphasises the fact that the liquid-air interface tension is involved.

10.7.3 Angle of Contact

The surface of liquid near the plane of contact, with another medium is in general curved. The angle between tangent to the liquid surface at the point of contact and solid surface inside the liquid is termed as angle of contact. It is denoted by θ. It is different at interfaces of different pairs of liquids and solids. The value of θ determines whether a liquid will spread on the surface of a solid or it will form droplets on it. For example, water forms droplets on lotus leaf as shown in Fig. 10.19 (a) while spreads over a clean plastic plate as shown in Fig. 10.19(b).

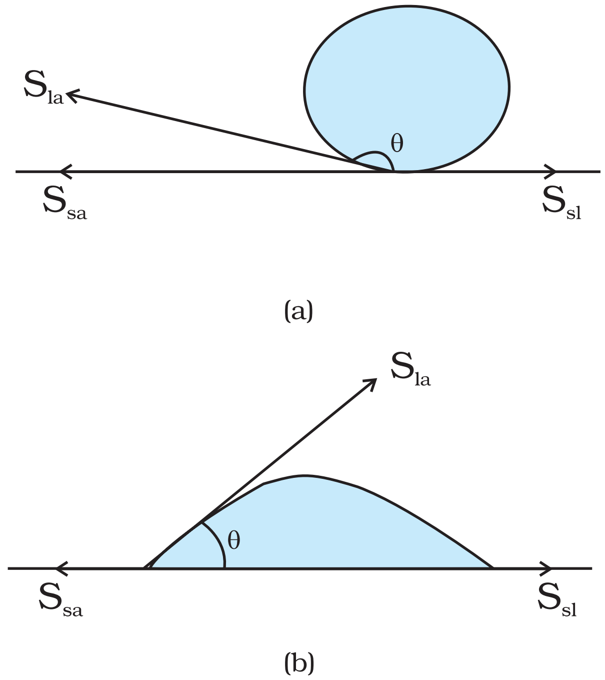


Fig. 10.19 *Different shapes of water drops with interfacial tensions (a) on a lotus leaf (b) on a clean plastic plate.*

We consider the three interfacial tensions at all the three interfaces, liquid-air, solid-air and solid-liquid denoted by S_{la}, S_{sa} & S_{sl} respectively as given in Fig. 10.19 (a) and (b). At the line of contact, the surface forces between the three media must be in equilibrium. From the Fig. 10.19(b) the following relation is easily derived.

$$S_{la} \cos\theta + S_{sl} = S_{sa} \quad (10.26)$$

The angle of contact is an obtuse angle if $S_{sl} > S_{la}$ as in the case of water-leaf interface while it is an acute angle if $S_{sl} < S_{la}$ as in the case of water-plastic interface. When θ is an obtuse angle then molecules of liquids are attracted strongly to themselves and weakly to those of solid, it costs a lot of energy to create a liquid-solid surface, and liquid then does not wet the solid. This is what happens with water on a waxy or oily surface, and with mercury on any surface. On the other hand, if the molecules of the liquid are strongly attracted to those of the solid, this will reduce S_{sl} and therefore, $\cos\theta$ may increase or θ may decrease. In this case θ is an acute angle. This is what happens for water on glass or on plastic and for kerosene oil on virtually anything (it just spreads). Soaps, detergents and dying substances are wetting agents. When they are added the angle of contact becomes small so that these may penetrate well and become effective. Water proofing agents on the other hand are added to create a large angle of contact between the water and fibres.

10.7.4 Drops and Bubbles

One consequence of surface tension is that free liquid drops and bubbles are spherical if effects of gravity can be neglected. You must have seen this especially clearly in small drops just formed in a high-speed spray or jet, and in soap bubbles blown by most of us in childhood. Why are drops and bubbles spherical? What keeps soap bubbles stable?

As we have been saying repeatedly, a liquid-air interface has energy, so for a given volume the surface with minimum energy is the one with the least area. The sphere has this property. Though it is out of the scope of this book, but you can check that a sphere is better than at least a cube in this respect! So, if gravity and other forces (e.g. air resistance) were ineffective, liquid drops would be spherical.

Another interesting consequence of surface tension is that the pressure inside a spherical drop Fig. 10.20(a) is more than the pressure outside. Suppose a spherical drop of radius r is in equilibrium. If its radius increase by Δr. The extra surface energy is

$$[4\pi(r + \Delta r)^2 - 4\pi r^2]\, S_{la} = 8\pi r\, \Delta r\, S_{la} \quad (10.27)$$

If the drop is in equilibrium this energy cost is balanced by the energy gain due to expansion under the pressure difference $(P_i - P_o)$ between the inside of the bubble and the outside. The work done is

$$W = (P_i - P_o)\, 4\pi r^2 \Delta r \quad (10.28)$$

so that

$$(P_i - P_o) = (2\, S_{la} / r) \quad (10.29)$$

In general, for a liquid-gas interface, the convex side has a higher pressure than the concave side. For example, an air bubble in a liquid, would have higher pressure inside it. See Fig 10.20 (b).

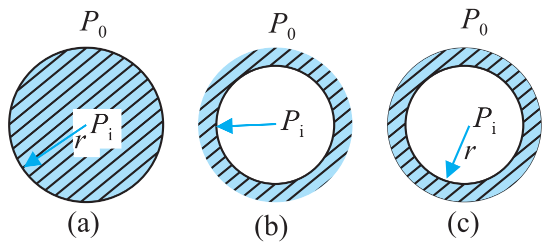


Fig. 10.20 *Drop, cavity and bubble of radius r.*

A bubble Fig 10.20 (c) differs from a drop and a cavity; in this it has two interfaces. Applying the above argument we have for a bubble

$$(P_i - P_o) = (4\, S_{la} / r) \quad (10.30)$$

This is probably why you have to blow hard, but not too hard, to form a soap bubble. A little extra air pressure is needed inside!

10.7.5 Capillary Rise

One consequence of the pressure difference across a curved liquid-air interface is the well-known effect that water rises up in a narrow tube in spite of gravity. The word capilla means

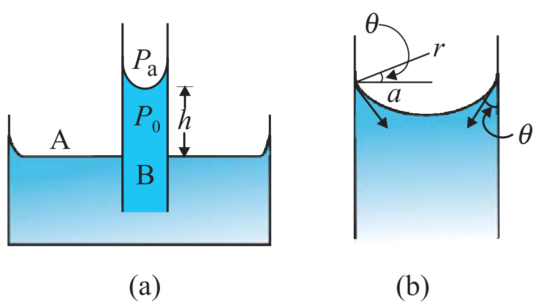


Fig. 10.21 *Capillary rise, (a) Schematic picture of a narrow tube immersed water. (b) Enlarged picture near interface.*

hair in Latin; if the tube were hair thin, the rise would be very large. To see this, consider a vertical capillary tube of circular cross section (radius a) inserted into an open vessel of water (Fig. 10.21). The contact angle between water and glass is acute. Thus the surface of water in **the capillary is concave. This means that there is a pressure difference between the two sides of the top surface. This is given by**

$(P_i - P_o) = (2S/r) = 2S/(a \sec \theta)$

$= (2S/a) \cos \theta$ (10.31)

Thus the pressure of the water inside the tube, just at the meniscus (air-water interface) is less than the atmospheric pressure. Consider the two points A and B in Fig. 10.21(a). They must be at the same pressure, namely

$P_0 + h \rho g = P_i = P_A$ (10.32)

where ρ is the density of water and h is called the capillary rise [Fig. 10.21(a)]. Using Eq. (10.31) and (10.32) we have

$h \rho g = (P_i - P_0) = (2S \cos \theta)/a$ (10.33)

The discussion here, and the Eqs. (10.28) and (10.29) make it clear that the capillary rise is due to surface tension. It is larger, for a smaller a. Typically it is of the order of a few cm for fine capillaries. For example, if a = 0.05 cm, using the value of surface tension for water (Table 10.3), we find that

$h = 2S/(\rho g a)$

$$\frac{2 \quad 0.073\ \text{Nm}^{-1}}{10^3\ \text{kg m}^{-3} \quad 9.8\ \text{m s}^{-2} \quad 5 \quad 10^{-4}\ \text{m}}$$

$= 2.98 \times 10^{-2}$ m = 2.98 cm

Notice that if the liquid meniscus is convex, as for mercury, i.e., if cos θ is negative then from Eq. (10.32) for example, it is clear that the liquid will be lower in the capillary !

10.7.6 Detergents and Surface Tension

We clean dirty clothes containing grease and oil stains sticking to cotton or other fabrics by adding detergents or soap to water, soaking clothes in it and shaking. Let us understand this process better.

Washing with water does not remove grease stains. This is because water does not wet greasy dirt; i.e., there is very little area of contact between them. If water could wet grease, the flow of water could carry some grease away. Something of this sort is achieved through detergents. The molecules of detergents are hairpin shaped, with one end attracted to water and the other to molecules of grease, oil or wax, thus tending to form water-oil interfaces. The result is shown in Fig. 10.22 as a sequence of figures.

In our language, we would say that addition of detergents, whose molecules attract at one end and say, oil on the other, reduces drastically the surface tension S (water-oil). It may even become energetically favourable to form such interfaces, i.e., globs of dirt surrounded by detergents and then by water. This kind of process using surface active detergents or surfactants is important not only for cleaning, but also in recovering oil, mineral ores etc.

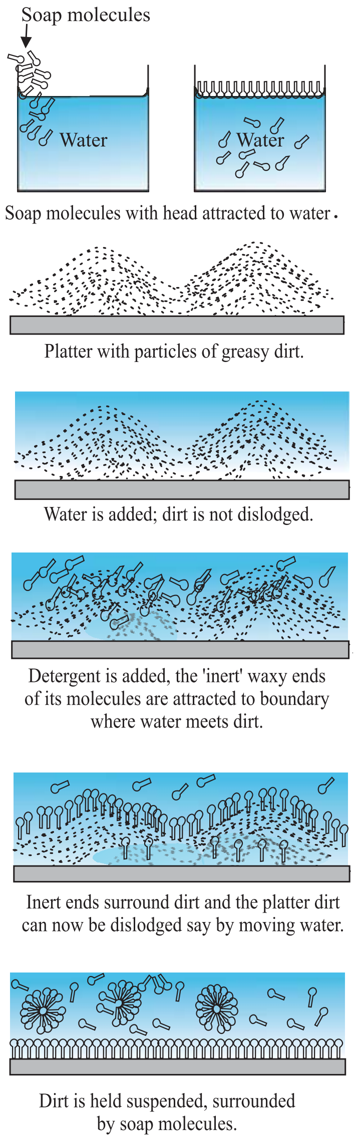


Fig. 10.22 *Detergent action in terms of what detergent molecules do.*

Example 10.12 The lower end of a capillary tube of diameter 2.00 mm is dipped 8.00 cm below the surface of water in a beaker. What is the pressure required in the tube in order to blow a hemispherical bubble at its end in water? The surface tension of water at temperature of the experiments is 7.30 $\times 10^{-2}$ Nm^{-1}. 1 atmospheric pressure = 1.01 × 10^5 Pa, density of water = 1000 kg/m^3, g = 9.80 m s^{-2}. Also calculate the excess pressure.

Answer The excess pressure in a bubble of gas in a liquid is given by $2S/r$, where S is the surface tension of the liquid-gas interface. You should note there is only one liquid surface in this case. (For a bubble of liquid in a gas, there are two liquid surfaces, so the formula for excess pressure in that case is $4S/r$.) The radius of the bubble is r. Now the pressure outside the bubble P_o equals atmospheric pressure plus the pressure due to 8.00 cm of water column. That is

$$P_o = (1.01 \times 10^5 \text{ Pa} + 0.08 \text{ m} \times 1000 \text{ kg m}^{-3} \times 9.80 \text{ m s}^{-2})$$

$$= 1.01784 \times 10^5 \text{ Pa}$$

Therefore, the pressure inside the bubble is

$$P_i = P_o + 2S/r$$

$$= 1.01784 \times 10^5 \text{ Pa} + (2 \times 7.3 \times 10^{-2} \text{ Pa m}/10^{-3} \text{ m})$$

$$= (1.01784 + 0.00146) \times 10^5 \text{ Pa}$$

$$= 1.02 \times 10^5 \text{ Pa}$$

where the radius of the bubble is taken to be equal to the radius of the capillary tube, since the bubble is hemispherical ! (The answer has been rounded off to three significant figures.) The excess pressure in the bubble is 146 Pa.

SUMMARY

1. The basic property of a fluid is that it can flow. The fluid does not have any resistance to change of its shape. Thus, the shape of a fluid is governed by the shape of its container.
2. A liquid is incompressible and has a free surface of its own. A gas is compressible and it expands to occupy all the space available to it.
3. If F is the normal force exerted by a fluid on an area A then the average pressure P_{av} is defined as the ratio of the force to area

   $$P_{av} = \frac{F}{A}$$

4. The unit of the pressure is the pascal (Pa). It is the same as N m^{-2}. Other common units of pressure are
   1 atm = 1.01×10^5 Pa
   1 bar = 10^5 Pa
   1 torr = 133 Pa = 0.133 kPa
   1 mm of Hg = 1 torr = 133 Pa
5. *Pascal's law* states that: Pressure in a fluid at rest is same at all points which are at the same height. A change in pressure applied to an enclosed fluid is transmitted undiminished to every point of the fluid and the walls of the containing vessel.
6. The pressure in a fluid varies with depth h according to the expression
   $P = P_a + \rho g h$
   where ρ is the density of the fluid, assumed uniform.
7. The volume of an incompressible fluid passing any point every second in a pipe of non uniform crossection is the same in the steady flow.
   $v A$ = constant (v is the velocity and A is the area of crossection)
   The equation is due to mass conservation in incompressible fluid flow.
8. *Bernoulli's principle* states that as we move along a streamline, the sum of the pressure (P), the kinetic energy per unit volume ($\rho v^2/2$) and the potential energy per unit volume ($\rho g y$) remains a constant.
   $P + \rho v^2/2 + \rho g y$ = constant

The equation is basically the conservation of energy applied to non viscuss fluid motion in steady state. There is no fluid which have zero viscosity, so the above statement is true only approximately. The viscosity is like friction and converts the kinetic energy to heat energy.

9. Though shear strain in a fluid does not require shear stress, when a shear stress is applied to a fluid, the motion is generated which causes a shear strain growing with time. The ratio of the shear stress to the time rate of shearing strain is known as coefficient of viscosity, η

   where symbols have their usual meaning and are defined in the text.

10. *Stokes' law* states that the viscous drag force **F** on a sphere of radius a moving with velocity **v** through a fluid of viscosity is, $\mathbf{F} = -6\pi\eta a\mathbf{v}$.

11. The onset of turbulence in a fluid is determined by a dimensionless parameter is called the *Reynolds number* given by

    $R_e = \rho v d/\eta$

    Where d is a typical geometrical length associated with the fluid flow and the other symbols have their usual meaning.

12. Surface tension is a force per unit length (or surface energy per unit area) acting in the plane of interface between the liquid and the bounding surface. It is the extra energy that the molecules at the interface have as compared to the interior.

POINTS TO PONDER

1. Pressure is a *scalar quantity*. The definition of the pressure as "force per unit area" may give one false impression that pressure is a vector. The "force" in the numerator of the definition is the component of the force normal to the area upon which it is impressed. While describing fluids as a conceptual shift from particle and rigid body mechanics is required. We are concerned with properties that vary from point to point in the fluid.

2. One should not think of pressure of a fluid as being exerted only on a solid like the walls of a container or a piece of solid matter immersed in the fluid. Pressure exists at all points in a fluid. An element of a fluid (such as the one shown in Fig. 10.2) is in equilibrium because the pressures exerted on the various faces are equal.

3. The expression for pressure

   $P = P_a + \rho g h$

   holds true if fluid is incompressible. Practically speaking it holds for liquids, which are largely incompressible and hence is a constant with height.

4. The gauge pressure is the difference of the actual pressure and the atmospheric pressure.

   $P - P_a = P_g$

   Many pressure-measuring devices measure the gauge pressure. These include the tyre pressure gauge and the blood pressure gauge (sphygmomanometer).

5. A streamline is a map of fluid flow. In a steady flow two streamlines do not intersect as it means that the fluid particle will have two possible velocities at the point.

6. Bernoulli's principal does not hold in presence of viscous drag on the fluid. The work done by this dissipative viscous force must be taken into account in this case, and P_2 [Fig. 10.9] will be lower than the value given by Eq. (10.12).

7. As the temperature rises the atoms of the liquid become more mobile and the coefficient of viscosity, η falls. In a gas the temperature rise increases the random motion of atoms and η increases.

8. The critical Reynolds number for the onset of turbulence is in the range 1000 to 10000, depending on the geometry of the flow. For most cases $R_e < 1000$ signifies laminar flow; $1000 < R_e < 2000$ is unsteady flow and $R_e > 2000$ implies turbulent flow.

9. Surface tension arises due to excess potential energy of the molecules on the surface in comparison to their potential energy in the interior. Such a surface energy is present at the interface separating two substances at least one of which is a fluid. It is not the property of a single fluid alone.

Physical Quantity	Symbol	Dimensions	Unit	Remarks
Pressure	P	$[M L^{-1} T^{-2}]$	pascal (Pa)	1 atm = 1.013×10^5 Pa, Scalar
Density	ρ	$[M L^{-3}]$	kg m^{-3}	Scalar
Specific Gravity		No	No	$\frac{\rho_{substance}}{\rho_{water}}$, Scalar
Co-efficient of viscosity	η	$[M L^{-1} T^{-1}]$	Pa s or poiseiulles (Pl)	Scalar
Reynold's Number	R_e	No	No	$R_e = \frac{\rho v d}{\eta}$ scalar
Surface Tension	S	$[M T^{-2}]$	N m^{-1}	Scalar

EXERCISES

10.1 Explain why
(a) The blood pressure in humans is greater at the feet than at the brain
(b) Atmospheric pressure at a height of about 6 km decreases to nearly half of its value at the sea level, though the height of the atmosphere is more than 100 km
(c) Hydrostatic pressure is a scalar quantity even though pressure is force divided by area.

10.2 Explain why
(a) The angle of contact of mercury with glass is obtuse, while that of water with glass is acute.
(b) Water on a clean glass surface tends to spread out while mercury on the same surface tends to form drops. (Put differently, water wets glass while mercury does not.)
(c) Surface tension of a liquid is independent of the area of the surface
(d) Water with detergent disolved in it should have small angles of contact.
(e) A drop of liquid under no external forces is always spherical in shape

10.3 Fill in the blanks using the word(s) from the list appended with each statement:
(a) Surface tension of liquids generally ... with temperatures (increases / decreases)
(b) Viscosity of gases ... with temperature, whereas viscosity of liquids ... with temperature (increases / decreases)
(c) For solids with elastic modulus of rigidity, the shearing force is proportional to ... , while for fluids it is proportional to ... (shear strain / rate of shear strain)
(d) For a fluid in a steady flow, the increase in flow speed at a constriction follows (conservation of mass / Bernoulli's principle)
(e) For the model of a plane in a wind tunnel, turbulence occurs at a ... speed for turbulence for an actual plane (greater / smaller)

10.4 Explain why
(a) To keep a piece of paper horizontal, you should blow over, not under, it
(b) When we try to close a water tap with our fingers, fast jets of water gush through the openings between our fingers
(c) The size of the needle of a syringe controls flow rate better than the thumb pressure exerted by a doctor while administering an injection
(d) A fluid flowing out of a small hole in a vessel results in a backward thrust on the vessel
(e) A spinning cricket ball in air does not follow a parabolic trajectory

10.5 A 50 kg girl wearing high heel shoes balances on a single heel. The heel is circular with a diameter 1.0 cm. What is the pressure exerted by the heel on the horizontal floor ?

10.6 Toricelli's barometer used mercury. Pascal duplicated it using French wine of density 984 kg m^{-3}. Determine the height of the wine column for normal atmospheric pressure.

10.7 A vertical off-shore structure is built to withstand a maximum stress of 10^9 Pa. Is the structure suitable for putting up on top of an oil well in the ocean ? Take the depth of the ocean to be roughly 3 km, and ignore ocean currents.

10.8 A hydraulic automobile lift is designed to lift cars with a maximum mass of 3000 kg. The area of cross-section of the piston carrying the load is 425 cm^2. What maximum pressure would the smaller piston have to bear ?

10.9 A U-tube contains water and methylated spirit separated by mercury. The mercury columns in the two arms are in level with 10.0 cm of water in one arm and 12.5 cm of spirit in the other. What is the specific gravity of spirit ?

10.10 In the previous problem, if 15.0 cm of water and spirit each are further poured into the respective arms of the tube, what is the difference in the levels of mercury in the two arms ? (Specific gravity of mercury = 13.6)

10.11 Can Bernoulli's equation be used to describe the flow of water through a rapid in a river ? Explain.

10.12 Does it matter if one uses gauge instead of absolute pressures in applying Bernoulli's equation ? Explain.

10.13 Glycerine flows steadily through a horizontal tube of length 1.5 m and radius 1.0 cm. If the amount of glycerine collected per second at one end is 4.0×10^{-3} kg s^{-1}, what is the pressure difference between the two ends of the tube ? (Density of glycerine = 1.3×10^3 kg m^{-3} and viscosity of glycerine = 0.83 Pa s). [You may also like to check if the assumption of laminar flow in the tube is correct].

10.14 In a test experiment on a model aeroplane in a wind tunnel, the flow speeds on the upper and lower surfaces of the wing are 70 m $s^{-1}$and 63 m s^{-1} respectively. What is the lift on the wing if its area is 2.5 m^2 ? Take the density of air to be 1.3 kg m^{-3}.

10.15 Figures 10.23(a) and (b) refer to the steady flow of a (non-viscous) liquid. Which of the two figures is incorrect ? Why ?

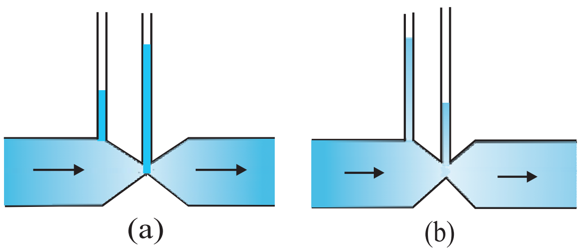


Fig. 10.23

10.16 The cylindrical tube of a spray pump has a cross-section of 8.0 cm^2 one end of which has 40 fine holes each of diameter 1.0 mm. If the liquid flow inside the tube is 1.5 m min^{-1}, what is the speed of ejection of the liquid through the holes ?

10.17 A U-shaped wire is dipped in a soap solution, and removed. The thin soap film formed between the wire and the light slider supports a weight of 1.5×10^{-2} N (which includes the small weight of the slider). The length of the slider is 30 cm. What is the surface tension of the film ?

10.18 Figure 10.24 (a) shows a thin liquid film supporting a small weight = 4.5×10^{-2} N. What is the weight supported by a film of the same liquid at the same temperature in Fig. (b) and (c) ? Explain your answer physically.

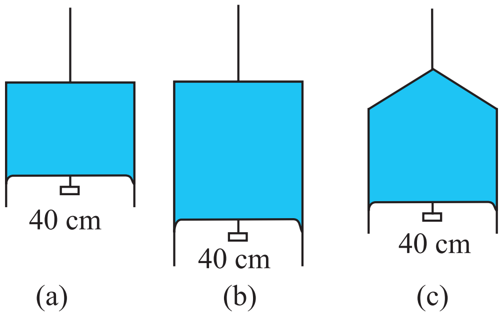


Fig. 10.24

10.19 What is the pressure inside the drop of mercury of radius 3.00 mm at room temperature ? Surface tension of mercury at that temperature (20 °C) is 4.65×10^{-1} N m^{-1}. The atmospheric pressure is 1.01×10^5 Pa. Also give the excess pressure inside the drop.

10.20 What is the excess pressure inside a bubble of soap solution of radius 5.00 mm, given that the surface tension of soap solution at the temperature (20 °C) is 2.50×10^{-2} N m^{-1}? If an air bubble of the same dimension were formed at depth of 40.0 cm inside a container containing the soap solution (of relative density 1.20), what would be the pressure inside the bubble ? (1 atmospheric pressure is 1.01×10^5 Pa).

Additional Exercises

10.21 A tank with a square base of area 1.0 m^2 is divided by a vertical partition in the middle. The bottom of the partition has a small-hinged door of area 20 cm^2. The tank is filled with water in one compartment, and an acid (of relative density 1.7) in the other, both to a height of 4.0 m. compute the force necessary to keep the door close.

10.22 A manometer reads the pressure of a gas in an enclosure as shown in Fig. 10.25 (a) When a pump removes some of the gas, the manometer reads as in Fig. 10.25 (b) The liquid used in the manometers is mercury and the atmospheric pressure is 76 cm of mercury.

(a) Give the absolute and gauge pressure of the gas in the enclosure for cases (a) and (b), in units of cm of mercury.

(b) How would the levels change in case (b) if 13.6 cm of water (immiscible with mercury) are poured into the right limb of the manometer ? (Ignore the small change in the volume of the gas).

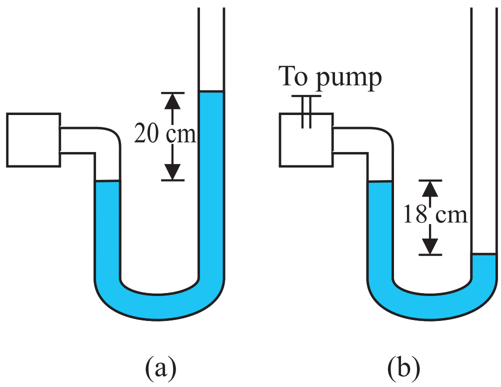


Fig. 10.25

10.23 Two vessels have the same base area but different shapes. The first vessel takes twice the volume of water that the second vessel requires to fill upto a particular common height. Is the force exerted by the water on the base of the vessel the same in the two cases ? If so, why do the vessels filled with water to that same height give different readings on a weighing scale ?

10.24 During blood transfusion the needle is inserted in a vein where the gauge pressure is 2000 Pa. At what height must the blood container be placed so that blood may just enter the vein ? [Use the density of whole blood from Table 10.1].

10.25 In deriving Bernoulli's equation, we equated the work done on the fluid in the tube to its change in the potential and kinetic energy. (a) What is the largest average velocity of blood flow in an artery of diameter 2×10^{-3} m if the flow must remain laminar ? (b) Do the dissipative forces become more important as the fluid velocity increases ? Discuss qualitatively.

10.26 (a) What is the largest average velocity of blood flow in an artery of radius 2×10^{-3}m if the flow must remain lanimar? (b) What is the corresponding flow rate ? (Take viscosity of blood to be 2.084×10^{-3} Pa s).

10.27 A plane is in level flight at constant speed and each of its two wings has an area of 25 m^2. If the speed of the air is 180 km/h over the lower wing and 234 km/h over the upper wing surface, determine the plane's mass. (Take air density to be 1 kg m^{-3}).

10.28 In Millikan's oil drop experiment, what is the terminal speed of an uncharged drop of radius 2.0×10^{-5} m and density 1.2×10^3 kg m^{-3}. Take the viscosity of air at the temperature of the experiment to be 1.8×10^{-5} Pa s. How much is the viscous force on the drop at that speed ? Neglect buoyancy of the drop due to air.

10.29 Mercury has an angle of contact equal to 140° with soda lime glass. A narrow tube of radius 1.00 mm made of this glass is dipped in a trough containing mercury. By what amount does the mercury dip down in the tube relative to the liquid surface outside ? Surface tension of mercury at the temperature of the experiment is 0.465 N m^{-1}. Density of mercury = 13.6×10^3 kg m^{-3}.

10.30 Two narrow bores of diameters 3.0 mm and 6.0 mm are joined together to form a U-tube open at both ends. If the U-tube contains water, what is the difference in its levels in the two limbs of the tube ? Surface tension of water at the temperature of the experiment is 7.3×10^{-2} N m^{-1}. Take the angle of contact to be zero and density of water to be 1.0×10^3 kg m^{-3} (g = 9.8 m s^{-2}) .

Calculator/Computer – Based Problem

10.31 (a) It is known that density ρ of air decreases with height y as

$$_0e^{\;y/y_o}$$

where ρ_0 = 1.25 kg m^{-3} is the density at sea level, and y_0 is a constant. This density variation is called the law of atmospheres. Obtain this law assuming that the temperature of atmosphere remains a constant (isothermal conditions). Also assume that the value of g remains constant.

(b) A large He balloon of volume 1425 m^3 is used to lift a payload of 400 kg. Assume that the balloon maintains constant radius as it rises. How high does it rise ?

[Take y_0 = 8000 m and ρ_{He} = 0.18 kg m^{-3}].

APPENDIX 10.1 : WHAT IS BLOOD PRESSURE ?

In evolutionary history there occurred a time when animals started spending a significant amount of time in the upright position. This placed a number of demands on the circulatory system. The venous system that returns blood from the lower extremities to the heart underwent changes. You will recall that veins are blood vessels through which blood returns to the heart. Humans and animals such as the giraffe have adapted to the problem of moving blood upward against gravity. But animals such as snakes, rats and rabbits will die if held upwards, since the blood remains in the lower extremities and the venous system is unable to move it towards the heart.

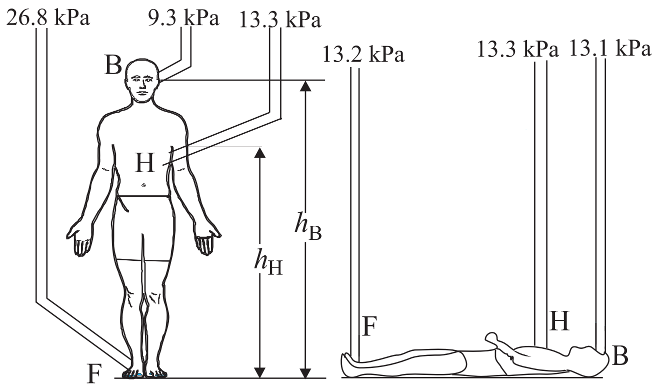


Fig. 10.26 *Schematic view of the gauge pressures in the arteries in various parts of the human body while standing or lying down. The pressures shown are averaged over a heart cycle.*

Figure 10.26 shows the average pressures observed in the arteries at various points in the human body. Since viscous effects are small, we can use Bernoulli's equation, Eq. (10.13),

$$P \quad \frac{1}{2} \quad v^2 \qquad gy \quad \text{Constant}$$

to understand these pressure values. The kinetic energy term ($\rho\, v^2/2$) can be ignored since the velocities in the three arteries are small ($\approx$ 0.1 m s^{-1}) and almost constant. Hence the gauge pressures at the brain P_B, the heart P_H, and the foot P_F are related by

$$P_F = P_H + \rho\, g\, h_H = P_B + \rho\, g\, h_B \tag{10.34}$$

where ρ is the density of blood.

Typical values of the heights to the heart and the brain are h_H = 1.3 m and h_B = 1.7 m. Taking $\rho = 1.06 \times 10^3$ kg m^{-3} we obtain that P_F = 26.8 kPa (kilopascals) and P_B = 9.3 kPa given that P_H = 13.3 kPa. Thus the pressures in the lower and upper parts of the body are so different when a person is standing, but are almost equal when he is lying down. As mentioned in the text the units for pressure more commonly employed in medicine and physiology are torr and mm of Hg. 1 mm of Hg = 1 torr = 0.133 kPa. Thus the average pressure at the heart is P_H = 13.3 kPa = 100 mm of Hg.

The human body is a marvel of nature. The veins in the lower extremities are equipped with valves, which open when blood flows towards the heart and close if it tends to drain down. Also, blood is returned at least partially by the pumping action associated with breathing and by the flexing of the skeletal muscles during walking. This explains why a soldier who is required to stand at attention may faint because of insufficient return of the blood to the heart. Once he is made to lie down, the pressures become equalized and he regains consciousness.

An instrument called the sphygmomanometer usually measures the blood pressure of humans. It is a fast, painless and non-invasive technique and gives the doctor a reliable idea about the patient's health. The measurement process is shown in Fig. 10.27. There are two reasons why the upper arm is used. First, it is at the same level as the heart and measurements here give values close to that at the heart. Secondly, the upper arm contains a single bone and makes the artery there (called the brachial artery) easy to compress. We have all measured pulse rates by placing our fingers over the wrist. Each pulse takes a little less than a second. During each pulse the pressure in the heart and the circulatory system goes through a

maximum as the blood is pumped by the heart (**systolic pressure**) and a minimum as the heart relaxes **(diastolic pressure)**. The sphygmomanometer is a device, which measures these extreme pressures. It works on the principle that **blood flow in the brachial (upper arm) artery can be made to go from laminar to turbulent by suitable compression. Turbulent flow is dissipative, and its sound can be picked up on the stethoscope**.

The gauge pressure in an air sack wrapped around the upper arm is measured using a manometer or a dial pressure gauge (Fig. 10.27). The pressure in the sack is first increased till the brachial artery is closed. The pressure in the sack is then slowly reduced while a stethoscope placed just below the sack is used to listen to noises arising in the brachial artery. When the pressure is just below the **systolic** (peak) pressure, the artery opens briefly. During this brief period, the blood velocity in the highly constricted artery is high and turbulent and hence noisy. The resulting noise is heard as a **tapping sound** on the stethoscope. When the pressure in the sack is lowered further, the artery remains open for a longer portion of the heart cycle. Nevertheless, it remains closed during the **diastolic** (minimum pressure) phase of the heartbeat. Thus the duration of the tapping sound is longer. When the pressure in the sack reaches the diastolic pressure the artery is open during the entire heart cycle. The flow is however, still turbulent and noisy. But instead of a tapping sound we hear a steady, continuous roar on the stethoscope.

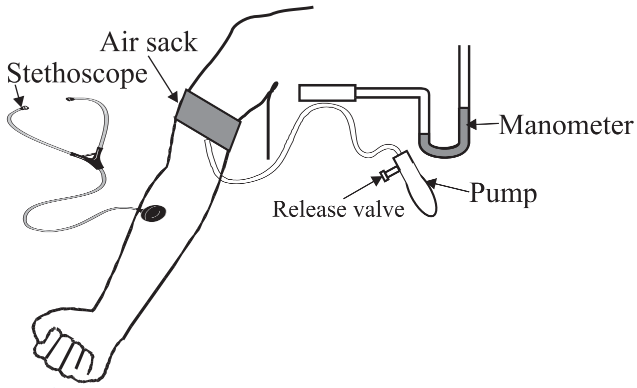


Fig. 10.27 *Blood pressure measurement using the sphygmomanometer and stethoscope.*

The blood pressure of a patient is presented as the ratio of systolic/diastolic pressures. For a resting healthy adult it is typically 120/80 mm of Hg (120/80 torr). Pressures above 140/90 require medical attention and advice. High blood pressures may seriously damage the heart, kidney and other organs and must be controlled.

CHAPTER ELEVEN

THERMAL PROPERTIES OF MATTER

11.1 INTRODUCTION

We all have common-sense notions of heat and temperature. Temperature is a measure of 'hotness' of a body. A kettle with boiling water is hotter than a box containing ice. In physics, we need to define the notion of heat, temperature, etc., more carefully. In this chapter, you will learn what heat is and how it is measured, and study the various proceses by which heat flows from one body to another. Along the way, you will find out why blacksmiths heat the iron ring before fitting on the rim of a wooden wheel of a bullock cart and why the wind at the beach often reverses direction after the sun goes down. You will also learn what happens when water boils or freezes, and its temperature does not change during these processes even though a great deal of heat is flowing into or out of it.

11.2 TEMPERATURE AND HEAT

We can begin studying thermal properties of matter with definitions of temperature and heat. Temperature is a relative measure, or indication of hotness or coldness. A hot utensil is said to have a high temperature, and ice cube to have a low temperature. An object that has a higher temperature than another object is said to be hotter. Note that hot and cold are relative terms, like tall and short. We can perceive temperature by touch. However, this temperature sense is somewhat unreliable and its range is too limited to be useful for scientific purposes.

We know from experience that a glass of ice-cold water left on a table on a hot summer day eventually warms up whereas a cup of hot tea on the same table cools down. It means that when the temperature of body, ice-cold water or hot tea in this case, and its surrounding medium are different, heat transfer takes place between the system and the surrounding medium, until the body and the surrounding medium are at the same temperature. We also know that in the case of glass tumbler of ice cold water, heat flows from the environment to

the glass tumbler, whereas in the case of hot tea, it flows from the cup of hot tea to the environment. So, we can say that **heat is the form of energy transferred between two (or more) systems or a system and its surroundings by virtue of temperature difference.** The SI unit of heat energy transferred is expressed in joule (J) while SI unit of temperature is kelvin (K), and °C is a commonly used unit of temperature. When an object is heated, many changes may take place. Its temperature may rise, it may expand or change state. We will study the effect of heat on different bodies in later sections.

11.3 MEASUREMENT OF TEMPERATURE

A measure of temperature is obtained using a thermometer. Many physical properties of materials change sufficiently with temperature to be used as the basis for constructing thermometers. The commonly used property is variation of the volume of a liquid with temperature. For example, a common thermometer (the liquid-in-glass type) with which you are familiar. Mercury and alcohol are the liquids used in most liquid-in-glass thermometers.

Thermometers are calibrated so that a numerical value may be assigned to a given temperature. For the definition of any standard scale, two fixed reference points are needed. Since all substances change dimensions with temperature, an absolute reference for expansion is not available. However, the necessary fixed points may be correlated to physical phenomena that always occur at the same temperature. The ice point and the steam point of water are two convenient fixed points and are known as the freezing and boiling points. These two points are the temperatures at which pure water freezes and boils under standard pressure. The two familiar temperature scales are the Fahrenheit temperature scale and the Celsius temperature scale. The ice and steam point have values 32 °F and 212 °F respectively, on the Fahrenheit scale and 0 °C and 100 °C on the Celsius scale. On the Fahrenheit scale, there are 180 equal intervals between two reference points, and on the celsius scale, there are 100.

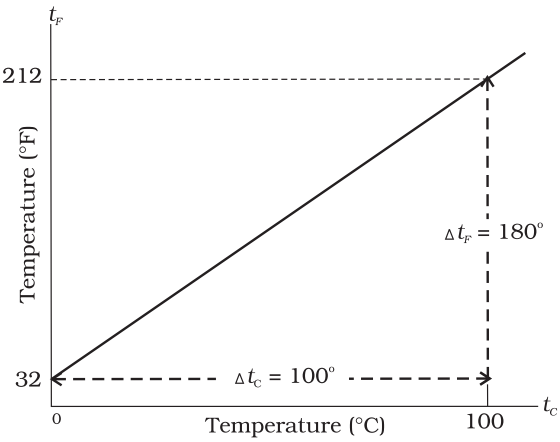


Fig. 11.1 *A plot of Fahrenheit temperature (t_F) versus Celsius temperature (t_C).*

A relationship for converting between the two scales may be obtained from a graph of Fahrenheit temperature (t_F) versus celsius temperature (t_C) in a straight line (Fig. 11.1), whose equation is

$$\frac{t_F - 32}{180} = \frac{t_C}{100} \qquad (11.1)$$

11.4 IDEAL-GAS EQUATION AND ABSOLUTE TEMPERATURE

Liquid-in-glass thermometers show different readings for temperatures other than the fixed points because of differing expansion properties. A thermometer that uses a gas, however, gives the same readings regardless of which gas is used. Experiments show that all gases at low densities exhibit same expansion behaviour. The variables that describe the behaviour of a given quantity (mass) of gas are pressure, volume, and temperature (P, V, and T)(where $T = t + 273.15$; t is the temperature in °C). When temperature is held constant, the pressure and volume of a quantity of gas are related as PV = constant. This relationship is known as Boyle's law, after Robert Boyle (1627-1691) the English Chemist who discovered it. When the pressure is held constant, the volume of a quantity of the gas is related to the temperature as V/T = constant. This relationship is known as Charles' law, after the French scientist Jacques Charles (1747-1823). Low density gases obey these laws, which may be combined into a single relationship.

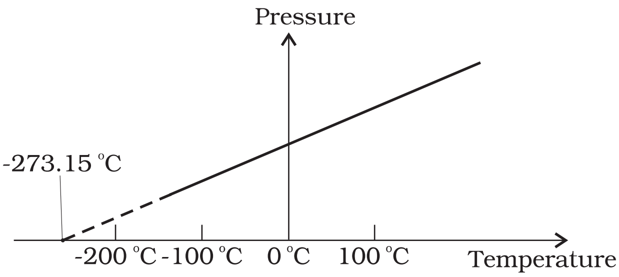


Fig. 11.2 *Pressure versus temperature of a low density gas kept at constant volume.*

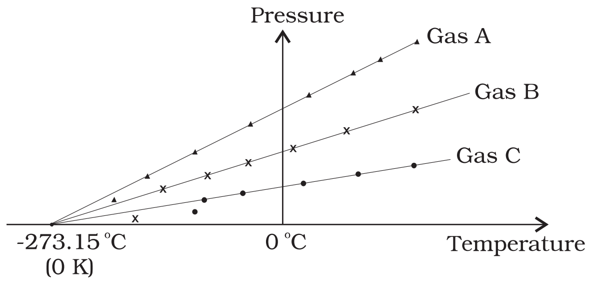


Fig. 11.3 *A plot of pressure versus temperature and extrapolation of lines for low density gases indicates the same absolute zero temperature.*

Notice that since PV = constant and V/T = constant for a given quantity of gas, then PV/T should also be a constant. This relationship is known as ideal gas law. It can be written in a more general form that applies not just to a given quantity of a single gas but to any quantity of any dilute gas and is known as **ideal-gas equation**:

$$\frac{PV}{T} = \mu R$$

or $PV = \mu RT$ (11.2)

where, μ is the number of moles in the sample of gas and R is called universal gas constant:

$R = 8.31$ J mol^{-1} K^{-1}

In Eq. 11.2, we have learnt that the pressure and volume are directly proportional to temperature : $PV \propto T$. This relationship allows a gas to be used to measure temperature in a constant volume gas thermometer. Holding the volume of a gas constant, it gives $P \propto T$. Thus, with a constant-volume gas thermometer, temperature is read in terms of pressure. A plot of pressure versus temperature gives a straight line in this case, as shown in Fig. 11.2.

However, measurements on real gases deviate from the values predicted by the ideal gas law at low temperature. But the relationship is linear over a large temperature range, and it looks as though the pressure might reach zero with decreasing temperature if the gas continued to be a gas. The absolute minimum temperature for an ideal gas, therefore, inferred by extrapolating the straight line to the axis, as in Fig. 11.3. This temperature is found to be – 273.15 °C and is designated as **absolute zero**. Absolute zero is the foundation of the Kelvin temperature scale or absolute scale temperature named after the British scientist Lord Kelvin. On this scale, – 273.15 °C is taken as the zero point, that is 0 K (Fig. 11.4).

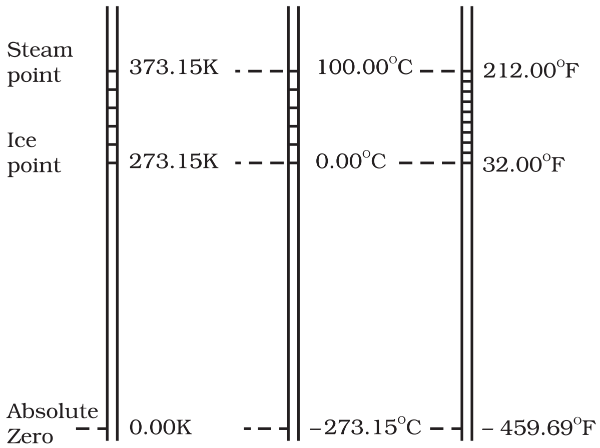


Fig. 11.4 *Comparision of the Kelvin, Celsius and Fahrenheit temperature scales.*

The size of the unit for Kelvin temperature is the same celsius degree, so temperature on these scales are related by

$$T = t_C + 273.15 \qquad (11.3)$$

11.5 THERMAL EXPANSION

You may have observed that sometimes sealed bottles with metallic lids are so tightly screwed that one has to put the lid in hot water for sometime to open the lid. This would allow the metallic cover to expand, thereby loosening it to unscrew easily. In case of liquids, you may have observed that mercury in a thermometer rises, when the thermometer is put in a slightly warm water. If we take out the thermometer from the

warm water the level of mercury falls again. Similarly, in the case of gases, a balloon partially inflated in a cool room may expand to full size when placed in warm water. On the other hand, a fully inflated balloon when immersed in cold water would start shrinking due to contraction of the air inside.

It is our common experience that most substances expand on heating and contract on cooling. A change in the temperature of a body causes change in its dimensions. The increase in the dimensions of a body due to the increase in its temperature is called thermal expansion. The expansion in length is called **linear expansion**. The expansion in area is called **area expansion**. The expansion in volume is called **volume expansion** (Fig. 11.5).

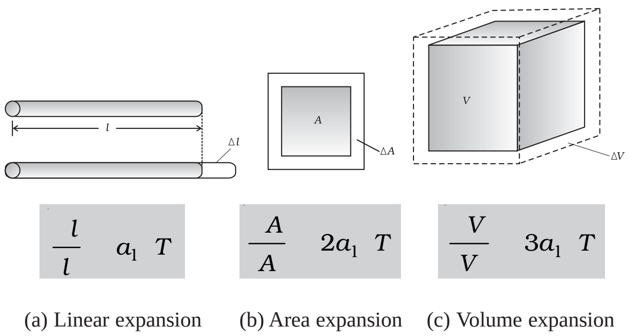


Fig. 11.5 *Thermal Expansion.*

If the substance is in the form of a long rod, then for small change in temperature, ΔT, the fractional change in length, $\Delta l/l$, is directly proportional to ΔT.

$$\frac{\Delta l}{l} = \alpha_1 \Delta T \quad (11.4)$$

where α_1 is known as the **coefficient of linear expansion** and is characteristic of the material of the rod. In Table 11.1 are given typical average values of the coefficient of linear expansion for some materials in the temperature range 0 °C to 100 °C. From this Table, compare the value of α_l for glass and copper. We find that copper expands about five times more than glass for the same rise in temperature. Normally, metals expand more and have relatively high values of α_l.

Table 11.1 Values of coefficient of linear expansion for some materials

Materials	α_l (10^{-5} K^{-1})
Aluminium	2.5
Brass	1.8
Iron	1.2
Copper	1.7
Silver	1.9
Gold	1.4
Glass (pyrex)	0.32
Lead	0.29

Similarly, we consider the fractional change in volume, $\frac{\Delta V}{V}$, of a substance for temperature change ΔT and define the **coefficient of volume expansion**, α_V as

$$\alpha_V = \left(\frac{\Delta V}{V}\right)\frac{1}{\Delta T} \quad (11.5)$$

Here α_V is also a characteristic of the substance but is not strictly a constant. It depends in general on temperature (Fig 11.6). It is seen that α_V becomes constant only at a high temperature.

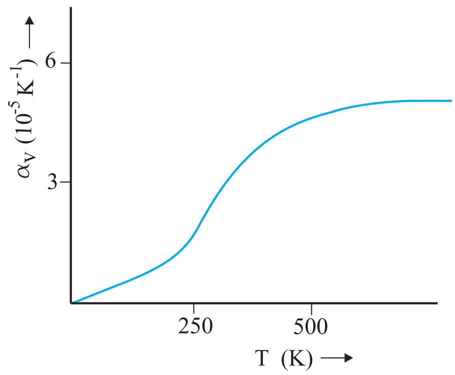


Fig. 11.6 *Coefficient of volume expansion of copper as a function of temperature.*

Table 11.2 gives the values of co-efficient of volume expansion of some common substances in the temperature range 0 –100 °C. You can see that thermal expansion of these substances (solids and liquids) is rather small, with

materials like pyrex glass and invar (a special iron-nickel alloy) having particularly low values of α_V. From this Table we find that the value of α_v for alcohol (ethyl) is more than mercury and expands more than mercury for the same rise in temperature.

Table 11.2 Values of coefficient of volume expansion for some substances

Materials	α_v (K^{-1})
Aluminium	7×10^{-5}
Brass	6×10^{-5}
Iron	3.55×10^{-5}
Paraffin	58.8×10^{-5}
Glass (ordinary)	2.5×10^{-5}
Glass (pyrex)	1×10^{-5}
Hard rubber	2.4×10^{-4}
Invar	2×10^{-6}
Mercurry	18.2×10^{-5}
Water	20.7×10^{-5}
Alcohol (ethyl)	110×10^{-5}

Water exhibits an anomalous behaviour; it contracts on heating between 0 °C and 4 °C. The volume of a given amount of water decreases as it is cooled from room temperature, until its temperature reaches 4 °C, [Fig. 11.7(a)]. Below 4 °C, the volume increases, and therefore the density decreases [Fig. 11.7(b)].

This means that water has a maximum density at 4 °C. This property has an important environmental effect: Bodies of water, such as lakes and ponds, freeze at the top first. As a lake cools toward 4 °C, water near the surface loses energy to the atmosphere, becomes denser, and sinks; the warmer, less dense water near the bottom rises. However, once the colder water on top reaches temperature below 4 °C, it becomes less dense and remains at the surface, where it freezes. If water did not have this property, lakes and ponds would freeze from the bottom up, which would destroy much of their animal and plant life.

Gases at ordinary temperature expand more than solids and liquids. For liquids, the coefficient of volume expansion is relatively independent of the temperature. However, for gases it is dependent on temperature. For an ideal gas, the coefficient of volume expansion at constant pressure can be found from the ideal gas equation :

$PV = \mu RT$

At constant pressure

$P\Delta V = \mu R \, \Delta T$

$$\frac{\Delta V}{V} = \frac{\Delta T}{T}$$

$$\text{i.e. } \alpha_v = \frac{1}{T} \text{ for ideal gas} \qquad (11.6)$$

At 0 °C, $\alpha_v = 3.7 \times 10^{-3}\ K^{-1}$, which is much larger than that for solids and liquids. Equation (11.6) shows the temperature dependence of α_v; it decreases with increasing temperature. For a gas at room temperature and constant pressure α_v is about $3300 \times 10^{-6} K^{-1}$, as

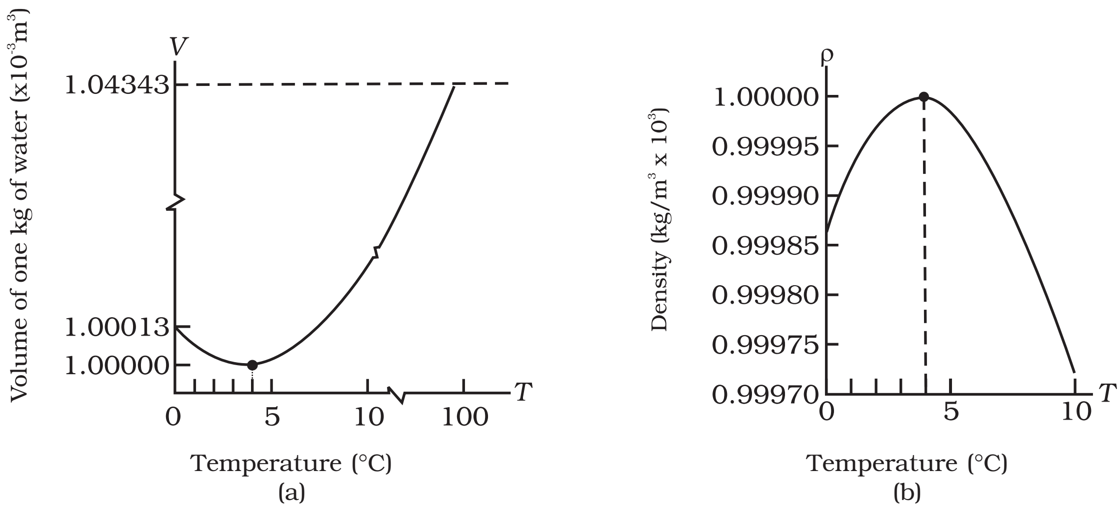


Fig. 11.7 *Thermal expansion of water.*

much as order(s) of magnitude larger than the coefficient of volume expansion of typical liquids.

There is a simple relation between the coefficient of volume expansion (α_v) and coefficient of linear expansion (α_l). Imagine a cube of length, l, that expands equally in all directions, when its temperature increases by ΔT. We have

$\Delta l = \alpha_l \, l \, \Delta T$

so, $\Delta V = (l+\Delta l)^3 - l^3 \simeq 3l^2 \, \Delta l$ (11.7)

In equation (11.7), terms in $(\Delta l)^2$ and $(\Delta l)^3$ have been neglected since Δl is small compared to l. So

$$\Delta V = \frac{3V \, \Delta l}{l} = 3V\alpha_l \, \Delta T \qquad (11.8)$$

which gives

$$\alpha_v = 3\alpha_l \qquad (11.9)$$

What happens by preventing the thermal expansion of a rod by fixing its ends rigidly? Clearly, the rod acquires a compressive strain due to the external forces provided by the rigid support at the ends. The corresponding stress set up in the rod is called **thermal stress**. For example, consider a steel rail of length 5 m and area of cross section 40 cm² that is prevented from expanding while the temperature rises by 10 °C. The coefficient of linear expansion of steel is $\alpha_{l(steel)} = 1.2 \times 10^{-5}$ K^{-1}. Thus, the compressive strain is $\frac{\Delta l}{l} = \alpha_{l(steel)} \Delta T = 1.2 \times 10^{-5} \times 10 = 1.2 \times 10^{-4}$.

Youngs modulus of steel is $Y_{(steel)} = 2 \times 10^{11}$ N m^{-2}. Therefore, the thermal stress developed is $\frac{\Delta F}{A} = Y_{steel}\left(\frac{\Delta l}{l}\right) = 2.4 \times 10^7$ N m^{-2}, which corresponds to an external force of

$$\Delta F = AY_{steel}\left(\frac{\Delta l}{l}\right) = 2.4 \times 10^7 \times 40 \times 10^{-4} \simeq 10^5 \text{N}.$$

If two such steel rails, fixed at their outer ends, are in contact at their inner ends, a force of this magnitude can easily bend the rails.

▶ ***Example 11.1*** Show that the coefficient of area expansions, $(\Delta A/A)/\Delta T$, of a rectangular sheet of the solid is twice its linear expansivity, α_l.

Answer

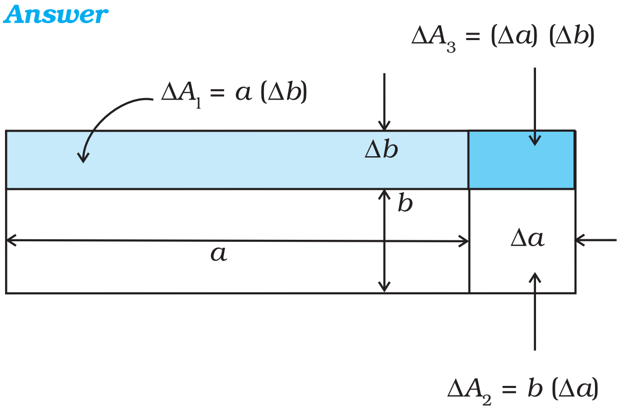


Fig. 11.8

Consider a rectangular sheet of the solid material of length a and breadth b (Fig. 11.8). When the temperature increases by ΔT, a increases by $\Delta a = \alpha_1 a \Delta T$ and b increases by $\Delta b = \alpha_1 b \, \Delta T$. From Fig. 11.8, the increase in area

$$\Delta A = \Delta A_1 + \Delta A_2 + \Delta A_3$$
$$\Delta A = a\,\Delta b + b\,\Delta a + (\Delta a)\ (\Delta b)$$
$$= a\,\alpha_1 b\,\Delta T + b\,\alpha_1\,a\,\Delta T + (\alpha_1)^2\,ab\,(\Delta T)^2$$
$$= \alpha_1\,ab\,\Delta T\,(2 + \alpha_1\,\Delta T) = \alpha_1\,A\,\Delta T\,(2 + \alpha_1\,\Delta T)$$

Since $\alpha_1 \simeq 10^{-5}$ K^{-1}, from Table 11.1, the product $\alpha_1 \Delta T$ for fractional temperature is small in comparision with 2 and may be neglected. Hence,

$$\left(\frac{\Delta A}{A}\right)\frac{1}{\Delta T} \simeq 2\alpha_l$$ ◀

▶ ***Example 11.2*** A blacksmith fixes iron ring on the rim of the wooden wheel of a bullock cart. The diameter of the rim and the iron ring are 5.243 m and 5.231 m respectively at 27 °C. To what temperature should the ring be heated so as to fit the rim of the wheel?

Given, $T_1 = 27$ °C
$L_{T1} = 5.231$ m
$L_{T2} = 5.243$ m

So,
$L_{T2} = L_{T1}\ [1+\alpha_1\ (T_2 - T_1)]$

5.243 m = 5.231 m [1 + 1.20×10^{-5} K^{-1} (T_2–27 °C)]

or T_2 = 218 °C. ◀

11.6 SPECIFIC HEAT CAPACITY

Take some water in a vessel and start heating it on a burner. Soon you will notice that bubbles begin to move upward. As the temperature is raised the motion of water particles increases till it becomes turbulent as water starts boiling. What are the factors on which the quantity of heat required to raise the temperature of a substance depend? In order to answer this question in the first step, heat a given quantity of water to raise its temperature by, say 20 °C and note the time taken. Again take the same amount of water and raise its temperature by 40 °C using the same source of heat. Note the time taken by using a stopwatch. You will find it takes about twice the time and therefore, double the quantity of heat required raising twice the temperature of same amount of water.

In the second step, now suppose you take double the amount of water and heat it, using the same heating arrangement, to raise the temperature by 20 °C, you will find the time taken is again twice that required in the first step.

In the third step, in place of water, now heat the same quantity of some oil, say mustard oil, and raise the temperature again by 20 °C. Now note the time by the same stopwatch. You will find the time taken will be shorter and therefore, the quantity of heat required would be less than that required by the same amount of water for the same rise in temperature.

The above observations show that the quantity of heat required to warm a given substance depends on its mass, m, the change in temperature, ΔT and the nature of substance. The change in temperature of a substance, when a given quantity of heat is absorbed or rejected by it, is characterised by a quantity called the **heat capacity** of that substance. We define heat capacity, S of a substance as

$$S = \frac{\Delta Q}{\Delta T} \qquad (11.10)$$

where ΔQ is the amount of heat supplied to the substance to change its temperature from T to $T + \Delta T$.

You have observed that if equal amount of heat is added to equal masses of different substances, the resulting temperature changes will not be the same. It implies that every substance has a unique value for the amount of heat absorbed or rejected to change the temperature of unit mass of it by one unit. This quantity is referred to as the **specific heat capacity** of the substance.

If ΔQ stands for the amount of heat absorbed or rejected by a substance of mass m when it undergoes a temperature change ΔT, then the specific heat capacity, of that substance is given by

$$s = \frac{S}{m} = \frac{1}{m}\frac{\Delta Q}{\Delta T} \qquad (11.11)$$

The **specific heat capacity** is the property of the substance which determines the change in the temperature of the substance (undergoing no phase change) when a given quantity of heat is absorbed (or rejected) by it. It is defined as the amount of heat per unit mass absorbed or rejected by the substance to change its temperature by one unit. It depends on the nature of the substance and its temperature. The SI unit of specific heat capacity is J kg^{-1} K^{-1}.

If the amount of substance is specified in terms of moles μ, instead of mass m in kg, we can define heat capacity per mole of the substance by

$$C = \frac{S}{\mu} = \frac{1}{\mu}\frac{\Delta Q}{\Delta T} \qquad (11.12)$$

where C is known as **molar specific heat capacity** of the substance. Like S, C also depends on the nature of the substance and its temperature. The SI unit of molar specific heat capacity is J mol^{-1} K^{-1}.

However, in connection with specific heat capacity of gases, additional conditions may be needed to define C. In this case, heat transfer can be achieved by keeping either pressure or volume constant. If the gas is held under constant pressure during the heat transfer, then it is called the **molar specific heat capacity at constant pressure** and is denoted by C_p. On the other hand, if the volume of the gas is maintained during the heat transfer, then the corresponding molar specific heat capacity is called **molar specific heat capacity at constant volume** and is denoted by C_v. For details see Chapter 12. Table 11.3 lists measured specific heat capacity of some substances at atmospheric pressure and ordinary temperature while Table 11.4 lists molar specific heat capacities of some gases. From Table 11.3 you can note that water

Table 11.3 Specific heat capacity of some substances at room temperature and atmospheric pressure

Substance	Specific heat capacity ($J\ kg^{-1}\ K^{-1}$)	Substance	Specific heat capacity ($J\ kg^{-1}\ K^{-1}$)
Aluminium	900.0	Ice	2060
Carbon	506.5	Glass	840
Copper	386.4	Iron	450
Lead	127.7	Kerosene	2118
Silver	236.1	Edible oil	1965
Tungesten	134.4	Mercury	140
Water	4186.0		

has the highest specific heat capacity compared to other substances. For this reason water is used as a coolant in automobile radiators as well as a heater in hot water bags. Owing to its high specific heat capacity, the water warms up much more slowly than the land during summer and consequently wind from the sea has a cooling effect. Now, you can tell why in desert areas, the earth surface warms up quickly during the day and cools quickly at night.

Table 11.4 Molar specific heat capacities of some gases

Gas	C_p ($J\ mol^{-1}K^{-1}$)	C_v($J\ mol^{-1}K^{-1}$)
He	20.8	12.5
H_2	28.8	20.4
N_2	29.1	20.8
O_2	29.4	21.1
CO_2	37.0	28.5

11.7 CALORIMETRY

A system is said to be isolated if no exchange or transfer of heat occurs between the system and its surroundings. When different parts of an isolated system are at different temperature, a quantity of heat transfers from the part at higher temperature to the part at lower temperature. The heat lost by the part at higher temperature is equal to the heat gained by the part at lower temperature.

Calorimetry means measurement of heat. When a body at higher temperature is brought in contact with another body at lower temperature, the heat lost by the hot body is equal to the heat gained by the colder body, provided no heat is allowed to escape to the surroundings. A device in which heat measurement can be made is called a **calorimeter**. It consists a metallic vessel and stirrer of the same material like copper or alumiunium. The vessel is kept inside a wooden jacket which contains heat insulating materials like glass wool etc. The outer jacket acts as a heat shield and reduces the heat loss from the inner vessel. There is an opening in the outer jacket through which a mercury thermometer can be inserted into the calorimeter. The following example provides a method by which the specific heat capacity of a given solid can be determinated by using the principle, heat gained is equal to the heat lost.

Example 11.3 A sphere of aluminium of 0.047 kg placed for sufficient time in a vessel containing boiling water, so that the sphere is at 100 °C. It is then immediately transfered to 0.14 kg copper calorimeter containing 0.25 kg of water at 20 °C. The temperature of water rises and attains a steady state at 23 °C. Calculate the specific heat capacity of aluminium.

Answer In solving this example we shall use the fact that at a steady state, heat given by an aluminium sphere will be equal to the heat absorbed by the water and calorimeter.

Mass of aluminium sphere (m_1) = 0.047 kg
Initial temp. of aluminium sphere = 100 °C
Final temp. = 23 °C
Change in temp (ΔT) = (100 °C - 23 °C) = 77 °C
Let specific heat capacity of aluminium be s_{Al}.

The amount of heat lost by the aluminium sphere = $m_1 s_{Al} \Delta T = 0.047 \text{kg} \times s_{Al} \times 77$ °C

Mass of water (m_2) = 0.25 kg

Mass of calorimeter (m_3) = 0.14 kg

Initial temp. of water and calorimeter = 20 °C

Final temp. of the mixture = 23 °C

Change in temp. (ΔT_2) = 23 °C – 20 °C = 3 °C

Specific heat capacity of water (s_w)

= 4.18 × 10^3 J kg^{-1} K^{-1}

Specific heat capacity of copper calorimeter

= 0.386 × 10^3 J kg^{-1} K^{-1}

The amount of heat gained by water and calorimeter = $m_2\, s_w\, \Delta T_2 + m_3 s_{cu} \Delta T_2$

= $(m_2 s_w + m_3 s_{cu})\ (\Delta T_2)$

= 0.25 kg ×4.18 × 10^3 J kg^{-1} K^{-1} + 0.14 kg × 0.386 × 10^3 J kg^{-1} K^{-1}) (23 °C – 20 °C)

In the steady state heat lost by the aluminium sphere = heat gained by water + heat gained by calorimeter.

So, 0.047 kg × s_{Al} × 77 °C

= (0.25 kg × 4.18 × 10^3 J kg^{-1} K^{-1}+ 0.14 kg × 0.386 × 10^3 J kg^{-1} K^{-1})(3 °C)

s_{Al} = 0.911 kJ kg^{-1} K^{-1} ◀

11.8 CHANGE OF STATE

Matter normally exists in three states: solid, liquid, and gas. A transition from one of these states to another is called a change of state. Two common changes of states are solid to liquid and liquid to gas (and vice versa). These changes can occur when the exchange of heat takes place between the substance and its surroundings. To study the change of state on heating or cooling, let us perform the following activity.

Take some cubes of ice in a beaker. Note the temperature of ice (0 °C). Start heating it slowly on a constant heat source. Note the temperature after every minute. Continuously stir the mixture of water and ice. Draw a graph between temperature and time (Fig. 11.9). You will observe no change in the temperature so long as there is ice in the beaker. In the above process, the temperature of the system does not change even though heat is being continuously supplied. The heat supplied is being utilised in changing the state from solid (ice) to liquid (water).

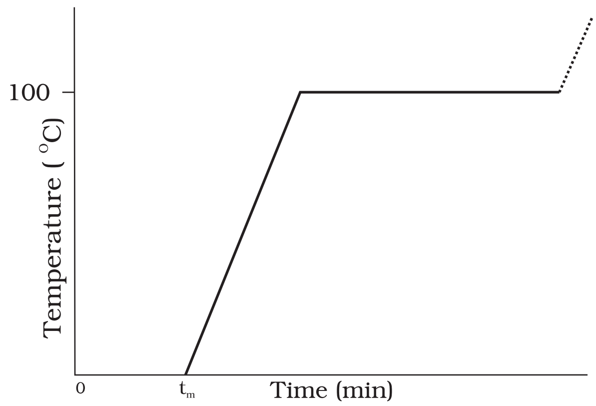


Fig. 11.9 *A plot of temperature versus time showing the changes in the state of ice on heating (not to scale).*

The change of state from solid to liquid is called **melting** and from liquid to solid is called **fusion**. It is observed that the temperature remains constant until the entire amount of the solid substance melts. That is, **both the solid and liquid states of the substance coexist in thermal equilibrium during the change of states from solid to liquid.** The temperature at which the solid and the liquid states of the substance in thermal equilibrium with each other is called its **melting point**. It is characteristic of the substance. It also depends on pressure. The melting point of a substance at standard atomspheric pressure is called its **normal melting point**. Let us do the following activity to understand the process of melting of ice.

Take a slab of ice. Take a metallic wire and fix two blocks, say 5 kg each, at its ends. Put the wire over the slab as shown in Fig. 11.10. You will observe that the wire passes through the ice slab. This happens due to the fact that just below the wire, ice melts at lower temperature due to increase in pressure. When the wire has passed, water above the wire freezes again. Thus the wire passes through the slab and the slab does not split. This phenomenon of refreezing is called **regelation**. Skating is possible on snow due to the formation of water below the skates. Water is formed due to the increase of pressure and it acts as a lubricant.

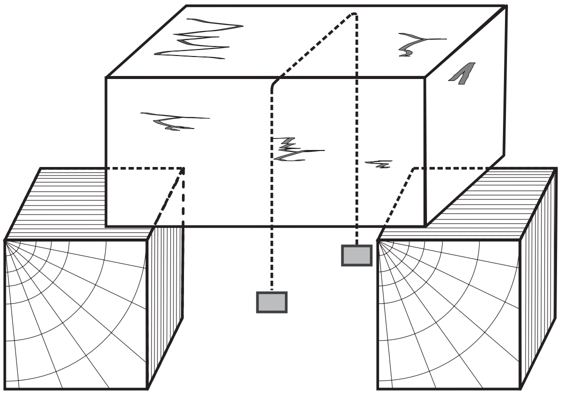


Fig. 11.10

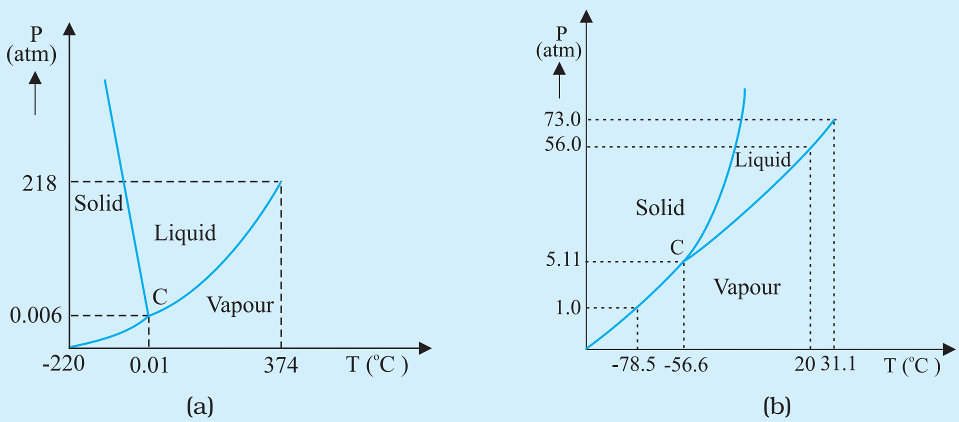
After the whole of ice gets converted into water and as we continue further heating, we shall see that temperature begins to rise. The temperature keeps on rising till it reaches nearly 100 °C when it again becomes steady. The heat supplied is now being utilised to change water from liquid state to vapour or gaseous state.

The change of state from liquid to vapour (or gas) is called **vaporisation**. It is observed that the temperature remains constant until the entire amount of the liquid is converted into vapour. That is, both the liquid and vapour states of the substance coexist in thermal equilibrium, during the change of state from liquid to vapour. The temperature at which the liquid and the vapour states of the substance coexist is called its **boiling point**. Let us do the following activity to understand the process of boiling of water.

Take a round-bottom flask, more than half filled with water. Keep it over a burner and fix a

Triple Point

The temperature of a substance remains constant during its change of state (phase change). A graph between the temperature *T* and the Pressure *P* of the substance is called a phase diagram or *P* – *T* diagram. The following figure shows the phase diagram of water and CO_2. Such a phase diagram divides the *P* – *T* plane into a solid-region, the vapour-region and the liquid-region. The regions are separated by the curves such as sublimation curve (BO), **fusion curve** (AO) and **vaporisation curve** (CO). The points on **sublimation curve** represent states in which solid and vapour phases coexist. The point on the sublimation curve BO represent states in which the solid and vapour phases co-exist. Points on the fusion curve AO represent states in which solid and liquid phase coexist. Points on the vapourisation curve CO represent states in which the liquid and vapour phases coexist. The temperature and pressure at which the fusion curve, the vaporisation curve and the sublimation curve meet and all the three phases of a substance coexist is called the **triple point** of the substance. For example the triple point of water is represented by the temperature 273.16 K and pressure 6.11×10^{-3} *Pa*.



Pressure-temperature phase diagrams for (a) water and (b) CO_2 (not to the scale).

thermometer and steam outlet through the cork of the flask (Fig. 11.11). As water gets heated in the flask, note first that the air, which was dissolved in the water, will come out as small bubbles. Later, bubbles of steam will form at the bottom but as they rise to the cooler water near the top, they condense and disappear. Finally, as the temperature of the entire mass of the water reaches 100 °C, bubbles of steam reach the surface and boiling is said to occur. The steam in the flask may not be visible but as it comes out of the flask, it condenses as tiny droplets of water, giving a foggy appearance.

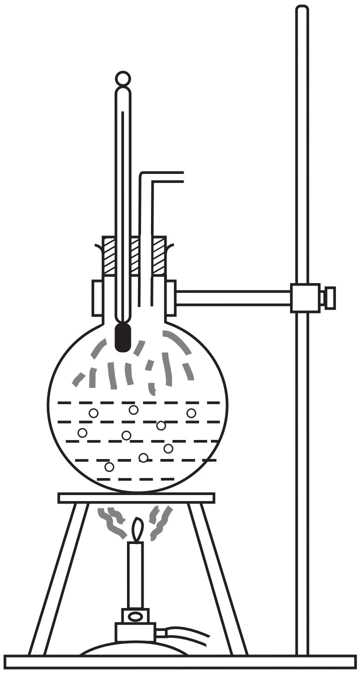


Fig. 11.11 *Boiling process.*

If now the steam outlet is closed for a few seconds to increase the pressure in the flask, you will notice that boiling stops. More heat would be required to raise the temperature (depending on the increase in pressure) before boiling begins again. Thus boiling point increases with increase in pressure.

Let us now remove the burner. Allow water to cool to about 80 °C. Remove the thermometer and steam outlet. Close the flask with the airtight cork. Keep the flask turned upside down on the stand. Pour ice-cold water on the flask. Water vapours in the flask condense reducing the pressure on the water surface inside the flask. Water begins to boil again, now at a lower temperature. Thus boiling point decreases with decrease in pressure.

This explains why cooking is difficult on hills. At high altitudes, atmospheric pressure is lower, reducing the boiling point of water as compared to that at sea level. On the other hand, boiling point is increased inside a pressure cooker by increasing the pressure. Hence cooking is faster. The boiling point of a substance at standard atmospheric pressure is called its **normal boiling point**.

However, all substances do not pass through the three states: solid-liquid-gas. There are certain substances which normally pass from the solid to the vapour state directly and vice versa. The change from solid state to vapour state without passing through the liquid state is called **sublimation**, and the substance is said to sublime. Dry ice (solid CO_2) sublimes, so also iodine. During the sublimation process both the solid and vapour states of a substance coexist in thermal equilibrium.

11.8.1 Latent Heat

In Section 11.8, we have learnt that certain amount of heat energy is transferred between a substance and its surroundings when it undergoes a change of state. The amount of heat per unit mass transferred during change of state of the substance is called latent heat of the substance for the process. For example, if heat is added to a given quantity of ice at –10 °C, the temperature of ice increases until it reaches its melting point (0 °C). At this temperature, the addition of more heat does not increase the temperature but causes the ice to melt, or changes its state. Once the entire ice melts, adding more heat will cause the temperature of the water to rise. A similar situation occurs during liquid gas change of state at the boiling point. Adding more heat to boiling water causes vaporisation, without increase in temperature.

Table 11.5 **Temperatures of the change of state and latent heats for various substances at 1 atm pressure**

Substance	Melting Point (°C)	L_f (10^5J kg^{-1})	Boiling Point (°C)	L_v (10^5J kg^{-1})
Ethyl alcohol	–114	1.0	78	8.5
Gold	1063	0.645	2660	15.8
Lead	328	0.25	1744	8.67
Mercury	–39	0.12	357	2.7
Nitrogen	–210	0.26	–196	2.0
Oxygen	–219	0.14	–183	2.1
Water	0	3.33	100	22.6

The heat required during a change of state depends upon the heat of transformation and the mass of the substance undergoing a change of state. Thus, if mass m of a substance undergoes a change from one state to the other, then the quantity of heat required is given by

$$Q = m L$$

or $$L = Q/m \qquad (11.13)$$

where L is known as latent heat and is a characteristic of the substance. Its SI unit is J kg^{-1}. The value of L also depends on the pressure. Its value is usually quoted at standard atmospheric pressure. The latent heat for a solid-liquid state change is called the **latent heat of fusion** (L_f), and that for a liquid-gas state change is called the **latent heat of vaporisation** (L_v). These are often referred to as the heat of fusion and the heat of vaporisation. A plot of temperature versus heat energy for a quantity of water is shown in Fig. 11.12. The latent heats of some substances, their freezing and boiling points, are given in Table 11.5.

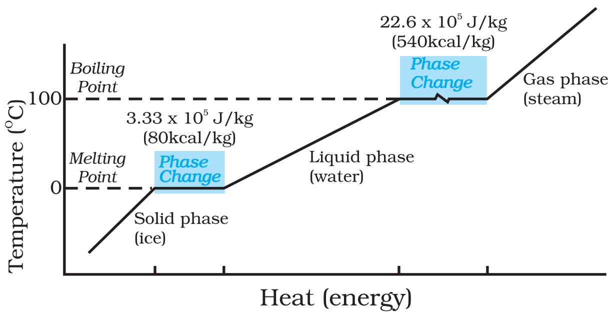


Fig. 11.12 *Temperature versus heat for water at 1 atm pressure (not to scale).*

Note that when heat is added (or removed) during a change of state, the temperature remains constant. Note in Fig. 11.12 that the slopes of the phase lines are not all the same, which indicates that specific heats of the various states are not equal. For water, the latent heat of fusion and vaporisation are L_f = 3.33 × 10^5 J kg^{-1} and L_v = 22.6 × 10^5 J kg^{-1} respectively. That is 3.33 × 10^5 J of heat are needed to melt 1 kg of ice at 0 °C, and 22.6 × 10^5 J of heat are needed to convert 1 kg of water to steam at 100 °C. So, steam at 100 °C carries 22.6 × 10^5 J kg^{-1} more heat than water at 100 °C. This is why burns from steam are usually more serious than those from boiling water.

Example 11.4 When 0.15 kg of ice of 0 °C mixed with 0.30 kg of water at 50 °C in a container, the resulting temperature is 6.7 °C. Calculate the heat of fusion of ice. (s_{water} = 4186 J kg^{-1} K^{-1})

Answer

Heat lost by water = $ms_w (\theta_f - \theta_i)_w$

= (0.30 kg) (4186 J kg^{-1} K^{-1}) (50.0 °C – 6.7 °C)

= 54376.14 J

Heat required to melt ice = $m_2 L_f$ = (0.15 kg) L_f

Heat required to raise temperature of ice water to final temperature = $m_I s_w (\theta_f - \theta_i)_I$

= (0.15 kg) (4186 J kg^{-1} K^{-1}) (6.7 °C – 0 °C)

= 4206.93 J

Heat lost = heat gained

54376.14 J = (0.15 kg) L_f + 4206.93 J

L_f = 3.34×10^5 J kg^{-1}.

Example 11.5 Calculate the heat required to convert 3 kg of ice at –12 °C kept in a calorimeter to steam at 100 °C at atmospheric pressure. Given specific heat capacity of ice = 2100 J kg^{-1} K^{-1}, specific heat capacity of water = 4186 J kg^{-1} K^{-1}, latent heat of fusion of ice = 3.35×10^5 J kg^{-1} and latent heat of steam = 2.256×10^6 J kg^{-1}.

Answer We have

Mass of the ice, $m = 3$ kg

specific heat capacity of ice, s_{ice}
= 2100 J kg^{-1} K^{-1}

specific heat capacity of water, s_{water}
= 4186 J kg^{-1} K^{-1}

latent heat of fusion of ice, $L_{f\,ice}$
= 3.35×10^5 J kg^{-1}

latent heat of steam, L_{steam}
= 2.256×10^6 J kg^{-1}

Now, Q = heat required to convert 3 kg of ice at –12 °C to steam at 100 °C,

Q_1 = heat required to convert ice at –12 °C to ice at 0 °C.

= $m\, s_{ice}\, \Delta T_1$ = (3 kg) (2100 J kg^{-1} K^{-1}) [0–(–12)]°C = 75600 J

Q_2 = heat required to melt ice at 0 °C to water at 0 °C

= $m L_{f\,ice}$ = (3 kg) (3.35×10^5 J kg^{-1})

= 1005000 J

Q_3 = heat required to convert water at 0 °C to water at 100 °C.

= $m s_w\, \Delta T_2$ = (3kg) (4186J kg^{-1} K^{-1}) (100 °C)

= 1255800 J

Q_4 = heat required to convert water at 100 °C to steam at 100 °C.

= $m\, L_{steam}$ = (3 kg) (2.256×10^6 J kg^{-1})

= 6768000 J

So, $Q = Q_1 + Q_2 + Q_3 + Q_4$

= 75600J + 1005000 J + 1255800 J + 6768000 J

= 9.1×10^6 J

11.9 HEAT TRANSFER

We have seen that heat is energy transfer from one system to another or from one part of a system to another part, arising due to temperature difference. What are the different ways by which this energy transfer takes place? There are three distinct modes of heat transfer : conduction, convection and radiation (Fig. 11.13).

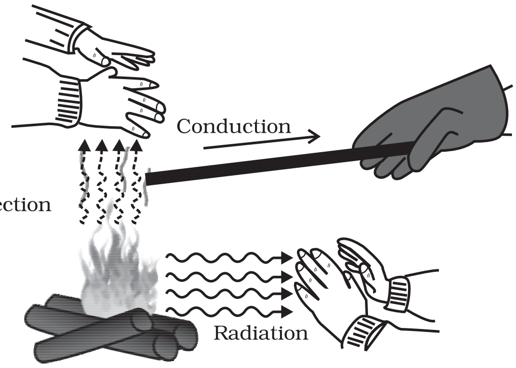


Fig. 11.13 *Heating by conduction, convection and radiation.*

11.9.1 Conduction

Conduction is the mechanism of transfer of heat between two adjacent parts of a body because of their temperature difference. Suppose one end of a metallic rod is put in a flame, the other end of the rod will soon be so hot that you cannot hold it by your bare hands. Here heat transfer takes place by conduction from the hot end of the rod through its different parts to the other end. Gases are poor thermal conductors while liquids have conductivities intermediate between solids and gases.

Heat conduction may be described quantitatively as the time rate of heat flow in a material for a given temperature difference. Consider a metallic bar of length L and uniform cross section A with its two ends maintained at different temperatures. This can be done, for example, by putting the ends in thermal contact with large reservoirs at temperatures, say, T_C and T_D respectively (Fig. 11.14). Let us assume the ideal condition that the sides of the bar are fully insulated so that no heat is exchanged between the sides and the surroundings.

After sometime, a steady state is reached; the temperature of the bar decreases uniformly with distance from T_C to T_D; (T_C>T_D). The reservoir at C supplies heat at a constant rate, which transfers through the bar and is given out at the same rate to the reservoir at D. It is found

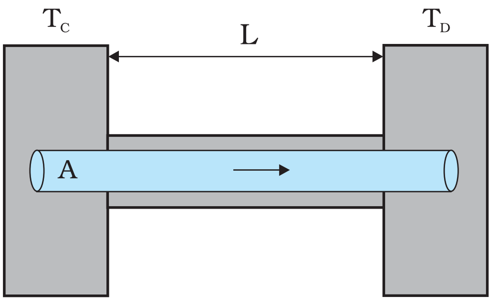


Fig. 11.14 *Steady state heat flow by conduction in a bar with its two ends maintained at temperatures T_C and T_D; ($T_C > T_D$).*

experimentally that in this steady state, the rate of flow of heat (or heat current) H is proportional to the temperature difference ($T_C - T_D$) and the area of cross section A and is inversely proportional to the length L :

$$H = KA\,\frac{T_C - T_D}{L} \qquad (11.14)$$

The constant of proportionality K is called the **thermal conductivity** of the material. The greater the value of K for a material, the more rapidly will it conduct heat. The SI unit of K is J S^{-1} m^{-1} K^{-1} or W m^{-1} K^{-1}. The thermal conductivities of various substances are listed in Table 11.5. These values vary slightly with temperature, but can be considered to be constant over a normal temperature range.

Compare the relatively large thermal conductivities of the good thermal conductors, the metals, with the relatively small thermal conductivities of some good thermal insulators, such as wood and glass wool. You may have noticed that some cooking pots have copper coating on the bottom. Being a good conductor of heat, copper promotes the distribution of heat over the bottom of a pot for uniform cooking. Plastic foams, on the other hand, are good insulators, mainly because they contain pockets of air. Recall that gases are poor conductors, and note the low thermal conductivity of air in the Table 11.5. Heat retention and transfer are important in many other applications. Houses made of concrete roofs get very hot during summer days, because thermal conductivity of concrete (though much smaller than that of a metal) is still not small enough. Therefore, people usually prefer to give a layer of earth or foam insulation on the ceiling so that heat transfer is prohibited and keeps the room cooler. In some situations, heat transfer is critical. In a nuclear reactor, for example, elaborate heat transfer systems need to be installed so that the enormous energy produced by nuclear fission in the core transits out sufficiently fast, thus preventing the core from overheating.

Table 11.6 Thermal conductivities of some materials

Materials	Thermal conductivity ($J\ s^{-1}\ m^{-1}\ K^{-1}$)
Metals	
Silver	406
Copper	385
Aluminium	205
Brass	109
Steel	50.2
Lead	34.7
Mercury	8.3
Non-metals	
Insulating brick	0.15
Concrete	0.8
Body fat	0.20
Felt	0.04
Glass	0.8
Ice	1.6
Glass wool	0.04
Wood	0.12
Water	0.8
Gases	
Air	0.024
Argon	0.016
Hydrogen	0.14

Example 11.6 What is the temperature of the steel-copper junction in the steady state of the system shown in Fig. 11.15. Length of the steel rod = 15.0 cm, length of the copper rod = 10.0 cm, temperature of the furnace = 300 °C, temperature of the other end = 0 °C. The area of cross section of the steel rod is twice that of the copper rod. (Thermal conductivity of steel = 50.2 J s^{-1} $m^{-1}K^{-1}$; and of copper = 385 J $s^{-1}m^{-1}K^{-1}$).

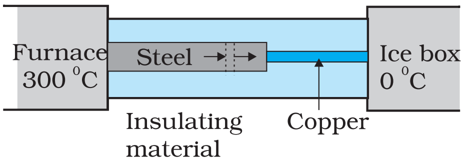


Fig. 11.15

Answer The insulating material around the rods reduces heat loss from the sides of the rods. Therefore, heat flows only along the length of the rods. Consider any cross section of the rod. In the steady state, heat flowing into the element must equal the heat flowing out of it; otherwise there would be a net gain or loss of heat by the element and its temperature would not be steady. Thus in the steady state, rate of heat flowing across a cross section of the rod is the same at every point along the length of the combined steel-copper rod. Let T be the temperature of the steel-copper junction in the steady state. Then,

$$\frac{K_1\ A_1\ \ 300\ \ \ T}{L_1} = \frac{K_2\ A_2\ \ T-0}{L_2}$$

where 1 and 2 refer to the steel and copper rod respectively. For A_1 = 2 A_2, L_1 = 15.0 cm, L_2 = 10.0 cm, K_1 = 50.2 J s^{-1} m^{-1} K^{-1}, K_2 = 385 J s^{-1} m^{-1} K^{-1}, we have

$$\frac{50.2\times 2\,(300-T)}{15} = \frac{385T}{10}$$

which gives T = 44.4 °C

Example 11.7 An iron bar (L_1 = 0.1 m, A_1 = 0.02 m^2, K_1 = 79 W m^{-1} K^{-1}) and a brass bar (L_2 = 0.1 m, A_2 = 0.02 m^2, K_2 = 109 W m^{-1}K^{-1}) are soldered end to end as shown in Fig. 11.16. The free ends of the iron bar and brass bar are maintained at 373 K and 273 K respectively. Obtain expressions for and hence compute (i) the temperature of the junction of the two bars, (ii) the equivalent thermal conductivity of the compound bar, and (iii) the heat current through the compound bar.

Answer

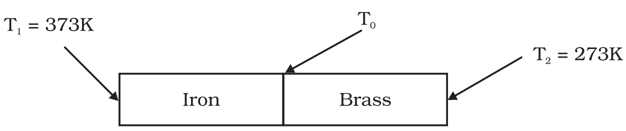


Fig 11.16

Given, $L_1 = L_2 = L$ = 0.1 m, $A_1 = A_2 = A$ = 0.02 m^2 K_1 = 79 W m^{-1} K^{-1}, K_2 = 109 W m^{-1} K^{-1}, T_1 = 373 K, and T_2 = 273 K.

Under steady state condition, the heat current (H_1) through iron bar is equal to the heat current (H_2) through brass bar.

So, $H = H_1 = H_2$

$$= \frac{K_1 A_1\ \ T_1 - T_0}{L_1} \quad \frac{K_2 A_2 (T_0 - T_2)}{L_2}$$

For $A_1 = A_2 = A$ and $L_1 = L_2 = L$, this equation leads to

$K_1\ (T_1 - T_0) = K_2\ (T_0 - T_2)$

Thus the junction temperature T_0 of the two bars is

$$T_0 = \frac{K_1 T_1 \quad K_2 T_2}{K_1 \quad K_2}$$

Using this equation, the heat current H through either bar is

$$H = \frac{K_1 A\ \ T_1 - T_0}{L} \quad \frac{K_2 A (T_0 - T_2)}{L}$$

$$\frac{K_1 K_2}{K_1 \quad K_2} \quad \frac{A\ \ T_1 - T_0}{L} \qquad \frac{A\ \ T_1 - T_2}{L\ \ \frac{1}{K_1}\ \ \frac{1}{K_2}}$$

Using these equations, the heat current H' through the compound bar of length $L_1 + L_2 = 2L$ and the equivalent thermal conductivity K', of the compound bar are given by

$$H \quad \frac{K \quad A \quad T_1 - T_2}{2\ L} \quad H$$

$$K \quad \frac{2\ K_1\ K_2}{K_1 \quad K_2}$$

(i) $$T_0 \quad \frac{K_1 T_1 \quad K_2 T_2}{K_1 \quad K_2}$$

$$\frac{79\,W\,m^{-1}K^{-1} \quad 373\,K \qquad 109\,W\,m^{-1}K^{-1} \quad 273\,K}{79\,W\,m^{-1}K^{-1} \quad 109\ W\,m^{-1}K^{-1}}$$

= 315 K

(ii) $$K = \frac{2K_1\ K_2}{K_1 \quad K_2}$$

$$= \frac{2\times(79\,W\,m^{-1}\,K^{-1})\times(109\,W\,m^{-1}\,K^{-1})}{79\,W\,m^{-1}\,K^{-1}+109\,W\,m^{-1}\,K^{-1}}$$

= 91.6 W m^{-1} K^{-1}

(iii) $H \quad H \quad \frac{K \quad A \quad T_1 - T_2}{2 \ L}$

$$\frac{91.6 \text{ W m}^{-1}\text{K}^{-1} \times 0.02 \text{ m}^2 \times 373\text{K}–273\text{K}}{2 \times 0.1 \text{ m}}$$

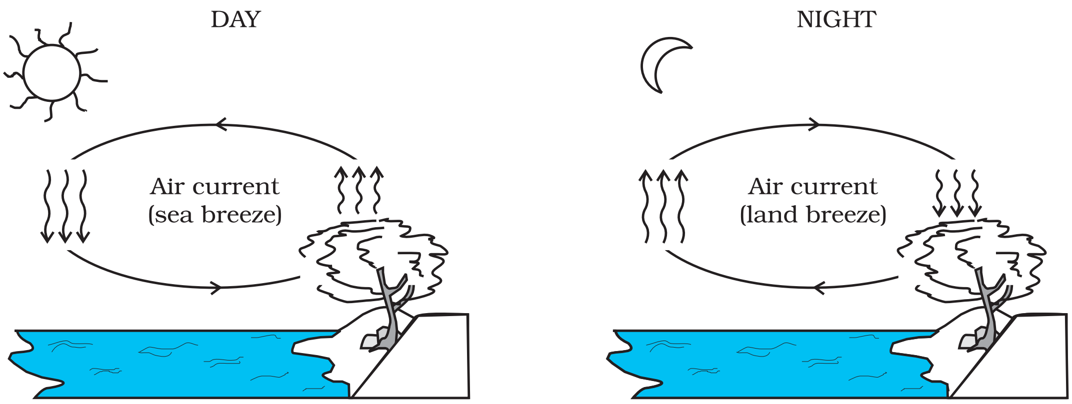
= 916.1 W ◀

11.9.2 Convection

Convection is a mode of heat transfer by actual motion of matter. It is possible only in fluids. Convection can be natural or forced. In natural convection, gravity plays an important part. When a fluid is heated from below, the hot part expands and, therefore, becomes less dense. Because of buoyancy, it rises and the upper colder part replaces it. This again gets heated, rises up and is replaced by the colder part of the fluid. The process goes on. This mode of heat transfer is evidently different from conduction. Convection involves bulk transport of different parts of the fluid. In forced convection, material is forced to move by a pump or by some other physical means. The common examples of forced convection systems are forced-air heating systems in home, the human circulatory system, and the cooling system of an automobile engine. In the human body, the heart acts as the pump that circulates blood through different parts of the body, transferring heat by forced convection and maintaining it at a uniform temperature.

Natural convection is responsible for many familiar phenomena. During the day, the ground heats up more quickly than large bodies of water do. This occurs both because the water has a greater specific heat and because mixing currents disperse the absorbed heat throughout the great volume of water. The air in contact with the warm ground is heated by conduction. It expands, becoming less dense than the surrounding cooler air. As a result, the warm air rises (air currents) and other air moves (winds) to fill the space-creating a sea breeze near a large body of water. Cooler air descends, and a thermal convection cycle is set up, which transfers heat away from the land. At night, the ground loses its heat more quickly, and the water surface is warmer than the land. As a result, the cycle is reveresed (Fig. 11.17).

The other example of natural convection is the steady surface wind on the earth blowing in from north-east towards the equator, the so called trade wind. A resonable explanation is as follows : the equatorial and polar regions of the earth receive unequal solar heat. Air at the earth's surface near the equator is hot while the air in the upper atmosphere of the poles is cool. In the absence of any other factor, a convection current would be set up, with the air at the equatorial surface rising and moving out towards the poles, descending and streaming in towards the equator. The rotation of the earth, however, modifies this convection current. Because of this, air close to the equator has an eastward speed of 1600 km/h, while it is zero close to the poles. As a result, the air descends not at the poles but at 30° N (North) latitude and returns to the equator. This is called **trade wind**.



Land warmer than water   Water warmer than land

Fig. 11.17 *Convection cycles.*

11.9.3 Radiation

Conduction and convection require some material as a transport medium. These modes of heat transfer cannot operate between bodies separated by a distance in vacuum. But the earth does receive heat from the sun across a huge distance and we quickly feel the warmth of the fire nearby even though air conducts poorly and before convection can set in. The third mechanism for heat transfer needs no medium; it is called radiation and the energy so radiated by electromagnetic waves is called radiant energy. In an electromagnetic wave electric and magnetic fields oscillate in space and time. Like any wave, electromagnetic waves can have different wavelengths and can travel in vacuum with the same speed, namely the speed of light i.e., 3×10^8 m s^{-1}. You will learn these matters in more details later, but you now know why heat transfer by radiation does not need any medium and why it is so fast. This is how heat is transfered to the earth from the sun through empty space. All bodies emit radiant energy, whether they are solid, liquid or gases. The electromagnetic radiation emitted by a body by virtue of its temperature like the radiation by a red hot iron or light from a filament lamp is called thermal radiation.

When this thermal radiation falls on other bodies, it is partly reflected and partly absorbed. The amount of heat that a body can absorb by radiation depends on the colour of the body.

We find that black bodies absorb and emit radiant energy better than bodies of lighter colours. This fact finds many applications in our daily life. We wear white or light coloured clothes in summer so that they absorb the least heat from the sun. However, during winter, we use dark coloured clothes which absorb heat from the sun and keep our body warm. The bottoms of the utensils for cooking food are blackened so that they absorb maximum heat from the fire and give it to the vegetables to be cooked.

Similarly, a Dewar flask or thermos bottle is a device to minimise heat transfer between the contents of the bottle and outside. It consists of a double-walled glass vessel with the inner and outer walls coated with silver. Radiation from the inner wall is reflected back into the contents of the bottle. The outer wall similarly reflects back any incoming radiation. The space between the walls is evacuated to reduce conduction and convection losses and the flask is supported on an insulator like cork. The device is, therefore, useful for preventing hot contents (like milk) from getting cold, or alternatively to store cold contents (like ice).

11.10 NEWTON'S LAW OF COOLING

We all know that hot water or milk when left on a table begins to cool gradually. Ultimately it attains the temperature of the surroundings. To study how a given body can cool on exchanging heat with its surroundings, let us perform the following activity.

Take some water, say 300 ml, in a calorimeter with a stirrer and cover it with two holed lid. Fix a thermometer through a hole in the lid and make sure that the bulb of thermometer is immersed in the water. Note the reading of the thermometer. This reading T_1 is the temperature of the surroundings. Heat the water kept in the calorimeter till it attains a temperature, say, 40 °C above room temperature (i.e., temperature of the surroundings). Then stop heating the water by removing the heat source. Start the stop-watch and note the reading of the thermometer after fixed interval of time, say after every one minute of stirring gently with the stirrer. Continue to note the temperature (T_2) of water till it attains a temperature about 5 °C above that of the surroundings. Then plot a graph by taking each value of temperature $\Delta T = T_2 - T_1$ along y axis and the coresponding value of t along x-axis (Fig. 11.18).

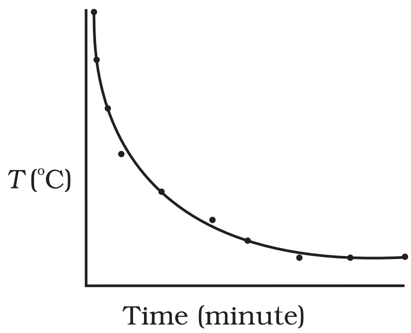


Fig. 11.18 *Curve showing cooling of hot water with time.*

From the graph you will infer how the cooling of hot water depends on the difference of its temperature from that of the surroundings. You will also notice that initially the rate of cooling is higher and decreases as the temperature of the body falls.

The above activity shows that a hot body loses heat to its surroundings in the form of heat radiation. The rate of loss of heat depends on the difference in temperature between the body and its surroundings. Newton was the first to study, in a systematic manner, the relation between the heat lost by a body in a given enclosure and its temperature.

According to Newton's law of cooling, the rate of loss of heat, $-\,dQ/dt$ of the body is directly proportional to the difference of temperature $\Delta T = (T_2 - T_1)$ of the body and the surroundings. The law holds good only for small difference of temperature. Also, the loss of heat by radiation depends upon the nature of the surface of the body and the area of the exposed surface. We can write

$$-\frac{dQ}{dt} = k(T_2 \quad T_1) \qquad (11.15)$$

where k is a positive constant depending upon the area and nature of the surface of the body. Suppose a body of mass m and specific heat capacity s is at temperature T_2. Let T_1 be the temperature of the surroundings. If the temperature falls by a small amount dT_2 in time dt, then the amount of heat lost is

$dQ = ms\, dT_2$

∴Rate of loss of heat is given by

$$\frac{dQ}{dt} = ms\frac{dT_2}{dt} \qquad (11.16)$$

From Eqs. (11.15) and (11.16) we have

$$-m\,s\frac{dT_2}{dt} = k(T_2 - T_1)$$

$$\frac{dT_2}{T_2 - T_1} = -\frac{k}{ms}dt = -K\,dt \qquad (11.17)$$

where $K = k/m\,s$

On integrating,

$$\log_e (T_2 - T_1) = -K\,t + c \qquad (11.18)$$

$$\text{or } T_2 = T_1 + C'\,e^{-Kt}; \text{ where } C' = e^c \qquad (11.19)$$

Equation (11.19) enables you to calculate the time of cooling of a body through a particular range of temperature.

For small temperature differences, the rate of cooling, due to conduction, convection, and radiation combined, is proportional to the difference in temperature. It is a valid approximation in the transfer of heat from a radiator to a room, the loss of heat through the wall of a room, or the cooling of a cup of tea on the table.

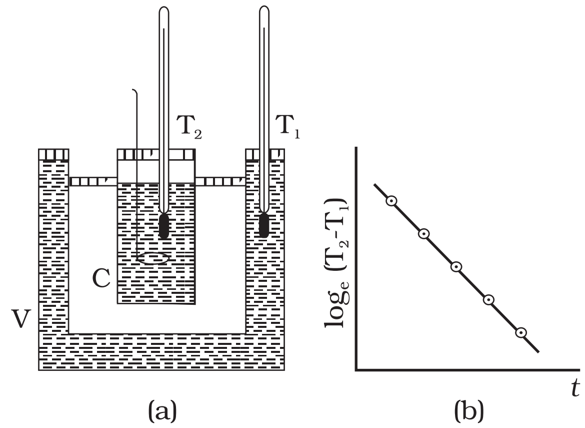


***Fig. 11.19** Verification of Newton's Law of cooling.*

Newton's law of cooling can be verified with the help of the experimental set-up shown in Fig. 11.19(a). The set-up consists of a double walled vessel (V) containing water in between the two walls. A copper calorimeter (C) containing hot water is placed inside the double walled vessel. Two thermometers through the corks are used to note the temperatures T_2 of water in calorimeter and T_1 of hot water in between the double walls respectively. Temperature of hot water in the calorimeter is noted after equal intervals of time. A graph is plotted between $\log_e (T_2 - T_1)$ and time (t). The nature of the graph is observed to be a straight line having a negative slope as shown in Fig. 11.19(b). This is in support of Eq. (11.18).

Example 11.8 A pan filled with hot food cools from 94 °C to 86 °C in 2 minutes when the room temperature is at 20 °C. How long will it take to cool from 71 °C to 69 °C?

Answer The average temperature of 94 °C and 86 °C is 90 °C, which is 70 °C above the room temperature. Under these conditions the pan cools 8 °C in 2 minutes.

Using Eq. (11.17), we have

$$\frac{\text{Change in temperature}}{\text{Time}} \quad K \quad T$$

$$\frac{8\,°\text{C}}{2\text{ min}} = K(70\,°\text{C})$$

The average of 69 °C and 71 °C is 70 °C, which is 50 °C above room temperature. K is the same for this situation as for the original.

$$\frac{2\,°\text{C}}{\text{Time}} = K\,(50\,°\text{C})$$

When we divide above two equations, we have

$$\frac{8\,°\text{C}/2\text{ min}}{2\,°\text{C}/\text{time}} = \frac{K\,(70\,°\text{C})}{K\,(50\,°\text{C})}$$

Time = 0.7 min

= 42 s ◀

SUMMARY

1. Heat is a form of energy that flows between a body and its surrounding medium by virtue of temperature difference between them. The degree of hotness of the body is quantitatively represented by temperature.
2. A temperature-measuring device (thermometer) makes use of some measurable property (called thermometric property) that changes with temperature. Different thermometers lead to different temperature scales. To construct a temperature scale, two fixed points are chosen and assigned some arbitrary values of temperature. The two numbers fix the origin of the scale and the size of its unit.
3. The Celsius temperature (t_C) and the Farenheit temperare (t_F)are related by

   $$t_F = (9/5)\ t_C + 32$$

4. The ideal gas equation connecting pressure (P), volume (V) and absolute temperature (T) is :

   $$PV = \mu RT$$

   where μ is the number of moles and R is the universal gas constant.
5. In the absolute temperature scale, the zero of the scale is the absolute zero of temperature – the temperature where every substance in nature has the least possible molecular activity. The Kelvin absolute temperature scale (T) has the same unit size as the Celsius scale (T_c), but differs in the origin :

   $$T_C = T - 273.15$$

6. The coefficient of linear expansion (α_l) and volume expansion (α_V) are defined by the relations :

   $$\frac{\Delta l}{l} = \alpha_l\ \Delta T$$

   $$\frac{\Delta V}{V} = \alpha_V\ \Delta T$$

   where Δl and ΔV denote the change in length l and volume V for a change of temperature ΔT. The relation between them is :

   $$\alpha_V = 3\ \alpha_l$$

7. The specific heat capacity of a substance is defined by

   $$s = \frac{1}{m}\frac{\Delta Q}{\Delta T}$$

where m is the mass of the substance and ΔQ is the heat required to change its temperature by ΔT. The molar specific heat capacity of a substance is defined by

$$C \quad \frac{1}{\ } \frac{Q}{T}$$

where μ is the number of moles of the substance.

8. The latent heat of fusion (L_f) is the heat per unit mass required to change a substance from solid into liquid at the same temperature and pressure. The latent heat of vaporisation (L_v) is the heat per unit mass required to change a substance from liquid to the vapour state without change in the temperature and pressure.

9. The three modes of heat transfer are conduction, convection and radiation.

10. In conduction, heat is transferred between neighbouring parts of a body through molecular collisions, without any flow of matter. For a bar of length L and uniform cross section A with its ends maintained at temperatures T_C and T_D, the rate of flow of heat H is :

$$H = K\ A\frac{T_C - T_D}{L}$$

where K is the thermal conductivity of the material of the bar.

11. Newton's Law of Cooling says that the rate of cooling of a body is proportional to the excess temperature of the body over the surroundings :

$$\frac{dQ}{dt} = -k\,(T_2 - T_1)$$

Where T_1 is the temperature of the surrounding medium and T_2 is the temperature of the body.

Quantity	Symbol	Dimensions	Unit	Remark
Amount of substance	μ	[mol]	mol	
Celsius temperature	t_c	[K]	°C	
Kelvin absolute temperature	T	[K]	K	$t_c = T - 273.15$
Co-efficient of linear expansion	α_l	[K^{-1}]	K^{-1}	
Co-efficient of volume expansion	α_v	[K^{-1}]	K^{-1}	$\alpha_v = 3\,\alpha_l$
Heat supplied to a system	ΔQ	[$ML^2\,T^{-2}$]	J	Q is not a state variable
Specific heat	s	[$L^2\,T^{-2}\,K^{-1}$]	J kg^{-1} K^{-1}	
Thermal Conductivity	K	[$M\,LT^{-3}K^{-1}$]	J $s^{-1}K^{-1}$	$H = -KA\,\frac{dT}{dx}$

POINTS TO PONDER

1. The relation connecting Kelvin temperature (T) and the Celsius temperature t_c

$$T = t_c + 273.15$$

and the assignment $T = 273.16$ K for the triple point of water are exact relations (by choice). With this choice, the Celsius temperature of the melting point of water and boiling point of water (both at 1 atm pressure) are very close to, but not exactly equal to 0 °C and 100 °C respectively. In the original Celsius scale, these latter fixed points were exactly at 0 °C and 100 °C (by choice), but now the triple point of water is the preferred choice for the fixed point, because it has a unique temperature.

2. A liquid in equilibrium with vapour has the same pressure and temperature throughout the system; the two phases in equilibrium differ in their molar volume (i.e. density). This is true for a system with any number of phases in equilibrium.
3. Heat transfer always involves temperature difference between two systems or two parts of the same system. Any energy transfer that does not involve temperature difference in some way is not heat.
4. Convection involves flow of matter *within a fluid* due to unequal temperatures of its parts. A hot bar placed under a running tap loses heat by conduction between the surface of the bar and water and not by convection within water.

EXERCISES

11.1 The triple points of neon and carbon dioxide are 24.57 K and 216.55 K respectively. Express these temperatures on the Celsius and Fahrenheit scales.

11.2 Two absolute scales A and B have triple points of water defined to be 200 A and 350 B. What is the relation between T_A and T_B ?

11.3 The electrical resistance in ohms of a certain thermometer varies with temperature according to the approximate law :

$$R = R_o \, [1 + \alpha \, (T - T_o)]$$

The resistance is 101.6 Ω at the triple-point of water 273.16 K, and 165.5 Ω at the normal melting point of lead (600.5 K). What is the temperature when the resistance is 123.4 Ω ?

11.4 Answer the following :

(a) The triple-point of water is a standard fixed point in modern thermometry. Why ? What is wrong in taking the melting point of ice and the boiling point of water as standard fixed points (as was originally done in the Celsius scale) ?

(b) There were two fixed points in the original Celsius scale as mentioned above which were assigned the number 0 °C and 100 °C respectively. On the absolute scale, one of the fixed points is the triple-point of water, which on the Kelvin absolute scale is assigned the number 273.16 K. What is the other fixed point on this (Kelvin) scale ?

(c) The absolute temperature (Kelvin scale) T is related to the temperature t_c on the Celsius scale by

$$t_c = T - 273.15$$

Why do we have 273.15 in this relation, and not 273.16 ?

(d) What is the temperature of the triple-point of water on an absolute scale whose unit interval size is equal to that of the Fahrenheit scale ?

11.5 Two ideal gas thermometers *A* and *B* use oxygen and hydrogen respectively. The following observations are made :

Temperature	Pressure thermometer A	Pressure thermometer B
Triple-point of water	1.250×10^5 Pa	0.200×10^5 Pa
Normal melting point of sulphur	1.797×10^5 Pa	0.287×10^5 Pa

(a) What is the absolute temperature of normal melting point of sulphur as read by thermometers *A* and *B* ?

(b) What do you think is the reason behind the slight difference in answers of thermometers *A* and *B* ? (The thermometers are not faulty). What further procedure is needed in the experiment to reduce the discrepancy between the two readings ?

11.6 A steel tape 1m long is correctly calibrated for a temperature of 27.0 °C. The length of a steel rod measured by this tape is found to be 63.0 cm on a hot day when the temperature is 45.0 °C. What is the actual length of the steel rod on that day ? What is the length of the same steel rod on a day when the temperature is 27.0 °C ? Coefficient of linear expansion of steel = 1.20×10^{-5} K^{-1} .

11.7 A large steel wheel is to be fitted on to a shaft of the same material. At 27 °C, the outer diameter of the shaft is 8.70 cm and the diameter of the central hole in the wheel is 8.69 cm. The shaft is cooled using 'dry ice'. At what temperature of the shaft does the wheel slip on the shaft? Assume coefficient of linear expansion of the steel to be constant over the required temperature range :
$\alpha_{steel} = 1.20 \times 10^{-5}$ K^{-1}.

11.8 A hole is drilled in a copper sheet. The diameter of the hole is 4.24 cm at 27.0 °C. What is the change in the diameter of the hole when the sheet is heated to 227 °C? Coefficient of linear expansion of copper = 1.70×10^{-5} K^{-1}.

11.9 A brass wire 1.8 m long at 27 °C is held taut with little tension between two rigid supports. If the wire is cooled to a temperature of –39 °C, what is the tension developed in the wire, if its diameter is 2.0 mm ? Co-efficient of linear expansion of brass = 2.0×10^{-5} K^{-1}; Young's modulus of brass = 0.91×10^{11} Pa.

11.10 A brass rod of length 50 cm and diameter 3.0 mm is joined to a steel rod of the same length and diameter. What is the change in length of the combined rod at 250 °C, if the original lengths are at 40.0 °C? Is there a 'thermal stress' developed at the junction ? The ends of the rod are free to expand (Co-efficient of linear expansion of brass = 2.0×10^{-5} K^{-1}, steel = 1.2×10^{-5} K^{-1}).

11.11 The coefficient of volume expansion of glycerin is 49×10^{-5} K^{-1}. What is the fractional change in its density for a 30 °C rise in temperature ?

11.12 A 10 kW drilling machine is used to drill a bore in a small aluminium block of mass 8.0 kg. How much is the rise in temperature of the block in 2.5 minutes, assuming 50% of power is used up in heating the machine itself or lost to the surroundings. Specific heat of aluminium = 0.91 J g^{-1} K^{-1}.

11.13 A copper block of mass 2.5 kg is heated in a furnace to a temperature of 500 °C and then placed on a large ice block. What is the maximum amount of ice that can melt? (Specific heat of copper = 0.39 J g^{-1} K^{-1}; heat of fusion of water = 335 J g^{-1}).

11.14 In an experiment on the specific heat of a metal, a 0.20 kg block of the metal at 150 °C is dropped in a copper calorimeter (of water equivalent 0.025 kg) containing 150 cm^3 of water at 27 °C. The final temperature is 40 °C. Compute the specific

heat of the metal. If heat losses to the surroundings are not negligible, is your answer greater or smaller than the actual value for specific heat of the metal ?

11.15 Given below are observations on molar specific heats at room temperature of some common gases.

Gas	**Molar specific heat (C_v) (cal mo1^{-1} K^{-1})**
Hydrogen	4.87
Nitrogen	4.97
Oxygen	5.02
Nitric oxide	4.99
Carbon monoxide	5.01
Chlorine	6.17

The measured molar specific heats of these gases are markedly different from those for monatomic gases. Typically, molar specific heat of a monatomic gas is 2.92 cal/mol K. Explain this difference. What can you infer from the somewhat larger (than the rest) value for chlorine ?

11.16 Answer the following questions based on the *P-T* phase diagram of carbon dioxide:

(a) At what temperature and pressure can the solid, liquid and vapour phases of CO_2 co-exist in equilibrium ?

(b) What is the effect of decrease of pressure on the fusion and boiling point of CO_2 ?

(c) What are the critical temperature and pressure for CO_2 ? What is their significance ?

(d) Is CO_2 solid, liquid or gas at (a) –70 °C under 1 atm, (b) –60 °C under 10 atm, (c) 15 °C under 56 atm ?

11.17 Answer the following questions based on the *P – T* phase diagram of CO_2:

(a) CO_2 at 1 atm pressure and temperature – 60 °C is compressed isothermally. Does it go through a liquid phase ?

(b) What happens when CO_2 at 4 atm pressure is cooled from room temperature at constant pressure ?

(c) Describe qualitatively the changes in a given mass of solid CO_2 at 10 atm pressure and temperature –65 °C as it is heated up to room temperature at constant pressure.

(d) CO_2 is heated to a temperature 70 °C and compressed isothermally. What changes in its properties do you expect to observe ?

11.18 A child running a temperature of 101°F is given an antipyrin (i.e. a medicine that lowers fever) which causes an increase in the rate of evaporation of sweat from his body. If the fever is brought down to 98 °F in 20 min, what is the average rate of extra evaporation caused, by the drug. Assume the evaporation mechanism to be the only way by which heat is lost. The mass of the child is 30 kg. The specific heat of human body is approximately the same as that of water, and latent heat of evaporation of water at that temperature is about 580 cal g^{-1}.

11.19 A 'thermacole' icebox is a cheap and efficient method for storing small quantities of cooked food in summer in particular. A cubical icebox of side 30 cm has a thickness of 5.0 cm. If 4.0 kg of ice is put in the box, estimate the amount of ice remaining after 6 h. The outside temperature is 45 °C, and co-efficient of thermal conductivity of thermacole is 0.01 J s^{-1} m^{-1} K^{-1}. [Heat of fusion of water = 335 × 10^{3} J kg^{-1}]

11.20 A brass boiler has a base area of 0.15 m^2 and thickness 1.0 cm. It boils water at the rate of 6.0 kg/min when placed on a gas stove. Estimate the temperature of the part

of the flame in contact with the boiler. Thermal conductivity of brass = 109 J s^{-1} m^{-1} K^{-1} ; Heat of vaporisation of water = 2256 × 10^3 J kg^{-1}.

11.21 Explain why :

(a) a body with large reflectivity is a poor emitter

(b) a brass tumbler feels much colder than a wooden tray on a chilly day

(c) an optical pyrometer (for measuring high temperatures) calibrated for an ideal black body radiation gives too low a value for the temperature of a red hot iron piece in the open, but gives a correct value for the temperature when the same piece is in the furnace

(d) the earth without its atmosphere would be inhospitably cold

(e) heating systems based on circulation of steam are more efficient in warming a building than those based on circulation of hot water

11.22 A body cools from 80 °C to 50 °C in 5 minutes. Calculate the time it takes to cool from 60 °C to 30 °C. The temperature of the surroundings is 20 °C.

CHAPTER TWELVE

THERMODYNAMICS

12.1 INTRODUCTION

In previous chapter we have studied thermal properties of matter. In this chapter we shall study laws that govern thermal energy. We shall study the processes where work is converted into heat and vice versa. In winter, when we rub our palms together, we feel warmer; here work done in rubbing produces the 'heat'. Conversely, in a steam engine, the 'heat' of the steam is used to do useful work in moving the pistons, which in turn rotate the wheels of the train.

In physics, we need to define the notions of heat, temperature, work, etc. more carefully. Historically, it took a long time to arrive at the proper concept of 'heat'. Before the modern picture, heat was regarded as a fine invisible fluid filling in the pores of a substance. On contact between a hot body and a cold body, the fluid (called caloric) flowed from the colder to the hotter body ! This is similar to what happens when a horizontal pipe connects two tanks containing water up to different heights. The flow continues until the levels of water in the two tanks are the same. Likewise, in the 'caloric' picture of heat, heat flows until the 'caloric levels' (i.e., the temperatures) equalise.

In time, the picture of heat as a fluid was discarded in favour of the modern concept of heat as a form of energy. An important experiment in this connection was due to Benjamin Thomson (also known as Count Rumford) in 1798. He observed that boring of a brass cannon generated a lot of heat, indeed enough to boil water. More significantly, the amount of heat produced depended on the work done (by the horses employed for turning the drill) but not on the sharpness of the drill. In the caloric picture, a sharper drill would scoop out more heat fluid from the pores; but this was not observed. A most natural explanation of the observations was that heat was a form of energy and the experiment demonstrated conversion of energy from one form to another–from work to heat.

Thermodynamics is the branch of physics that deals with the concepts of heat and temperature and the inter-conversion of heat and other forms of energy. Thermodynamics is a macroscopic science. It deals with bulk systems and does not go into the molecular constitution of matter. In fact, its concepts and laws were formulated in the nineteenth century before the molecular picture of matter was firmly established. Thermodynamic description involves relatively few macroscopic variables of the system, which are suggested by common sense and can be usually measured directly. A microscopic description of a gas, for example, would involve specifying the co-ordinates and velocities of the huge number of molecules constituting the gas. The description in kinetic theory of gases is not so detailed but it does involve molecular distribution of velocities. Thermodynamic description of a gas, on the other hand, avoids the molecular description altogether. Instead, the state of a gas in thermodynamics is specified by macroscopic variables such as pressure, volume, temperature, mass and composition that are felt by our sense perceptions and are measurable*.

The distinction between mechanics and thermodynamics is worth bearing in mind. In mechanics, our interest is in the motion of particles or bodies under the action of forces and torques. Thermodynamics is not concerned with the motion of the system as a whole. It is concerned with the internal macroscopic state of the body. When a bullet is fired from a gun, what changes is the mechanical state of the bullet (its kinetic energy, in particular), not its temperature. When the bullet pierces a wood and stops, the kinetic energy of the bullet gets converted into heat, changing the temperature of the bullet and the surrounding layers of wood. Temperature is related to the energy of the internal (disordered) motion of the bullet, not to the motion of the bullet as a whole.

12.2 THERMAL EQUILIBRIUM

Equilibrium in mechanics means that the net external force and torque on a system are zero. The term 'equilibrium' in thermodynamics appears in a different context : we say the state of a system is an equilibrium state if the macroscopic variables that characterise the system do not change in time. For example, a gas inside a closed rigid container, completely insulated from its surroundings, with fixed values of pressure, volume, temperature, mass and composition that do not change with time, is in a state of thermodynamic equilibrium.

In general, whether or not a system is in a state of equilibrium depends on the surroundings and the nature of the wall that separates the system from the surroundings. Consider two gases A and B occupying two different containers. We know experimentally that pressure and volume of a given mass of gas can be chosen to be its two independent variables. Let the pressure and volume of the gases be (P_A, V_A) and (P_B, V_B) respectively. Suppose first that the two systems are put in proximity but are separated by an **adiabatic wall** – an insulating wall (can be movable) that does not allow flow of energy (heat) from one to another. The systems are insulated from the rest of the surroundings also by similar adiabatic walls. The situation is shown schematically in Fig. 12.1 (a). In this case, it is found that any possible pair of values (P_A, V_A) will be in equilibrium with any possible pair of values (P_B, V_B). Next, suppose that the adiabatic wall is replaced by a **diathermic wall** – a conducting wall that allows energy flow (heat) from one to another. It is then found that the macroscopic variables of the systems A and B change spontaneously until both the systems attain equilibrium states. After that there is no change in their states. The situation is shown in Fig. 12.1(b). The pressure and volume variables of the two gases change to $(P_B{'}, V_B{'})$ and $(P_A{'}, V_A{'})$ such that the new states of A and B are in equilibrium with each other**. There is no more energy flow from one to another. We then say that the system A is in thermal equilibrium with the system B.

What characterises the situation of thermal equilibrium between two systems ? You can guess the answer from your experience. In thermal equilibrium, the temperatures of the two systems

* *Thermodynamics may also involve other variables that are not so obvious to our senses e.g. entropy, enthalpy, etc., and they are all macroscopic variables.*

** *Both the variables need not change. It depends on the constraints. For instance, if the gases are in containers of fixed volume, only the pressures of the gases would change to achieve thermal equilibrium.*

are equal. We shall see how does one arrive at the concept of temperature in thermodynamics? The Zeroth law of thermodynamics provides the clue.

12.3 ZEROTH LAW OF THERMODYNAMICS

Imagine two systems *A* and *B*, separated by an adiabatic wall, while each is in contact with a third system *C*, via a conducting wall [Fig. 12.2(a)]. The states of the systems (i.e., their macroscopic variables) will change until both *A* and *B* come to thermal equilibrium with *C*. After this is achieved, suppose that the adiabatic wall between *A* and *B* is replaced by a conducting wall and *C* is insulated from *A* and *B* by an adiabatic wall [Fig.12.2(b)]. It is found that the states of *A* and *B* change no further i.e. they are found **to be in thermal equilibrium with each other**. This observation forms the basis of the **Zeroth Law of Thermodynamics**, which states that **'two systems in thermal equilibrium with a third system separately are in thermal equilibrium with each other'**. R.H. Fowler formulated this law in 1931 long after the first and second Laws of thermodynamics were stated and so numbered.

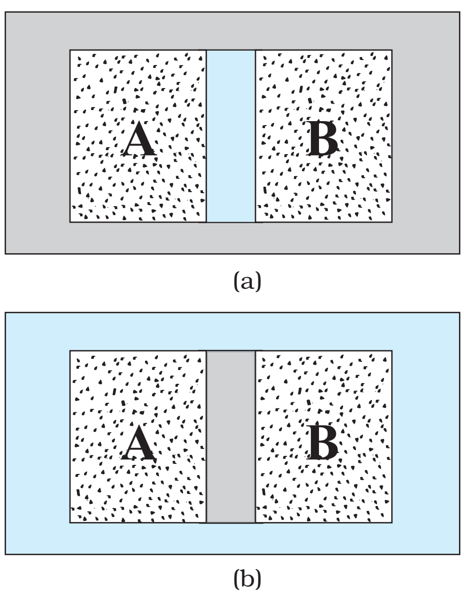


Fig. 12.1 *(a) Systems A and B (two gases) separated by an adiabatic wall – an insulating wall that does not allow flow of heat. (b) The same systems A and B separated by a diathermic wall – a conducting wall that allows heat to flow from one to another. In this case, thermal equilibrium is attained in due course.*

The Zeroth Law clearly suggests that when two systems *A* and *B*, are in thermal equilibrium, there must be a physical quantity that has the same value for both. This thermodynamic variable whose value is equal for two systems in thermal equilibrium is called temperature (T). Thus, if *A* and *B* are separately in equilibrium with *C*, $T_A = T_C$ and $T_B = T_C$. This implies that $T_A = T_B$ i.e. the systems *A* and *B* are also in thermal equilibrium.

We have arrived at the concept of temperature formally via the Zeroth Law. The next question is : how to assign numerical values to temperatures of different bodies ? In other words, how do we construct a scale of temperature ? Thermometry deals with this basic question to which we turn in the next section.

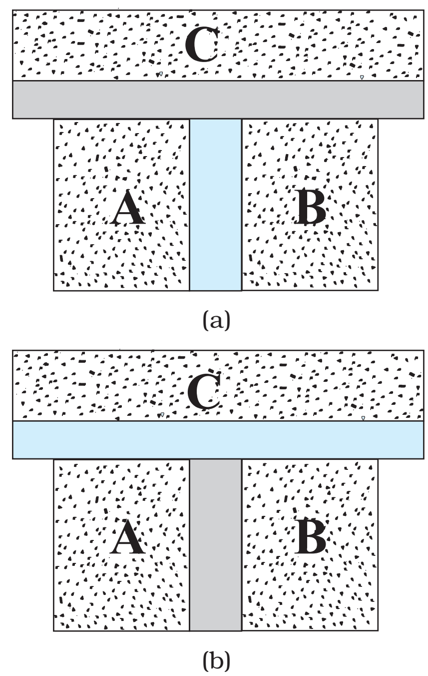


Fig. 12.2 *(a) Systems A and B are separated by an adiabatic wall, while each is in contact with a third system C via a conducting wall. (b) The adiabatic wall between A and B is replaced by a conducting wall, while C is insulated from A and B by an adiabatic wall.*

12.4 HEAT, INTERNAL ENERGY AND WORK

The Zeroth Law of Thermodynamics led us to the concept of temperature that agrees with our commonsense notion. Temperature is a marker

of the 'hotness' of a body. It determines the direction of flow of heat when two bodies are placed in thermal contact. Heat flows from the body at a higher temperature to the one at lower temperature. The flow stops when the temperatures equalise; the two bodies are then in thermal equilibrium. We saw in some detail how to construct temperature scales to assign temperatures to different bodies. We now describe the concepts of heat and other relevant quantities like internal energy and work.

The concept of internal energy of a system is not difficult to understand. We know that every bulk system consists of a large number of molecules. Internal energy is simply the sum of the kinetic energies and potential energies of these molecules. We remarked earlier that in thermodynamics, the kinetic energy of the system, as a whole, is not relevant. Internal energy is thus, the sum of molecular kinetic and potential energies in the frame of reference relative to which the centre of mass of the system is at rest. Thus, it includes only the (disordered) energy associated with the random motion of molecules of the system. We denote the internal energy of a system by *U*.

Though we have invoked the molecular picture to understand the meaning of internal energy, as far as thermodynamics is concerned, *U* is simply a macroscopic variable of the system. The important thing about internal energy is that it depends only on the state of the system, not on how that state was achieved. Internal energy *U* of a system is an example of a thermodynamic 'state variable' – its value depends only on the given state of the system, not on history i.e. not on the 'path' taken to arrive at that state. Thus, the internal energy of a given mass of gas depends on its state described by specific values of pressure, volume and temperature. It does not depend on how this state of the gas came about. Pressure, volume, temperature, and internal energy are thermodynamic state variables of the system (gas) (see section 12.7). If we neglect the small intermolecular forces in a gas, the internal energy of a gas is just the sum of kinetic energies associated with various random motions of its molecules. We will see in the next chapter that in a gas this motion is not only translational (i.e. motion from one point to another in the volume of the container); it also includes rotational and vibrational motion of the molecules (Fig. 12.3).

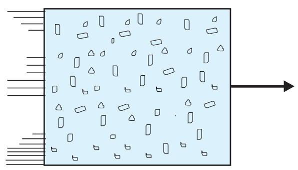
What are the ways of changing internal energy of a system ? Consider again, for simplicity, the system to be a certain mass of gas contained in a cylinder with a movable piston as shown in Fig. 12.4. Experience shows there are two ways of changing the state of the gas (and hence its internal energy). One way is to put the cylinder in contact with a body at a higher temperature than that of the gas. The temperature difference will cause a flow of energy (heat) from the hotter body to the gas, thus increasing the internal energy of the gas. The other way is to push the piston down i.e. to do work on the system, which again results in increasing the internal energy of the gas. Of course, both these things could happen in the reverse direction. With surroundings at a lower temperature, heat would flow from the gas to the surroundings. Likewise, the gas could push the piston up and do work on the surroundings. In short, heat and work are two different modes of altering the state of a thermodynamic system and changing its internal energy.

The notion of heat should be carefully distinguished from the notion of internal energy. Heat is certainly energy, but it is the energy in transit. This is not just a play of words. The distinction is of basic significance. The state of a thermodynamic system is characterised by its internal energy, not heat. A statement like **'a gas in a given state has a certain amount of heat'** is as meaningless as the statement that **'a gas in a given state has a certain amount of work'**. In contrast, **'a gas in a given state has a certain amount of internal energy'** is a perfectly meaningful statement. Similarly, the statements **'a certain amount of heat is supplied to the system' or 'a certain amount of work was done by the system'** are perfectly meaningful.

To summarise, heat and work in thermodynamics are not state variables. They are modes of energy transfer to a system resulting in change in its internal energy, which, as already mentioned, is a state variable.

In ordinary language, we often confuse heat with internal energy. The distinction between them is sometimes ignored in elementary

physics books. For proper understanding of thermodynamics, however, the distinction is crucial.



(a) (b)

Fig. 12.3 *(a) Internal energy U of a gas is the sum of the kinetic and potential energies of its molecules when the box is at rest. Kinetic energy due to various types of motion (translational, rotational, vibrational) is to be included in U. (b) If the same box is moving as a whole with some velocity, the kinetic energy of the box is not to be included in U.*

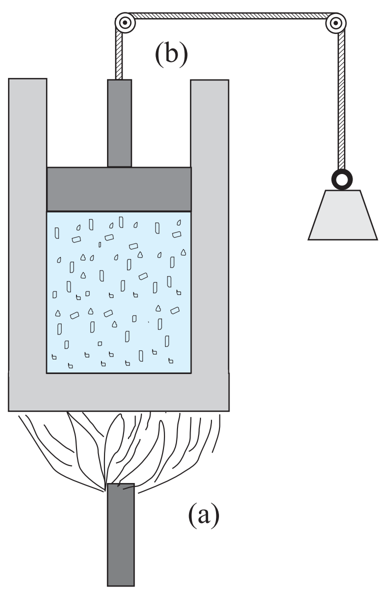


Fig. 12.4 *Heat and work are two distinct modes of energy transfer to a system that results in change in its internal energy. (a) Heat is energy transfer due to temperature difference between the system and the surroundings. (b) Work is energy transfer brought about by means (e.g. moving the piston by raising or lowering some weight connected to it) that do not involve such a temperature difference.*

12.5 FIRST LAW OF THERMODYNAMICS

We have seen that the internal energy U of a system can change through two modes of energy transfer : heat and work. Let

ΔQ = Heat supplied *to* the system *by* the surroundings

ΔW = Work done *by* the system *on* the surroundings

ΔU = Change in internal energy of the system

The general principle of conservation of energy then implies that

$$\Delta Q = \Delta U + \Delta W \qquad (12.1)$$

i.e. the energy (ΔQ) supplied to the system goes in partly to increase the internal energy of the system (ΔU) and the rest in work on the environment (ΔW). Equation (12.1) is known as the **First Law of Thermodynamics**. It is simply the general law of conservation of energy applied to any system in which the energy transfer from or to the surroundings is taken into account.

Let us put Eq. (12.1) in the alternative form

$$\Delta Q - \Delta W = \Delta U \qquad (12.2)$$

Now, the system may go from an initial state to the final state in a number of ways. For example, to change the state of a gas from (P_1, V_1) to (P_2, V_2), we can first change the volume of the gas from V_1 to V_2, keeping its pressure constant i.e. we can first go the state (P_1, V_2) and then change the pressure of the gas from P_1 to P_2, keeping volume constant, to take the gas to (P_2, V_2). Alternatively, we can first keep the volume constant and then keep the pressure constant. Since U is a state variable, ΔU depends only on the initial and final states and not on the path taken by the gas to go from one to the other. However, ΔQ and ΔW will, in general, depend on the path taken to go from the initial to final states. From the First Law of Thermodynamics, Eq. (12.2), it is clear that the combination $\Delta Q - \Delta W$, is however, path independent. This shows that if a system is taken through a process in which $\Delta U = 0$ (for example, isothermal expansion of an ideal gas, see section 12.8),

$$\Delta Q = \Delta W$$

i.e., heat supplied to the system is used up entirely by the system in doing work on the environment.

If the system is a gas in a cylinder with a movable piston, the gas in moving the piston

does work. Since force is pressure times area, and area times displacement is volume, work done by the system against a constant pressure P is

$$\Delta W = P\,\Delta V$$

where ΔV is the change in volume of the gas. Thus, for this case, Eq. (12.1) gives

$$\Delta Q = \Delta U + P\,\Delta V \qquad (12.3)$$

As an application of Eq. (12.3), consider the change in internal energy for 1 g of water when we go from its liquid to vapour phase. The measured latent heat of water is 2256 J/g. i.e., for 1 g of water ΔQ = 2256 J. At atmospheric pressure, 1 g of water has a volume 1 cm^3 in liquid phase and 1671 cm^3 in vapour phase.

Therefore,

$$\Delta W = P(V_g - V_l) = 1.013\times10^5 \times(1670)\times10^{-6} = 169.2\ \text{J}$$

Equation (12.3) then gives

$$\Delta U = 2256 - 169.2 = 2086.8\ \text{J}$$

We see that most of the heat goes to increase the internal energy of water in transition from the liquid to the vapour phase.

12.6 SPECIFIC HEAT CAPACITY

Suppose an amount of heat ΔQ supplied to a substance changes its temperature from T to $T + \Delta T$. We define heat capacity of a substance (see Chapter 11) to be

$$S = \frac{\Delta Q}{\Delta T} \qquad (12.4)$$

We expect ΔQ and, therefore, heat capacity S to be proportional to the mass of the substance. Further, it could also depend on the temperature, i.e., a different amount of heat may be needed for a unit rise in temperature at different temperatures. To define a constant characteristic of the substance and independent of its amount, we divide S by the mass of the substance m in kg :

$$s \quad \frac{S}{m} \quad \frac{1}{m}\frac{Q}{T} \qquad (12.5)$$

s is known as the specific heat capacity of the substance. It depends on the nature of the substance and its temperature. The unit of specific heat capacity is J kg^{-1} K^{-1}.

If the amount of substance is specified in terms of moles μ (instead of mass m in kg), we can define heat capacity per mole of the substance by

$$C \quad \frac{S}{} \quad \frac{1}{}\frac{Q}{T} \qquad (12.6)$$

C is known as molar specific heat capacity of the substance. Like s, C is independent of the amount of substance. C depends on the nature of the substance, its temperature and the conditions under which heat is supplied. The unit of C is J mol^{-1} K^{-1}. As we shall see later (in connection with specific heat capacity of gases), additional conditions may be needed to define C or s. The idea in defining C is that simple predictions can be made in regard to molar specific heat capacities.

Table 12.1 lists measured specific and molar heat capacities of solids at atmospheric pressure and ordinary room temperature.

We will see in Chapter 13 that predictions of specific heats of gases generally agree with experiment. We can use the same law of equipartition of energy that we use there to predict molar specific heat capacities of solids. Consider a solid of N atoms, each vibrating about its mean position. An oscillator in one dimension has average energy of 2 × ½ $k_BT = k_BT$. In three dimensions, the average energy is 3 k_BT. For a mole of a solid, the total energy is

$$U = 3\,k_BT \times N_A = 3\,RT$$

Now, at constant pressure, $\Delta Q = \Delta U + P\,\Delta V \cong \Delta U$, since for a solid ΔV is negligible. Therefore,

$$C = \frac{\Delta Q}{\Delta T} = \frac{\Delta U}{\Delta T} = 3R \qquad (12.7)$$

Table 12.1 **Specific and molar heat capacities of some solids at room temperature and atmospheric pressure**

Substance	Specific heat (J kg^{-1}K^{-1})	Molar specific Heat(J mol^{-1}K^{-1})
Aluminium	900.0	24.4
Carbon	506.5	6.1
Copper	386.4	24.5
Lead	127.7	26.5
Silver	236.1	25.5
Tungsten	134.4	24.9

As Table 12.1 shows, the experimentally measured values which generally agrees with

predicted value 3R at ordinary temperatures. (Carbon is an exception.) The agreement is known to break down at low temperatures.

Specific heat capacity of water

The old unit of heat was calorie. One calorie was earlier defined to be the amount of heat required to raise the temperature of 1g of water by 1°C. With more precise measurements, it was found that the specific heat of water varies slightly with temperature. Figure 12.5 shows this variation in the temperature range 0 to 100 °C.

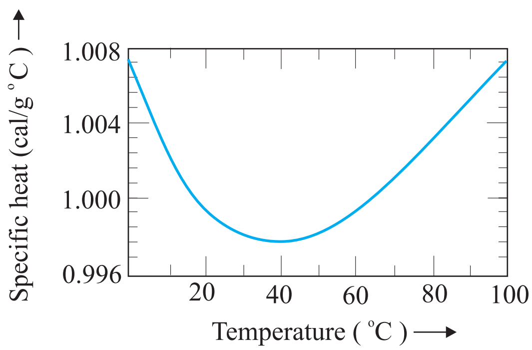


Fig. 12.5 *Variation of specific heat capacity of water with temperature.*

For a precise definition of calorie, it was, therefore, necessary to specify the unit temperature interval. One calorie is defined to be the amount of heat required to raise the temperature of 1g of water from 14.5 °C to 15.5 °C. Since heat is just a form of energy, it is preferable to use the unit joule, J. In SI units, the specific heat capacity of water is 4186 J kg^{-1} K^{-1} i.e. 4.186 J g^{-1} K^{-1}. The so called mechanical equivalent of heat defined as the amount of work needed to produce 1 cal of heat is in fact just a conversion factor between two different units of energy : calorie to joule. Since in SI units, we use the unit joule for heat, work or any other form of energy, the term mechanical equivalent is now superfluous and need not be used.

As already remarked, the specific heat capacity depends on the process or the conditions under which heat capacity transfer takes place. For gases, for example, we can define two specific heats : **specific heat capacity at constant volume and specific heat capacity at constant pressure**. For an ideal gas, we have a simple relation.

$$C_p - C_v = R \qquad (12.8)$$

where C_p and C_v are molar specific heat capacities of an ideal gas at constant pressure and volume respectively and R is the universal gas constant. To prove the relation, we begin with Eq. (12.3) for 1 mole of the gas :

$$\Delta Q = \Delta U + P\,\Delta V$$

If ΔQ is absorbed at constant volume, $\Delta V = 0$

$$C_v \quad \frac{Q}{T}_{v} \quad \frac{U}{T}_{v} \quad \frac{U}{T} \qquad (12.9)$$

where the subscript v is dropped in the last step, since U of an ideal gas depends only on temperature. (The subscript denotes the quantity kept fixed.) If, on the other hand, ΔQ is absorbed at constant pressure,

$$C_p \quad \frac{Q}{T}_{p} \quad \frac{U}{T}_{p} \quad P \quad \frac{V}{T}_{p} \qquad (12.10)$$

The subscript p can be dropped from the first term since U of an ideal gas depends only on T. Now, for a mole of an ideal gas

$$PV = RT$$

which gives

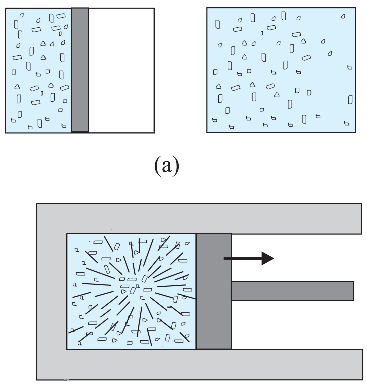
$$P \quad \frac{V}{T}_{p} \quad R \qquad (12.11)$$

Equations (12.9) to (12.11) give the desired relation, Eq. (12.8).

12.7 THERMODYNAMIC STATE VARIABLES AND EQUATION OF STATE

Every **equilibrium state** of a thermodynamic system is completely described by specific values of some macroscopic variables, also called state variables. For example, an equilibrium state of a gas is completely specified by the values of pressure, volume, temperature, and mass (and composition if there is a mixture of gases). A thermodynamic system is not always in equilibrium. For example, a gas allowed to expand freely against vacuum is not an equilibrium state [Fig. 12.6(a)]. During the rapid expansion, pressure of the gas may

not be uniform throughout. Similarly, a mixture of gases undergoing an explosive chemical reaction (e.g. a mixture of petrol vapour and air when ignited by a spark) is not an equilibrium state; again its temperature and pressure are not uniform [Fig. 12.6(b)]. Eventually, the gas attains a uniform temperature and pressure and comes to thermal and mechanical equilibrium with its surroundings.



(a)

(b)

Fig. 12.6 *(a) The partition in the box is suddenly removed leading to free expansion of the gas. (b) A mixture of gases undergoing an explosive chemical reaction. In both situations, the gas is not in equilibrium and cannot be described by state variables.*

In short, thermodynamic state variables describe equilibrium states of systems. The various state variables are not necessarily independent. The connection between the state variables is called the equation of state. For example, for an ideal gas, the equation of state is the ideal gas relation

$$P V = \mu R T$$

For a fixed amount of the gas i.e. given μ, there are thus, only two independent variables, say P and V or T and V. The pressure-volume curve for a fixed temperature is called an **isotherm**. Real gases may have more complicated equations of state.

The thermodynamic state variables are of two kinds: **extensive** and **intensive**. Extensive variables indicate the 'size' of the system. Intensive variables such as pressure and temperature do not. To decide which variable is extensive and which intensive, think of a relevant system in equilibrium, and imagine that it is divided into two equal parts. The variables that remain unchanged for each part are intensive. The variables whose values get halved in each part are extensive. It is easily seen, for example, that internal energy U, volume V, total mass M are extensive variables. Pressure P, temperature T, and density ρ are intensive variables. It is a good practice to check the consistency of thermodynamic equations using this classification of variables. For example, in the equation

$$\Delta Q = \Delta U + P \Delta V$$

quantities on both sides are extensive*. (The product of an intensive variable like P and an extensive quantity ΔV is extensive.)

12.8 THERMODYNAMIC PROCESSES

12.8.1 Quasi-static process

Consider a gas in thermal and mechanical equilibrium with its surroundings. The pressure of the gas in that case equals the external pressure and its temperature is the same as that of its surroundings. Suppose that the external pressure is suddenly reduced (say by lifting the weight on the movable piston in the container). The piston will accelerate outward. During the process, the gas passes through states that are not equilibrium states. The non-equilibrium states do not have well-defined pressure and temperature. In the same way, if a finite temperature difference exists between the gas and its surroundings, there will be a rapid exchange of heat during which the gas will pass through non-equilibrium states. In due course, the gas will settle to an equilibrium state with well-defined temperature and pressure equal to those of the surroundings. The free expansion of a gas in vacuum and a mixture of gases undergoing an explosive chemical reaction, mentioned in section 12.7 are also examples where the system goes through non-equilibrium states.

Non-equilibrium states of a system are difficult to deal with. It is, therefore, convenient to imagine an idealised process in which at every stage the system is an equilibrium state. Such a

* *As emphasised earlier, Q is not a state variable. However, ΔQ is clearly proportional to the total mass of system and hence is extensive.*

process is, in principle, infinitely slow-hence the name quasi-static (meaning nearly static). The system changes its variables (P, T, V) so slowly that it remains in thermal and mechanical equilibrium with its surroundings throughout. In a quasi-static process, at every stage, the difference in the pressure of the system and the external pressure is infinitesimally small. The same is true of the temperature difference between the system and its surroundings. To take a gas from the state (P, T) to another state (P', T') via a quasi-static process, we change the external pressure by a very small amount, allow the system to equalise its pressure with that of the surroundings and continue the process infinitely slowly until the system achieves the pressure P'. Similarly, to change the temperature, we introduce an infinitesimal temperature difference between the system and the surrounding reservoirs and by choosing reservoirs of progressively different temperatures T to T', the system achieves the temperature T'.

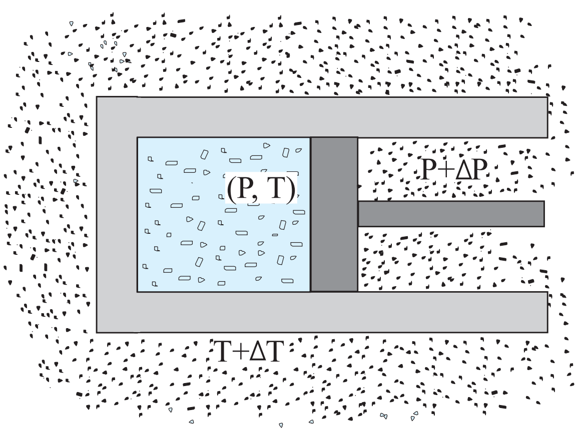


Fig. 12.7 *In a quasi-static process, the temperature of the surrounding reservoir and the external pressure differ only infinitesimally from the temperature and pressure of the system.*

A quasi-static process is obviously a hypothetical construct. In practice, processes that are sufficiently slow and do not involve accelerated motion of the piston, large temperature gradient, etc. are reasonably approximation to an ideal quasi-static process. We shall from now on deal with quasi-static processes only, except when stated otherwise.

A process in which the temperature of the system is kept fixed throughout is called an **isothermal process**. The expansion of a gas in a metallic cylinder placed in a large reservoir of fixed temperature is an example of an isothermal process. (Heat transferred from the reservoir to the system does not materially affect the temperature of the reservoir, because of its very large heat capacity.) In **isobaric processes** the pressure is constant while in **isochoric processes** the volume is constant. Finally, if the system is insulated from the surroundings and no heat flows between the system and the surroundings, the process is **adiabatic**. The definitions of these special processes are summarised in Table. 12.2

Table 12.2 **Some special thermodynamic processes**

Type of processes	Feature
Isothermal	Temperature constant
Isobaric	Pressure constant
Isochoric	Volume constant
Adiabatic	No heat flow between the system and the surroundings ($\Delta Q = 0$)

We now consider these processes in some detail :

Isothermal process

For an isothermal process (T fixed), the ideal gas equation gives

$$PV = \text{constant}$$

i.e., pressure of a given mass of gas varies inversely as its volume. This is nothing but Boyle's Law.

Suppose an ideal gas goes isothermally (at temperature T) from its initial state (P_1, V_1) to the final state (P_2, V_2). At any intermediate stage with pressure P and volume change from V to $V + \Delta V$ (ΔV small)

$$\Delta W = P \Delta V$$

Taking ($\Delta V \rightarrow 0$) and summing the quantity ΔW over the entire process,

$$W = \int_{V_1}^{V_2} P \, dV$$

$$= RT \int_{V_1}^{V_2} \frac{dV}{V} = RT \ \ln \frac{V_2}{V_1} \qquad (12.12)$$

where in the second step we have made use of the ideal gas equation $PV = \mu RT$ and taken the constants out of the integral. For an ideal gas, internal energy depends only on temperature. Thus, there is no change in the internal energy of an ideal gas in an isothermal process. The First Law of Thermodynamics then implies that heat supplied to the gas equals the work done by the gas : $Q = W$. Note from Eq. (12.12) that for $V_2 > V_1$, $W > 0$; and for $V_2 < V_1$, $W < 0$. That is, in an isothermal expansion, the gas absorbs heat and does work while in an isothermal compression, work is done on the gas by the environment and heat is released.

Adiabatic process

In an adiabatic process, the system is insulated from the surroundings and heat absorbed or released is zero. From Eq. (12.1), we see that work done by the gas results in decrease in its internal energy (and hence its temperature for an ideal gas). We quote without proof (the result that you will learn in higher courses) that for an adiabatic process of an ideal gas.

$$P V^{\gamma} = \text{const} \qquad (12.13)$$

where γ is the ratio of specific heats (ordinary or molar) at constant pressure and at constant volume.

$$\gamma = \frac{C_p}{C_v}$$

Thus if an ideal gas undergoes a change in its state adiabatically from (P_1, V_1) to (P_2, V_2) :

$$P_1 V_1^{\gamma} = P_2 V_2^{\gamma} \qquad (12.14)$$

Figure12.8 shows the *P-V* curves of an ideal gas for two adiabatic processes connecting two isotherms.

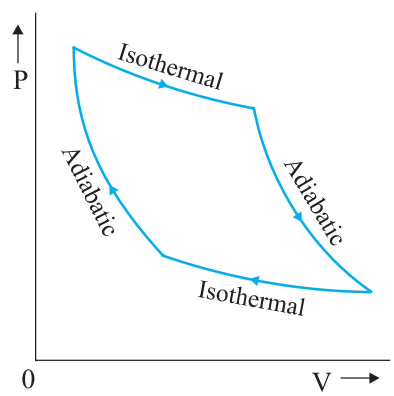


Fig. 12.8 *P-V curves for isothermal and adiabatic processes of an ideal gas.*

We can calculate, as before, the work done in an adiabatic change of an ideal gas from the state (P_1, V_1, T_1) to the state (P_2, V_2, T_2).

$$W = \int_{V_1}^{V_2} P \, dV$$

$$= \text{constant} \int_{V_1}^{V_2} \frac{dV}{V^{\gamma}} = \text{constant} \left. \frac{V^{-\gamma+1}}{1-\gamma} \right|_{V_1}^{V_2}$$

$$= \frac{\text{constant}}{(1-\gamma)} \times \left[\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_1^{\gamma-1}} \right] \qquad (12.15)$$

From Eq. (12.34), the constant is $P_1 V_1^{\gamma}$ or $P_2 V_2^{\gamma}$

$$W = \frac{1}{1-\gamma} \left[\frac{P_2 V_2^{\gamma}}{V_2^{\gamma-1}} - \frac{P_1 V_1^{\gamma}}{V_1^{\gamma-1}} \right]$$

$$= \frac{1}{1-\gamma} \left[P_2 V_2 - P_1 V_1 \right] = \frac{\mu R(T_1 - T_2)}{\gamma - 1} \qquad (12.16)$$

As expected, if work is done by the gas in an adiabatic process ($W > 0$), from Eq. (12.16), $T_2 < T_1$. On the other hand, if work is done on the gas ($W < 0$), we get $T_2 > T_1$ i.e., the temperature of the gas rises.

Isochoric process

In an isochoric process, V is constant. No work is done on or by the gas. From Eq. (12.1), the heat absorbed by the gas goes entirely to change its internal energy and its temperature. The change in temperature for a given amount of heat is determined by the specific heat of the gas at constant volume.

Isobaric process

In an isobaric process, P is fixed. Work done by the gas is

$$W = P(V_2 - V_1) = \mu R (T_2 - T_1) \qquad (12.17)$$

Since temperature changes, so does internal energy. The heat absorbed goes partly to increase internal energy and partly to do work. The change in temperature for a given amount of heat is determined by the specific heat of the gas at constant pressure.

Cyclic process

In a cyclic process, the system returns to its initial state. Since internal energy is a state variable, $\Delta U = 0$ for a cyclic process. From

Eq. (12.1), the total heat absorbed equals the work done by the system.

12.9 HEAT ENGINES

Heat engine is a device by which a system is made to undergo a cyclic process that results in conversion of heat to work.

(1) It consists of a **working substance**–the system. For example, a mixture of fuel vapour and air in a gasoline or diesel engine or steam in a steam engine are the working substances.

(2) The working substance goes through a cycle consisting of several processes. In some of these processes, it absorbs a total amount of heat Q_1 from an external reservoir at some high temperature T_1.

(3) In some other processes of the cycle, the working substance releases a total amount of heat Q_2 to an external reservoir at some lower temperature T_2.

(4) The work done (W) by the system in a cycle is transferred to the environment via some arrangement (e.g. the working substance may be in a cylinder with a moving piston that transfers mechanical energy to the wheels of a vehicle via a shaft).

The basic features of a heat engine are schematically represented in Fig. 12.9.

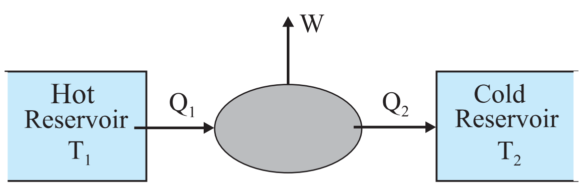


Fig. 12.9 *Schematic representation of a heat engine. The engine takes heat Q_1 from a hot reservoir at temperature T_1, releases heat Q_2 to a cold reservoir at temperature T_2 and delivers work W to the surroundings.*

The cycle is repeated again and again to get useful work for some purpose. The discipline of thermodynamics has its roots in the study of heat engines. A basic question relates to the efficiency of a heat engine. The efficiency (η) of a heat engine is defined by

$$\eta = \frac{W}{Q_1} \tag{12.18}$$

where Q_1 is the heat input i.e., the heat absorbed by the system in one complete cycle and W is the work done on the environment in a cycle. In a cycle, a certain amount of heat (Q_2) may also be rejected to the environment. Then, according to the First Law of Thermodynamics, over one complete cycle,

$$W = Q_1 - Q_2 \tag{12.19}$$

i.e.,

$$\eta = 1 - \frac{Q_2}{Q_1} \tag{12.20}$$

For $Q_2 = 0$, $\eta = 1$, i.e., the engine will have 100% efficiency in converting heat into work. Note that the First Law of Thermodynamics i.e., the energy conservation law does not rule out such an engine. But experience shows that such an ideal engine with $\eta = 1$ is never possible, even if we can eliminate various kinds of losses associated with actual heat engines. It turns out that there is a fundamental limit on the efficiency of a heat engine set by an independent principle of nature, called the Second Law of Thermodynamics (section 12.11).

The mechanism of conversion of heat into work varies for different heat engines. Basically, there are two ways : the system (say a gas or a mixture of gases) is heated by an external furnace, as in a steam engine; or it is heated internally by an exothermic chemical reaction as in an internal combustion engine. The various steps involved in a cycle also differ from one engine to another. For the purpose of general analysis, it is useful to conceptualise a heat engine as having the following essential ingredients.

12.10 REFRIGERATORS AND HEAT PUMPS

A refrigerator is the reverse of a heat engine. Here the working substance extracts heat Q_2 from the cold reservoir at temperature T_2, some external work W is done on it and heat Q_1 is released to the hot reservoir at temperature T_1 (Fig. 12.10).

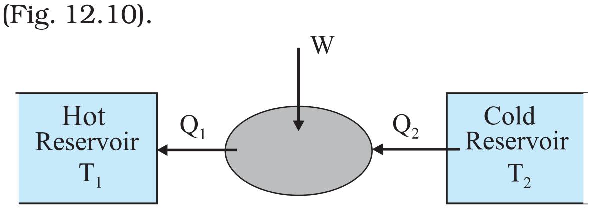
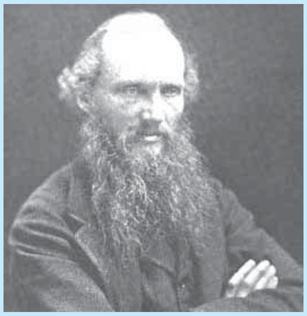


Fig. 12.10 *Schematic representation of a refrigerator or a heat pump, the reverse of a heat engine.*

Pioneers of Thermodynamics



Lord Kelvin (William Thomson) (1824-1907), born in Belfast, Ireland, is among the foremost British scientists of the nineteenth century. Thomson played a key role in the development of the law of conservation of energy suggested by the work of James Joule (1818-1889), Julius Mayer (1814-1878) and Hermann Helmholtz (1821-1894). He collaborated with Joule on the so-called Joule-Thomson effect : cooling of a gas when it expands into vacuum. He introduced the notion of the absolute zero of temperature and proposed the absolute temperature scale, now called the Kelvin scale in his honour. From the work of Sadi Carnot (1796-1832), Thomson arrived at a form of the Second Law of Thermodynamics. Thomson was a versatile physicist, with notable contributions to electromagnetic theory and hydrodynamics.

Rudolf Clausius (1822-1888), born in Poland, is generally regarded as the discoverer of the Second Law of Thermodynamics. Based on the work of Carnot and Thomson, Clausius arrived at the important notion of entropy that led him to a fundamental version of the Second Law of Thermodynamics that states that the entropy of an isolated system can never decrease. Clausius also worked on the kinetic theory of gases and obtained the first reliable estimates of molecular size, speed, mean free path, etc.

A heat pump is the same as a refrigerator. What term we use depends on the purpose of the device. If the purpose is to cool a portion of space, like the inside of a chamber, and higher temperature reservoir is surrounding, we call the device a refrigerator; if the idea is to pump heat into a portion of space (the room in a building when the outside environment is cold), the device is called a heat pump.

In a refrigerator the working substance (usually, in gaseous form) goes through the following steps : (a) sudden expansion of the gas from high to low pressure which cools it and converts it into a vapour-liquid mixture, (b) absorption by the cold fluid of heat from the region to be cooled converting it into vapour, (c) heating up of the vapour due to external work done on the system, and (d) release of heat by the vapour to the surroundings, bringing it to the initial state and completing the cycle.

The coefficient of performance (α) of a refrigerator is given by

$$\frac{Q_2}{W} \qquad (12.21)$$

where Q_2 is the heat extracted from the cold reservoir and W is the work done on the system–the refrigerant. (α for heat pump is defined as Q_1/W) Note that while η by definition can never exceed 1, α can be greater than 1. By energy conservation, the heat released to the hot reservoir is

$$Q_1 = W + Q_2$$

i.e.,
$$\frac{Q_2}{Q_1 - Q_2} \qquad (12.22)$$

In a heat engine, heat cannot be fully converted to work; likewise a refrigerator cannot work without some external work done on the system, i.e., the coefficient of performance in Eq. (12.21) cannot be infinite.

12.11 SECOND LAW OF THERMODYNAMICS

The First Law of Thermodynamics is the principle of conservation of energy. Common experience shows that there are many conceivable processes that are perfectly allowed by the First Law and yet are never observed. For example, nobody has ever seen a book lying on a table jumping to a height by itself. But such a thing

would be possible if the principle of conservation of energy were the only restriction. The table could cool spontaneously, converting some of its internal energy into an equal amount of mechanical energy of the book, which would then hop to a height with potential energy equal to the mechanical energy it acquired. But this never happens. Clearly, some additional basic principle of nature forbids the above, even though it satisfies the energy conservation principle. This principle, which disallows many phenomena consistent with the First Law of Thermodynamics is known as the Second Law of Thermodynamics.

The Second Law of Thermodynamics gives a fundamental limitation to the efficiency of a heat engine and the co-efficient of performance of a refrigerator. In simple terms, it says that efficiency of a heat engine can never be unity. According to Eq. (12.20), this implies that heat released to the cold reservoir can never be made zero. For a refrigerator, the Second Law says that the co-efficient of performance can never be infinite. According to Eq. (12.21), this implies that external work (W) can never be zero. The following two statements, one due to Kelvin and Planck denying the possibility of a perfect heat engine, and another due to Clausius denying the possibility of a perfect refrigerator or heat pump, are a concise summary of these observations.

Second Law of Thermodynamics

Kelvin-Planck statement

No process is possible whose sole result is the absorption of heat from a reservoir and the complete conversion of the heat into work.

Clausius statement

No process is possible whose sole result is the transfer of heat from a colder object to a hotter object.

It can be proved that the two statements above are completely equivalent.

12.12 REVERSIBLE AND IRREVERSIBLE PROCESSES

Imagine some process in which a thermodynamic system goes from an initial state i to a final state f. During the process the system absorbs heat Q from the surroundings and performs work W on it. Can we reverse this process and bring both the system and surroundings to their initial states with no other effect anywhere ? Experience suggests that for most processes in nature this is not possible. The spontaneous processes of nature are irreversible. Several examples can be cited. The base of a vessel on an oven is hotter than its other parts. When the vessel is removed, heat is transferred from the base to the other parts, bringing the vessel to a uniform temperature (which in due course cools to the temperature of the surroundings). The process cannot be reversed; a part of the vessel will not get cooler spontaneously and warm up the base. It will violate the Second Law of Thermodynamics, if it did. The free expansion of a gas is irreversible. The combustion reaction of a mixture of petrol and air ignited by a spark cannot be reversed. Cooking gas leaking from a gas cylinder in the kitchen diffuses to the entire room. The diffusion process will not spontaneously reverse and bring the gas back to the cylinder. The stirring of a liquid in thermal contact with a reservoir will convert the work done into heat, increasing the internal energy of the reservoir. The process cannot be reversed exactly; otherwise it would amount to conversion of heat entirely into work, violating the Second Law of Thermodynamics. Irreversibility is a rule rather an exception in nature.

Irreversibility arises mainly from two causes: one, many processes (like a free expansion, or an explosive chemical reaction) take the system to non-equilibrium states; two, most processes involve friction, viscosity and other dissipative effects (e.g., a moving body coming to a stop and losing its mechanical energy as heat to the floor and the body; a rotating blade in a liquid coming to a stop due to viscosity and losing its mechanical energy with corresponding gain in the internal energy of the liquid). Since dissipative effects are present everywhere and can be minimised but not fully eliminated, most processes that we deal with are irreversible.

A thermodynamic process (state $i \rightarrow$ state f) is reversible if the process can be turned back such that both the system and the surroundings return to their original states, with no other change anywhere else in the universe. From the preceding discussion, a reversible process is an idealised notion. A process is reversible only if it is quasi-static (system in equilibrium with the

surroundings at every stage) and there are no dissipative effects. For example, a quasi-static isothermal expansion of an ideal gas in a cylinder fitted with a frictionless movable piston is a reversible process.

Why is reversibility such a basic concept in thermodynamics ? As we have seen, one of the concerns of thermodynamics is the efficiency with which heat can be converted into work. The Second Law of Thermodynamics rules out the possibility of a perfect heat engine with 100% efficiency. But what is the highest efficiency possible for a heat engine working between two reservoirs at temperatures T_1 and T_2? It turns out that a heat engine based on idealised reversible processes achieves the highest efficiency possible. All other engines involving irreversibility in any way (as would be the case for practical engines) have lower than this limiting efficiency.

12.13 CARNOT ENGINE

Suppose we have a hot reservoir at temperature T_1 and a cold reservoir at temperature T_2. What is the maximum efficiency possible for a heat engine operating between the two reservoirs and what cycle of processes should be adopted to achieve the maximum efficiency ? Sadi Carnot, a French engineer, first considered this question in 1824. Interestingly, Carnot arrived at the correct answer, even though the basic concepts of heat and thermodynamics had yet to be firmly established.

We expect the ideal engine operating between two temperatures to be a reversible engine. Irreversibility is associated with dissipative effects, as remarked in the preceding section, and lowers efficiency. A process is reversible if it is quasi-static and non-dissipative. We have seen that a process is not quasi-static if it involves finite temperature difference between the system and the reservoir. This implies that in a reversible heat engine operating between two temperatures, heat should be absorbed (from the hot reservoir) isothermally and released (to the cold reservoir) isothermally. We thus have identified two steps of the reversible heat engine : isothermal process at temperature T_1 absorbing heat Q_1 from the hot reservoir, and another isothermal process at temperature T_2 releasing heat Q_2 to the cold reservoir. To complete a cycle, we need to take the system from temperature T_1 to T_2 and then back from temperature T_2 to T_1. Which processes should we employ for this purpose that are reversible? A little reflection shows that we can only adopt reversible adiabatic processes for these purposes, which involve no heat flow from any reservoir. If we employ any other process that is not adiabatic, say an isochoric process, to take the system from one temperature to another, we shall need a series of reservoirs in the temperature range T_2 to T_1 to ensure that at each stage the process is quasi-static. (Remember again that for a process to be quasi-static and reversible, there should be no finite temperature difference between the system and the reservoir.) But we are considering a reversible engine that operates between only two temperatures. Thus adiabatic processes must bring about the temperature change in the system from T_1 to T_2 and T_2 to T_1 in this engine.

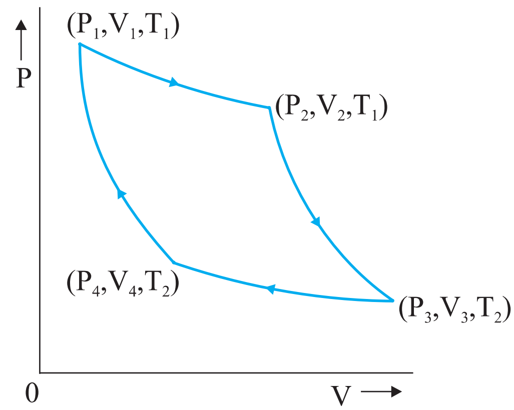


Fig. 12.11 *Carnot cycle for a heat engine with an ideal gas as the working substance.*

A reversible heat engine operating between two temperatures is called a Carnot engine. We have just argued that such an engine must have the following sequence of steps constituting one cycle, called the Carnot cycle, shown in Fig. 12.11. We have taken the working substance of the Carnot engine to be an ideal gas.

(a) *Step* 1 →2 Isothermal expansion of the gas taking its state from (P_1, V_1, T_1) to (P_2, V_2, T_1).

The heat absorbed by the gas (Q_1) from the reservoir at temperature T_1 is given by

Eq. (12.12). This is also the work done ($W_{1\rightarrow 2}$) by the gas on the environment.

$$W_{1\rightarrow 2} = Q_1 = \mu R T_1 \ln\left(\frac{V_2}{V_1}\right) \quad (12.23)$$

(b) *Step* 2 $\rightarrow$3 Adiabatic expansion of the gas from (P_2, V_2, T_1) to (P_3, V_3, T_2) Work done by the gas, using Eq. (12.16), is

$$W_{2 \quad 3} \quad \frac{R \quad T_1 \quad T_2}{1} \quad (12.24)$$

(c) *Step* 3 $\rightarrow$4 Isothermal compression of the gas from (P_3, V_3, T_2) to (P_4, V_4, T_2).

Heat released (Q_2) by the gas to the reservoir at temperature T_2 is given by Eq. (12.12). This is also the work done ($W_{3\rightarrow 4}$) on the gas by the environment.

$$W_{3 \quad 4} \quad Q_2 \quad RT_2 \ln \frac{V_3}{V_4} \quad (12.25)$$

(d) *Step* 4 $\rightarrow$1 Adiabatic compression of the gas from (P_4, V_4, T_2) to (P_1,V_1, T_1).

Work done on the gas, [using Eq.(12.16)], is

$$W_{4 \quad 1} \quad R \frac{T_1 \quad T_2}{-1} \quad (12.26)$$

From Eqs. (12.23) to (12.26) total work done by the gas in one complete cycle is

$$W = W_{1\rightarrow 2} + W_{2\rightarrow 3} - W_{3\rightarrow 4} - W_{4\rightarrow 1}$$

$$= \mu RT_1 \ln\left(\frac{V_2}{V_1}\right) - \mu RT_2 \ln\left(\frac{V_3}{V_4}\right) \quad (12.27)$$

The efficiency η of the Carnot engine is

$$\frac{W}{Q_1} \quad 1 \quad \frac{Q_2}{Q_1}$$

$$1 \quad \frac{T_2}{T_1} \quad \frac{\ln \frac{V_3}{V_4}}{\ln \frac{V_2}{V_1}} \quad (12.28)$$

Now since step 2 $\rightarrow$3 is an adiabatic process,

$$T_1 V_2^{\quad 1} \quad T_2 V_3^{\quad 1}$$

i.e. $$\frac{V_2}{V_3} = \frac{T_2}{T_1}^{1/(\quad 1)} \quad (12.29)$$

Similarly, since step 4 $\rightarrow$1 is an adiabatic process

$$T_2 V_4^{\quad 1} \quad T_1 V_1^{\quad 1}$$

i.e. $$\frac{V_1}{V_4} = \frac{T_2}{T_1}^{1/ \quad 1} \quad (12.30)$$

From Eqs. (12.29) and (12.30),

$$\frac{V_3}{V_4} = \frac{V_2}{V_1} \quad (12.31)$$

Using Eq. (12.31) in Eq. (12.28), we get

$$1 - \frac{T_2}{T_1} \quad \text{(Carnot engine)} \quad (12.32)$$

We have already seen that a Carnot engine is a reversible engine. Indeed it is the only reversible engine possible that works between two reservoirs at different temperatures. Each step of the Carnot cycle given in Fig. 12.11 can be reversed. This will amount to taking heat Q_2 from the cold reservoir at T_2, doing work W on the system, and transferring heat Q_1 to the hot reservoir. This will be a reversible refrigerator.

We next establish the important result (sometimes called Carnot's theorem) that (a) working between two given temperatures T_1 and T_2 of the hot and cold reservoirs respectively, no engine can have efficiency more than that of the Carnot engine and (b) the efficiency of the Carnot engine is independent of the nature of the working substance.

To prove the result (a), imagine a reversible (Carnot) engine R and an irreversible engine I working between the same source (hot reservoir) and sink (cold reservoir). Let us couple the engines, I and R, in such a way so that I acts like a heat engine and R acts as a refrigerator. Let I absorb heat Q_1 from the source, deliver work W' and release the heat Q_1- W' to the sink. We arrange so that R returns the same heat Q_1 to the source, taking heat Q_2 from the sink and requiring work $W = Q_1 - Q_2$ to be done on it.

Now suppose $\eta_R < \eta_I$ i.e. if R were to act as an engine it would give less work output than that of I i.e. $W < W'$ for a given Q_1. With R acting like a refrigerator, this would mean $Q_2 = Q_1 - W > Q_1 - W'$. Thus on the whole, the coupled I-R system extracts heat $(Q_1 - W) - (Q_1 - W') = (W' - W)$ from the cold reservoir and delivers the same amount of work in one cycle, without any change in the source or anywhere else. This is clearly against the Kelvin-Planck statement of the Second Law of Thermodynamics. Hence the assertion $\eta_I > \eta_R$ is wrong. No engine can have efficiency greater

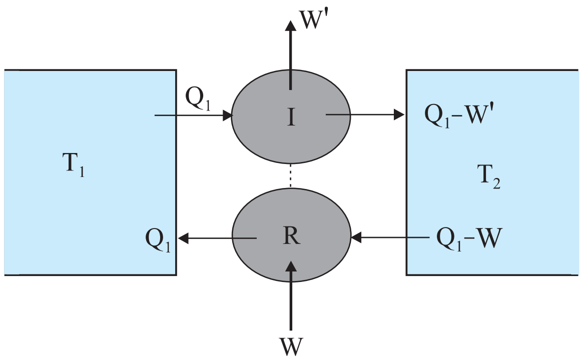


Fig. 12.12 *An irreversible engine (I) coupled to a reversible refrigerator (R). If W′ > W, this would amount to extraction of heat W′ – W from the sink and its full conversion to work, in contradiction with the Second Law of Thermodynamics.*

than that of the Carnot engine. A similar argument can be constructed to show that a reversible engine with one particular substance cannot be more efficient than the one using another substance. The maximum efficiency of a Carnot engine given by Eq. (12.32) is independent of the nature of the system performing the Carnot cycle of operations. Thus we are justified in using an ideal gas as a system in the calculation of efficiency η of a Carnot engine. The ideal gas has a simple equation of state, which allows us to readily calculate η but the final result for η [Eq. (12.32)], is true for any Carnot engine.

This final remark shows that in a Carnot cycle,

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \qquad (12.33)$$

is a universal relation independent of the nature of the system. Here Q_1 and Q_2 are respectively, the heat absorbed and released isothermally (from the hot and to the cold reservoirs) in a Carnot engine. Equation (12.33), can, therefore, be used as a relation to define a truly universal thermodynamic temperature scale that is independent of any particular properties of the system used in the Carnot cycle. Of course, for an ideal gas as a working substance, this universal temperature is the same as the ideal gas temperature introduced in section 12.11.

SUMMARY

1. The zeroth law of thermodynamics states that '*two systems in thermal equilibrium with a third system are in thermal equilibrium with each other*'. The Zeroth Law leads to the concept of temperature.
2. Internal energy of a system is the sum of kinetic energies and potential energies of the molecular constituents of the system. It does not include the over-all kinetic energy of the system. Heat and work are two modes of energy transfer to the system. Heat is the energy transfer arising due to temperature difference between the system and the surroundings. Work is energy transfer brought about by other means, such as moving the piston of a cylinder containing the gas, by raising or lowering some weight connected to it.
3. The first law of thermodynamics is the general law of conservation of energy applied to any system in which energy transfer from or to the surroundings (through heat and work) is taken into account. It states that

$$\Delta Q = \Delta U + \Delta W$$

where ΔQ is the heat supplied to the system, ΔW is the work done by the system and ΔU is the change in internal energy of the system.

4. The specific heat capacity of a substance is defined by

$$s = \frac{1}{m}\frac{\Delta Q}{\Delta T}$$

where m is the mass of the substance and ΔQ is the heat required to change its temperature by ΔT. The molar specific heat capacity of a substance is defined by

$$C \quad \frac{1}{\;} \frac{Q}{T}$$

where μ is the number of moles of the substance. For a solid, the law of equipartition of energy gives

$$C = 3R$$

which generally agrees with experiment at ordinary temperatures.

Calorie is the old unit of heat. 1 calorie is the amount of heat required to raise the temperature of 1 g of water from 14.5 °C to 15.5 °C. 1 cal = 4.186 J.

5. For an ideal gas, the molar specific heat capacities at constant pressure and volume satisfy the relation

$$C_p - C_v = R$$

where R is the universal gas constant.

6. Equilibrium states of a thermodynamic system are described by state variables. The value of a state variable depends only on the particular state, not on the path used to arrive at that state. Examples of state variables are pressure (P), volume (V), temperature (T), and mass (m). Heat and work are not state variables. An Equation of State (like the ideal gas equation $PV = \mu RT$) is a relation connecting different state variables.

7. A quasi-static process is an infinitely slow process such that the system remains in thermal and mechanical equilibrium with the surroundings throughout. In a quasi-static process, the pressure and temperature of the environment can differ from those of the system only infinitesimally.

8. In an isothermal expansion of an ideal gas from volume V_1 to V_2 at temperature T the heat absorbed (Q) equals the work done (W) by the gas, each given by

$$Q = W = \mu R T \ln\left(\frac{V_2}{V_1}\right)$$

9. In an adiabatic process of an ideal gas

$$PV^{\gamma} = \text{constant}$$

where $$\frac{C_p}{C_v}$$

Work done by an ideal gas in an adiabatic change of state from (P_1, V_1, T_1) to (P_2, V_2, T_2) is

$$W \quad \frac{R \; T_1 \quad T_2}{-1}$$

10. Heat engine is a device in which a system undergoes a cyclic process resulting in conversion of heat into work. If Q_1 is the heat absorbed from the source, Q_2 is the heat released to the sink, and the work output in one cycle is W, the efficiency η of the engine is:

$$\frac{W}{Q_1} \quad 1 \quad \frac{Q_2}{Q_1}$$

11. In a refrigerator or a heat pump, the system extracts heat Q_2 from the cold reservoir and releases Q_1 amount of heat to the hot reservoir, with work W done on the system. The co-efficient of performance of a refrigerator is given by

$$\alpha = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2}$$

12. The second law of thermodynamics disallows some processes consistent with the First Law of Thermodynamics. It states

    Kelvin-Planck statement

    No process is possible whose sole result is the absorption of heat from a reservoir and complete conversion of the heat into work.

    Clausius statement

    No process is possible whose sole result is the transfer of heat from a colder object to a hotter object.

    Put simply, the Second Law implies that no heat engine can have efficiency η equal to 1 or no refrigerator can have co-efficient of performance α equal to infinity.

13. A process is reversible if it can be reversed such that both the system and the surroundings return to their original states, with no other change anywhere else in the universe. Spontaneous processes of nature are irreversible. The idealised reversible process is a quasi-static process with no dissipative factors such as friction, viscosity, etc.

14. Carnot engine is a reversible engine operating between two temperatures T_1 (source) and T_2 (sink). The Carnot cycle consists of two isothermal processes connected by two adiabatic processes. The efficiency of a Carnot engine is given by

$$\eta = 1 - \frac{T_2}{T_1} \qquad \text{(Carnot engine)}$$

    No engine operating between two temperatures can have efficiency greater than that of the Carnot engine.

15. If $Q > 0$, heat is added to the system

    If $Q < 0$, heat is removed to the system

    If $W > 0$, Work is done by the system

    If $W < 0$, Work is done on the system

Quantity	Symbol	Dimensions	Unit	Remark
Co-efficienty of volume expansion	α_v	$[K^{-1}]$	K^{-1}	$\alpha_v = 3\,\alpha_l$
Heat supplied to a system	ΔQ	$[ML^2 T^{-2}]$	J	Q is not a state variable
Specific heat	s	$[L^2 T^{-2} K^{-1}]$	$J\ kg^{-1}\ K^{-1}$	
Thermal Conductivity	K	$[MLT^{-3} K^{-1}]$	$J\ s^{-1}\ K^{-1}$	$H = -KA\dfrac{dt}{dx}$

POINTS TO PONDER

1. Temperature of a body is related to its average internal energy, not to the kinetic energy of motion of its centre of mass. A bullet fired from a gun is not at a higher temperature because of its high speed.
2. Equilibrium in thermodynamics refers to the situation when macroscopic variables describing the thermodynamic state of a system do not depend on time. Equilibrium of a system in mechanics means the net external force and torque on the system are zero.
3. In a state of thermodynamic equilibrium, the microscopic constituents of a system are not in equilibrium (in the sense of mechanics).
4. Heat capacity, in general, depends on the process the system goes through when heat is supplied.
5. In isothermal quasi-static processes, heat is absorbed or given out by the system even though at every stage the gas has the same temperature as that of the surrounding reservoir. This is possible because of the infinitesimal difference in temperature between the system and the reservoir.

EXERCISES

12.1 A geyser heats water flowing at the rate of 3.0 litres per minute from 27 °C to 77 °C. If the geyser operates on a gas burner, what is the rate of consumption of the fuel if its heat of combustion is 4.0×10^4 J/g ?

12.2 What amount of heat must be supplied to 2.0×10^{-2} kg of nitrogen (at room temperature) to raise its temperature by 45 °C at constant pressure ? (Molecular mass of N_2 = 28; R = 8.3 J mol^{-1} K^{-1}.)

12.3 Explain why

(a) Two bodies at different temperatures T_1 and T_2 if brought in thermal contact do not necessarily settle to the mean temperature $(T_1 + T_2)/2$.

(b) The coolant in a chemical or a nuclear plant (i.e., the liquid used to prevent the different parts of a plant from getting too hot) should have high specific heat.

(c) Air pressure in a car tyre increases during driving.

(d) The climate of a harbour town is more temperate than that of a town in a desert at the same latitude.

12.4 A cylinder with a movable piston contains 3 moles of hydrogen at standard temperature and pressure. The walls of the cylinder are made of a heat insulator, and the piston is insulated by having a pile of sand on it. By what factor does the pressure of the gas increase if the gas is compressed to half its original volume ?

12.5 In changing the state of a gas adiabatically from an equilibrium state A to another equilibrium state B, an amount of work equal to 22.3 J is done on the system. If the gas is taken from state A to B via a process in which the net heat absorbed by the system is 9.35 cal, how much is the net work done by the system in the latter case ? (Take 1 cal = 4.19 J)

12.6 Two cylinders A and B of equal capacity are connected to each other via a stopcock. A contains a gas at standard temperature and pressure. B is completely evacuated. The entire system is thermally insulated. The stopcock is suddenly opened. Answer the following :

(a) What is the final pressure of the gas in A and B ?

(b) What is the change in internal energy of the gas ?

(c) What is the change in the temperature of the gas ?

(d) Do the intermediate states of the system (before settling to the final equilibrium state) lie on its P-V-T surface ?

12.7 A steam engine delivers 5.4×10^{8}J of work per minute and services 3.6×10^{9}J of heat per minute from its boiler. What is the efficiency of the engine? How much heat is wasted per minute?

12.8 An electric heater supplies heat to a system at a rate of 100W. If system performs work at a rate of 75 joules per second. At what rate is the internal energy increasing?

12.9 A thermodynamic system is taken from an original state to an intermediate state by the linear process shown in Fig. (12.13)

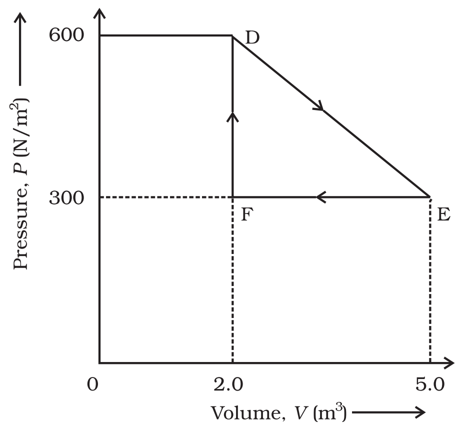


Fig. 12.13

Its volume is then reduced to the original value from E to F by an isobaric process. Calculate the total work done by the gas from D to E to F

12.10 A refrigerator is to maintain eatables kept inside at 9^{0}C. If room temperature is 36^{0}C, calculate the coefficient of performance.

CHAPTER THIRTEEN

KINETIC THEORY

13.1 INTRODUCTION

Boyle discovered the law named after him in 1661. Boyle, Newton and several others tried to explain the behaviour of gases by considering that gases are made up of tiny atomic particles. The actual atomic theory got established more than 150 years later. Kinetic theory explains the behaviour of gases based on the idea that the gas consists of rapidly moving atoms or molecules. This is possible as the inter-atomic forces, which are short range forces that are important for solids and liquids, can be neglected for gases. The kinetic theory was developed in the nineteenth century by Maxwell, Boltzmann and others. It has been remarkably successful. It gives a molecular interpretation of pressure and temperature of a gas, and is consistent with gas laws and Avogadro's hypothesis. It correctly explains specific heat capacities of many gases. It also relates measurable properties of gases such as viscosity, conduction and diffusion with molecular parameters, yielding estimates of molecular sizes and masses. This chapter gives an introduction to kinetic theory.

13.2 MOLECULAR NATURE OF MATTER

Richard Feynman, one of the great physicists of 20th century considers the discovery that "Matter is made up of atoms" to be a very significant one. Humanity may suffer annihilation (due to nuclear catastrophe) or extinction (due to environmental disasters) if we do not act wisely. If that happens, and all of scientific knowledge were to be destroyed then Feynman would like the 'Atomic Hypothesis' to be communicated to the next generation of creatures in the universe. Atomic Hypothesis: All things are made of atoms - little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another.

Speculation that matter may not be continuous, existed in many places and cultures. Kanada in India and Democritus

Atomic Hypothesis in Ancient India and Greece

Though John Dalton is credited with the introduction of atomic viewpoint in modern science, scholars in ancient India and Greece conjectured long before the existence of atoms and molecules. In the Vaiseshika school of thought in India founded by Kanada (Sixth century B.C.) the atomic picture was developed in considerable detail. Atoms were thought to be eternal, indivisible, infinitesimal and ultimate parts of matter. It was argued that if matter could be subdivided without an end, there would be no difference between a mustard seed and the Meru mountain. The four kinds of atoms (**Paramanu** — Sanskrit word for the smallest particle) postulated were Bhoomi (Earth), Ap (water), Tejas (fire) and Vayu (air) that have characteristic mass and other attributes, were propounded. Akasa (space) was thought to have no atomic structure and was continuous and inert. Atoms combine to form different molecules (e.g. two atoms combine to form a diatomic molecule dvyanuka, three atoms form a tryanuka or a triatomic molecule), their properties depending upon the nature and ratio of the constituent atoms. The size of the atoms was also estimated, by conjecture or by methods that are not known to us. The estimates vary. In Lalitavistara, a famous biography of the Buddha written mainly in the second century B.C., the estimate is close to the modern estimate of atomic size, of the order of 10^{-10} m.

In ancient Greece, Democritus (Fourth century B.C.) is best known for his atomic hypothesis. The word 'atom' means 'indivisible' in Greek. According to him, atoms differ from each other physically, in shape, size and other properties and this resulted in the different properties of the substances formed by their combination. The atoms of water were smooth and round and unable to 'hook' on to each other, which is why liquid /water flows easily. The atoms of earth were rough and jagged, so they held together to form hard substances. The atoms of fire were thorny which is why it caused painful burns. These fascinating ideas, despite their ingenuity, could not evolve much further, perhaps because they were intuitive conjectures and speculations not tested and modified by quantitative experiments - the hallmark of modern science.

in Greece had suggested that matter may consist of indivisible constituents. The scientific 'Atomic Theory' is usually credited to John Dalton. He proposed the atomic theory to explain the laws of definite and multiple proportions obeyed by elements when they combine into compounds. The first law says that any given compound has, a fixed proportion by mass of its constituents. The second law says that when two elements form more than one compound, for a fixed mass of one element, the masses of the other elements are in ratio of small integers.

To explain the laws Dalton suggested, about 200 years ago, that the smallest constituents of an element are atoms. Atoms of one element are identical but differ from those of other elements. A small number of atoms of each element combine to form a molecule of the compound. Gay Lussac's law, also given in early 19^{th} century, states: When gases combine chemically to yield another gas, their volumes are in the ratios of small integers. Avogadro's law (or hypothesis) says: Equal volumes of all gases at equal temperature and pressure have the same number of molecules. Avogadro's law, when combined with Dalton's theory explains Gay Lussac's law. Since the elements are often in the form of molecules, Dalton's atomic theory can also be referred to as the molecular theory of matter. The theory is now well accepted by scientists. However even at the end of the nineteenth century there were famous scientists who did not believe in atomic theory !

From many observations, in recent times we now know that molecules (made up of one or more atoms) constitute matter. Electron microscopes and scanning tunnelling microscopes enable us to even see them. The size of an atom is about an angstrom (10^{-10} m). In solids, which are tightly packed, atoms are spaced about a few angstroms (2 Å) apart. In liquids the separation between atoms is also about the same. In liquids the atoms are not as rigidly fixed as in solids, and can move around. This enables a liquid to flow. In gases the interatomic distances are in tens of angstroms. The average distance a molecule can travel without colliding is called the **mean free path**. The mean free path, in gases, is of the order of thousands of angstroms. The atoms are much freer in gases and can travel long distances without colliding. If they are not enclosed, gases disperse away. In solids and liquids the closeness makes the interatomic force important. The force has a long range attraction and a short range repulsion. The atoms attract when they are at a few angstroms but repel when they come closer. The static appearance of a gas

is misleading. The gas is full of activity and the equilibrium is a dynamic one. In dynamic equilibrium, molecules collide and change their speeds during the collision. Only the average properties are constant.

Atomic theory is not the end of our quest, but the beginning. We now know that atoms are not indivisible or elementary. They consist of a nucleus and electrons. The nucleus itself is made up of protons and neutrons. The protons and neutrons are again made up of quarks. Even quarks may not be the end of the story. There may be string like elementary entities. Nature always has surprises for us, but the search for truth is often enjoyable and the discoveries beautiful. In this chapter, we shall limit ourselves to understanding the behaviour of gases (and a little bit of solids), as a collection of moving molecules in incessant motion.

13.3 BEHAVIOUR OF GASES

Properties of gases are easier to understand than those of solids and liquids. This is mainly because in a gas, molecules are far from each other and their mutual interactions are negligible except when two molecules collide. Gases at low pressures and high temperatures much above that at which they liquefy (or solidify) approximately satisfy a simple relation between their pressure, temperature and volume given by (see Ch. 11)

$$PV = KT \qquad (13.1)$$

for a given sample of the gas. Here T is the temperature in kelvin or (absolute) scale. K is a constant for the given sample but varies with the volume of the gas. If we now bring in the idea of atoms or molecules then K is proportional to the number of molecules, (say) N in the sample. We can write $K = N\,k$. Observation tells us that this k is same for all gases. It is called Boltzmann constant and is denoted by k_B.

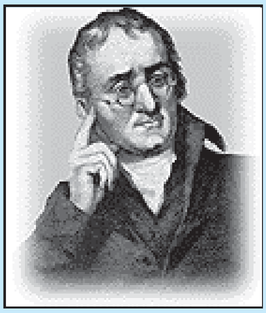
$$\text{As } \frac{P_1V_1}{N_1T_1} = \frac{P_2V_2}{N_2T_2} = \text{constant} = k_B \qquad (13.2)$$

if P, V and T are same, then N is also same for all gases. This is Avogadro's hypothesis, that the number of molecules per unit volume is same for all gases at a fixed temperature and pressure. The number in 22.4 litres of any gas is 6.02×10^{23}. This is known as Avogadro number and is denoted by N_A. The mass of 22.4 litres of any gas is equal to its molecular weight in grams at S.T.P (standard temperature 273 K and pressure 1 atm). This amount of substance is called a mole (see Chapter 2 for a more precise definition). Avogadro had guessed the equality of numbers in equal volumes of gas at a fixed temperature and pressure from chemical reactions. Kinetic theory justifies this hypothesis.

The perfect gas equation can be written as

$$PV = \mu\, RT \qquad (13.3)$$

where μ is the number of moles and $R = N_A k_B$ is a universal constant. The temperature T is absolute temperature. Choosing kelvin scale for



John Dalton (1766- 1844)

He was an English chemist. When different types of atoms combine, they obey certain simple laws. Dalton's atomic theory explains these laws in a simple way. He also gave a theory of colour blindness.

Amedeo Avogadro (1776 – 1856)

He made a brilliant guess that equal volumes of gases have equal number of molecules at the same temperature and pressure. This helped in understanding the combination of different gases in a very simple way. It is now called Avogadro's hypothesis (or law). He also suggested that the smallest constituent of gases like hydrogen, oxygen and nitrogen are not atoms but diatomic molecules.

absolute temperature, R = 8.314 J mol^{-1}K^{-1}. Here

$$\frac{M}{M_0} \quad \frac{N}{N_A} \qquad (13.4)$$

where M is the mass of the gas containing N molecules, M_0 is the molar mass and N_A the Avogadro's number. Using Eqs. (13.4) and (13.3) can also be written as

$PV = k_B\, NT$ or $P = k_B\, nT$

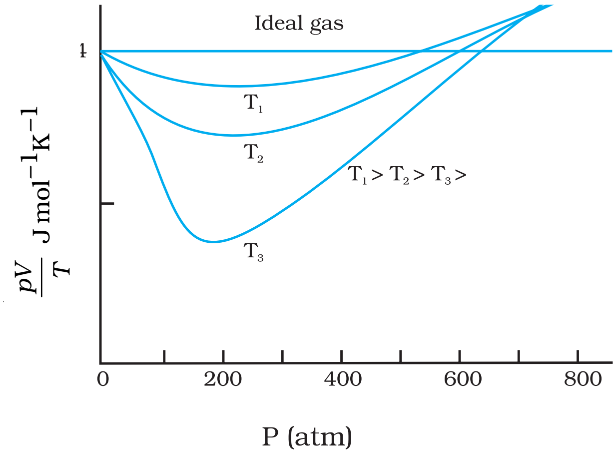


Fig.13.1 *Real gases approach ideal gas behaviour at low pressures and high temperatures.*

where n is the number density, i.e. number of molecules per unit volume. k_B is the Boltzmann constant introduced above. Its value in SI units is 1.38 × 10^{-23} J K^{-1}.

Another useful form of Eq. (13.3) is

$$P \quad \frac{RT}{M_0} \qquad (13.5)$$

where ρ is the mass density of the gas.

A gas that satisfies Eq. (13.3) exactly at all pressures and temperatures is defined to be an **ideal gas**. An ideal gas is a simple theoretical model of a gas. No real gas is truly ideal. Fig. 13.1 shows departures from ideal gas behaviour for a real gas at three different temperatures. Notice that all curves approach the ideal gas behaviour for low pressures and high temperatures.

At low pressures or high temperatures the molecules are far apart and molecular interactions are negligible. Without interactions the gas behaves like an ideal one.

If we fix μ and T in Eq. (13.3), we get

$$PV = \text{constant} \qquad (13.6)$$

i.e., keeping temperature constant, pressure of a given mass of gas varies inversely with volume. This is the famous **Boyle's law**. Fig. 13.2 shows comparison between experimental P-V curves and the theoretical curves predicted by Boyle's law. Once again you see that the agreement is good at high temperatures and low pressures. Next, if you fix P, Eq. (13.1) shows that $V \propto T$ i.e., for a fixed pressure, the volume of a gas is proportional to its absolute temperature T **(Charles' law)**. See Fig. 13.3.

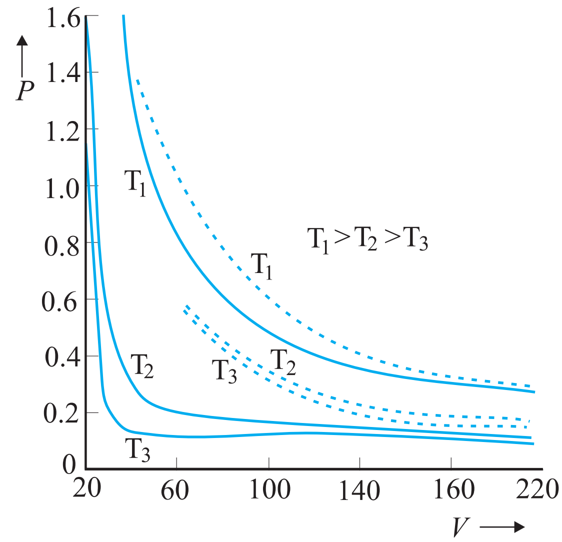


Fig.13.2 *Experimental P-V curves (solid lines) for steam at three temperatures compared with Boyle's law (dotted lines). P is in units of 22 atm and V in units of 0.09 litres.*

Finally, consider a mixture of non-interacting ideal gases: μ_1 moles of gas 1, μ_2 moles of gas 2, etc. in a vessel of volume V at temperature T and pressure P. It is then found that the equation of state of the mixture is :

$$PV = (\mu_1 + \mu_2 + \ldots)\, RT \qquad (13.7)$$

$$\text{i.e. } P \quad {}_1\frac{RT}{V} \quad {}_2\frac{RT}{V} \quad \ldots \qquad (13.8)$$

$$= P_1 + P_2 + \ldots \qquad (13.9)$$

Clearly $P_1 = \mu_1 RT/V$ is the pressure gas 1 would exert at the same conditions of volume and temperature if no other gases were present. This is called the partial pressure of the gas. Thus, the total pressure of a mixture of ideal gases is the sum of partial pressures. This is Dalton's law of partial pressures.

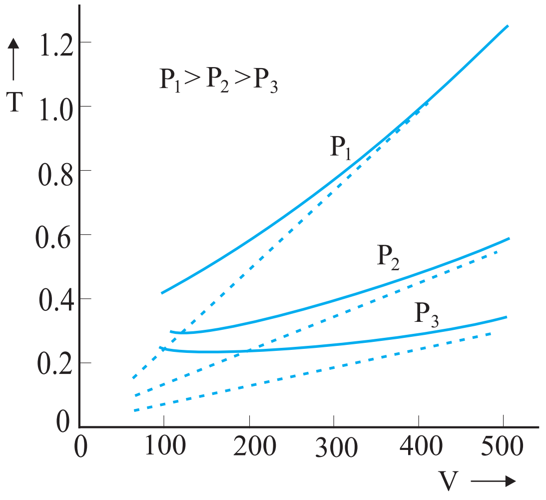


Fig. 13.3 *Experimental T-V curves (solid lines) for CO_2 at three pressures compared with Charles' law (dotted lines). T is in units of 300 K and V in units of 0.13 litres.*

We next consider some examples which give us information about the volume occupied by the molecules and the volume of a single molecule.

> ***Example 13.1*** The density of water is 1000 kg m^{-3}. The density of water vapour at 100 °C and 1 atm pressure is 0.6 kg m^{-3}. The volume of a molecule multiplied by the total number gives ,what is called, molecular volume. Estimate the ratio (or fraction) of the molecular volume to the total volume occupied by the water vapour under the above conditions of temperature and pressure.

Answer For a given mass of water molecules, the density is less if volume is large. So the volume of the vapour is 1000/0.6 = /(6×10^{-4}) times larger. If densities of bulk water and water molecules are same, then the fraction of molecular volume to the total volume in liquid state is 1. As volume in vapour state has increased, the fractional volume is less by the same amount, i.e. 6×10^{-4}. ◀

> **Example 13.2** Estimate the volume of a water molecule using the data in Example 13.1.

Answer In the liquid (or solid) phase, the molecules of water are quite closely packed. The density of water molecule may therefore, be regarded as roughly equal to the density of bulk water = 1000 kg m^{-3}. To estimate the volume of a water molecule, we need to know the mass of a single water molecule. We know that 1 mole of water has a mass approximately equal to

(2 + 16)g = 18 g = 0.018 kg.

Since 1 mole contains about 6×10^{23} molecules (Avogadro's number), the mass of a molecule of water is $(0.018)/(6 \times 10^{23})$ kg = 3×10^{-26} kg. Therefore, a rough estimate of the volume of a water molecule is as follows :

Volume of a water molecule

= (3×10^{-26} kg)/ (1000 kg m^{-3})

= 3×10^{-29} m^3

= (4/3) π (Radius)^3

Hence, Radius ≈ 2×10^{-10} m = 2 Å ◀

> **Example 13.3** What is the average distance between atoms (interatomic distance) in water? Use the data given in Examples 13.1 and 13.2.

Answer : A given mass of water in vapour state has 1.67×10^3 times the volume of the same mass of water in liquid state (Ex. 13.1). This is also the increase in the amount of volume available for each molecule of water. When volume increases by 10^3 times the radius increases by $V^{1/3}$ or 10 times, i.e., 10 × 2 Å = 20 Å. So the average distance is 2 × 20 = 40 Å. ◀

> **Example 13.4** A vessel contains two non-reactive gases : neon (monatomic) and oxygen (diatomic). The ratio of their partial pressures is 3:2. Estimate the ratio of (i) number of molecules and (ii) mass density of neon and oxygen in the vessel. Atomic mass of Ne = 20.2 u, molecular mass of O_2 = 32.0 u.

Answer Partial pressure of a gas in a mixture is the pressure it would have for the same volume and temperature if it alone occupied the vessel. (The total pressure of a mixture of non-reactive gases is the sum of partial pressures due to its constituent gases.) Each gas (assumed ideal) obeys the gas law. Since V and T are common to the two gases, we have $P_1 V = \mu_1 RT$ and $P_2 V = \mu_2 RT$, i.e. $(P_1/P_2) = (\mu_1 / \mu_2)$. Here 1 and 2 refer to neon and oxygen respectively. Since (P_1/P_2) = (3/2) (given), (μ_1/ μ_2) = 3/2.

(i) By definition $\mu_1 = (N_1/N_A)$ and $\mu_2 = (N_2/N_A)$ where N_1 and N_2 are the number of molecules of 1 and 2, and N_A is the Avogadro's number. Therefore, $(N_1/N_2) = (\mu_1 / \mu_2) = 3/2$.

(ii) We can also write $\mu_1 = (m_1/M_1)$ and $\mu_2 = (m_2/M_2)$ where m_1 and m_2 are the masses of 1 and 2; and M_1 and M_2 are their molecular masses. (Both m_1 and M_1; as well as m_2 and M_2 should be expressed in the same units). If ρ_1 and ρ_2 are the mass densities of 1 and 2 respectively, we have

$$\frac{1}{2} \quad \frac{m_1/V}{m_2/V} \quad \frac{m_1}{m_2} \quad \frac{1}{2} \quad \frac{M_1}{M_2}$$

$$\frac{3}{2} \quad \frac{20.2}{32.0} \quad 0.947$$

◀

13.4 KINETIC THEORY OF AN IDEAL GAS

Kinetic theory of gases is based on the molecular picture of matter. A given amount of gas is a collection of a large number of molecules (typically of the order of Avogadro's number) that are in incessant random motion. At ordinary pressure and temperature, the average distance between molecules is a factor of 10 or more than the typical size of a molecule (2 Å). Thus the interaction between the molecules is negligible and we can assume that they move freely in straight lines according to Newton's first law. However, occasionally, they come close to each other, experience intermolecular forces and their velocities change. These interactions are called collisions. The molecules collide incessantly against each other or with the walls and change their velocities. The collisions are considered to be elastic. We can derive an expression for the pressure of a gas based on the kinetic theory.

We begin with the idea that molecules of a gas are in incessant random motion, colliding against one another and with the walls of the container. All collisions between molecules among themselves or between molecules and the walls are elastic. This implies that total kinetic energy is conserved. The total momentum is conserved as usual.

13.4.1 Pressure of an Ideal Gas

Consider a gas enclosed in a cube of side l. Take the axes to be parallel to the sides of the cube, as shown in Fig. 13.4. A molecule with velocity

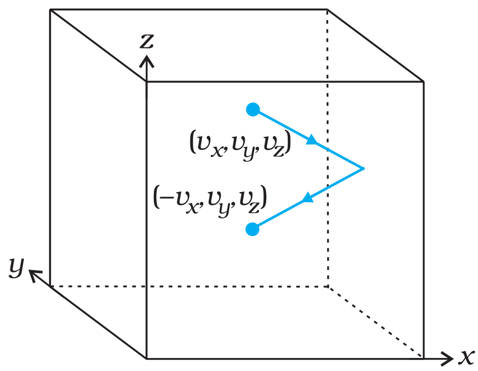


Fig. 13.4 *Elastic collision of a gas molecule with the wall of the container.*

(v_x, v_y, v_z) hits the planar wall parallel to *yz*-plane of area A (= l^2). Since the collision is elastic, the molecule rebounds with the same velocity; its y and z components of velocity do not change in the collision but the x-component reverses sign. That is, the velocity after collision is $(-v_x, v_y, v_z)$. The change in momentum of the molecule is : $-mv_x - (mv_x) = -2mv_x$. By the principle of conservation of momentum, the momentum imparted to the wall in the collision $= 2mv_x$.

To calculate the force (and pressure) on the wall, we need to calculate momentum imparted to the wall per unit time. In a small time interval Δt, a molecule with x-component of velocity v_x will hit the wall if it is within the distance $v_x \Delta t$ from the wall. That is, all molecules within the volume $Av_x \Delta t$ only can hit the wall in time Δt. But, on the average, half of these are moving towards the wall and the other half away from the wall. Thus the number of molecules with velocity (v_x, v_y, v_z) hitting the wall in time Δt is $\frac{1}{2}A\, v_x\, \Delta t\, n$ where n is the number of molecules per unit volume. The total momentum transferred to the wall by these molecules in time Δt is :

$$Q = (2mv_x)\,(\tfrac{1}{2}\, n\, A\, v_x\, \Delta t) \qquad (13.10)$$

The force on the wall is the rate of momentum transfer $Q/\Delta t$ and pressure is force per unit area :

$$P = Q\,/(A\,\Delta t) = n\, m\, v_x^2 \qquad (3.11)$$

Actually, all molecules in a gas do not have the same velocity; there is a distribution in velocities. The above equation therefore, stands for pressure due to the group of molecules with speed v_x in the x-direction and n stands for the number density of that group of molecules. The

total pressure is obtained by summing over the contribution due to all groups:

$$P = n\, m\, \overline{v_x^2} \qquad (13.12)$$

where $\overline{v_x^2}$ is the average of v_x^2. Now the gas is isotropic, i.e. there is no preferred direction of velocity of the molecules in the vessel. Therefore, by symmetry,

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$$

$$= (1/3)\,[\,\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}\,] = (1/3)\,\overline{v^2} \qquad (13.13)$$

where v is the speed and $\overline{v^2}$ denotes the mean of the squared speed. Thus

$$P = (1/3)\, n\, m\, \overline{v^2} \qquad (13.14)$$

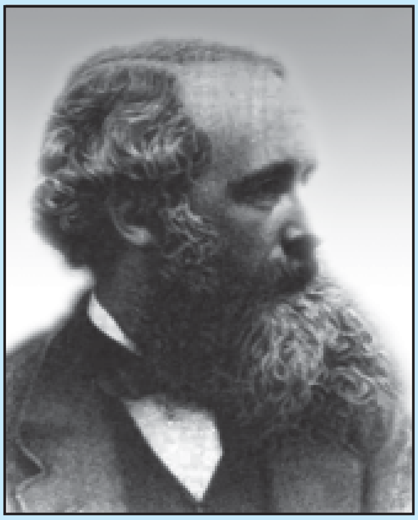
Some remarks on this derivation. First, though we choose the container to be a cube, the shape of the vessel really is immaterial. For a vessel of arbitrary shape, we can always choose a small infinitesimal (planar) area and carry through the steps above. Notice that both A and Δt do not appear in the final result. By Pascal's law, given in Ch. 10, pressure in one portion of the gas in equilibrium is the same as anywhere else. Second, we have ignored any collisions in the derivation. Though this assumption is difficult to justify rigorously, we can qualitatively see that it will not lead to erroneous results. The number of molecules hitting the wall in time Δt was found to be ½ $n A v_x \Delta t$. Now the collisions are random and the gas is in a steady state. Thus, if a molecule with velocity (v_x, v_y, v_z) acquires a different velocity due to collision with some molecule, there will always be some other molecule with a different initial velocity which after a collision acquires the velocity (v_x, v_y, v_z). If this were not so, the distribution of velocities would not remain steady. In any case we are finding $\overline{v_x^2}$. Thus, on the whole, molecular collisions (if they are not too frequent and the time spent in a collision is negligible compared to time between collisions) will not affect the calculation above.

13.4.2 Kinetic Interpretation of Temperature

Equation (13.14) can be written as

$$PV = (1/3)\, nV\, m\, \overline{v^2} \qquad (13.15a)$$

Founders of Kinetic Theory of Gases



James Clerk Maxwell (1831 – 1879), born in Edinburgh, Scotland, was among the greatest physicists of the nineteenth century. He derived the thermal velocity distribution of molecules in a gas and was among the first to obtain reliable estimates of molecular parameters from measurable quantities like viscosity, etc. Maxwell's greatest achievement was the unification of the laws of electricity and magnetism (discovered by Coulomb, Oersted, Ampere and Faraday) into a consistent set of equations now called Maxwell's equations. From these he arrived at the most important conclusion that light is an electromagnetic wave. Interestingly, Maxwell did not agree with the idea (strongly suggested by the Faraday's laws of electrolysis) that electricity was particulate in nature.

Ludwig Boltzmann (1844 – 1906) born in Vienna, Austria, worked on the kinetic theory of gases independently of Maxwell. A firm advocate of atomism, that is basic to kinetic theory, Boltzmann provided a statistical interpretation of the Second Law of thermodynamics and the concept of entropy. He is regarded as one of the founders of classical statistical mechanics. The proportionality constant connecting energy and temperature in kinetic theory is known as Boltzmann's constant in his honour.

$$PV \; = \; (2/3)\, N\, x\, ½\, m\, \overline{v^2} \qquad (13.15b)$$

where N (= nV) is the number of molecules in the sample.

The quantity in the bracket is the average translational kinetic energy of the molecules in the gas. Since the internal energy E of an ideal gas is purely kinetic*,

$$E = N \times (1/2)\; m\; \overline{v^2} \qquad (13.16)$$

Equation (13.15) then gives :

$$PV = (2/3)\; E \qquad (13.17)$$

We are now ready for a kinetic interpretation of temperature. Combining Eq. (13.17) with the ideal gas Eq. (13.3), we get

$$E = (3/2)\;\; k_B\, NT \qquad (13.18)$$

or $E/\,N = ½\;\; m\; \overline{v^2} \;\; = \;\; (3/2)\; k_B T$ (13.19)

i.e., the average kinetic energy of a molecule is proportional to the absolute temperature of the gas; it is independent of pressure, volume or the nature of the ideal gas. This is a fundamental result relating temperature, a macroscopic measurable parameter of a gas (a thermodynamic variable as it is called) to a molecular quantity, namely the average kinetic energy of a molecule. The two domains are connected by the Boltzmann constant. We note in passing that Eq. (13.18) tells us that internal energy of an ideal gas depends only on temperature, not on pressure or volume. With this interpretation of temperature, kinetic theory of an ideal gas is completely consistent with the ideal gas equation and the various gas laws based on it.

For a mixture of non-reactive ideal gases, the total pressure gets contribution from each gas in the mixture. Equation (13.14) becomes

$$P = (1/3)\; [n_1 m_1 \overline{v_1^2} \; + n_2\, m_2 \; \overline{v_2^2} + \ldots\;] \qquad (13.20)$$

In equilibrium, the average kinetic energy of the molecules of different gases will be equal. That is,

$$½\;\; m_1\;\; \overline{v_1^2} \;\; = ½\; m_2\;\; \overline{v_2^2} = (3/2)\; k_B\; T$$

so that

$$P = (n_1 + n_2 + \ldots\;\;)\; k_B\; T \qquad (13.21)$$

which is Dalton's law of partial pressures.

From Eq. (13.19), we can get an idea of the typical speed of molecules in a gas. At a temperature T = 300 K, the mean square speed of a molecule in nitrogen gas is :

$$m \quad \frac{M_{N_2}}{N_A} \quad \frac{28}{6.02 \quad 10^{26}} \quad 4.65 \quad 10^{-26} \text{ kg.}$$

$$\overline{v^2} \;\; = \; 3\; k_B\; T \,/\; m \;\; = \;\; (516)^2 \text{ m}^2\text{s}^{-2}$$

The square root of $\overline{v^2}$ is known as root mean square (rms) speed and is denoted by v_{rms},

(We can also write $\overline{v^2}$ as $< v^2 >$.)

$$v_{rms} \;\; = \;\; 516 \text{ m s}^{-1}$$

The speed is of the order of the speed of sound in air. It follows from Eq. (13.19) that at the same temperature, lighter molecules have greater rms speed.

Example 13.5 A flask contains argon and chlorine in the ratio of 2:1 by mass. The temperature of the mixture is 27 °C. Obtain the ratio of (i) average kinetic energy per molecule, and (ii) root mean square speed v_{rms} of the molecules of the two gases. Atomic mass of argon = 39.9 u; Molecular mass of chlorine = 70.9 u.

Answer The important point to remember is that the average kinetic energy (per molecule) of any (ideal) gas (be it monatomic like argon, diatomic like chlorine or polyatomic) is always equal to (3/2) $k_B T$. It depends only on temperature, and is independent of the nature of the gas.

(i) Since argon and chlorine both have the same temperature in the flask, the ratio of average kinetic energy (per molecule) of the two gases is 1:1.

(ii) Now ½ $m\, v_{rms}{}^2$ = average kinetic energy per molecule = (3/2)) $k_B T$ where m is the mass of a molecule of the gas. Therefore,

$$\frac{(\mathbf{v}^2_{rms})_{Ar}}{(\mathbf{v}^2_{rms})_{Cl}} \quad \frac{(m)_{Cl}}{(m)_{Ar}} \quad \frac{(M)_{Cl}}{(M)_{Ar}} = \frac{70.9}{39.9} = 1.77$$

where M denotes the molecular mass of the gas. (For argon, a molecule is just an atom of argon.) Taking square root of both sides,

$$\frac{(\mathbf{v}_{rms})_{Ar}}{(\mathbf{v}_{rms})_{Cl}} \; = \; 1.33$$

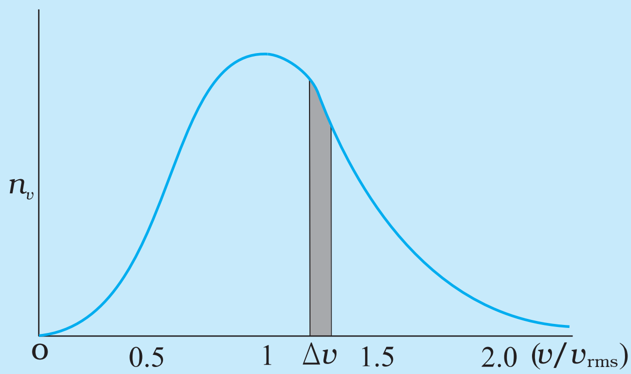
You should note that the composition of the mixture by mass is quite irrelevant to the above

* *E denotes the translational part of the internal energy U that may include energies due to other degrees of freedom also. See section 13.5.*

Maxwell Distribution Function

In a given mass of gas, the velocities of all molecules are not the same, even when bulk parameters like pressure, volume and temperature are fixed. Collisions change the direction and the speed of molecules. However in a state of equilibrium, the distribution of speeds is constant or fixed.

Distributions are very important and useful when dealing with systems containing large number of objects. As an example consider the ages of different persons in a city. It is not feasible to deal with the age of each individual. We can divide the people into groups: children up to age 20 years, adults between ages of 20 and 60, old people above 60. If we want more detailed information we can choose smaller intervals, 0-1, 1-2,..., 99-100 of age groups. When the size of the interval becomes smaller, say half year, the number of persons in the interval will also reduce, roughly half the original number in the one year interval. The number of persons d$N(x)$ in the age interval x and x+dx is proportional to dx or d$N(x)$ = n_x dx. We have used n_x to denote the number of persons at the value of x.



Maxwell distribution of molecular speeds

In a similar way the molecular speed distribution gives the number of molecules between the speeds v and v+ dv. d$N(v)$ = $4p\ N\ a^3 e^{-bv^2} v^2$ dv = n_vdv. This is called Maxwell distribution. The plot of n_v against v is shown in the figure. The fraction of the molecules with speeds v and v+dv is equal to the area of the strip shown. The average of any quantity like v^2 is defined by the integral $<v^2> = (1/N) \int v^2$ d$N(v)$ = Å$(3k_B\ T/m)$ which agrees with the result derived from more elementary considerations.

calculation. Any other proportion by mass of argon and chlorine would give the same answers to (i) and (ii), provided the temperature remains unaltered. ◀

▶ ***Example 13.6*** Uranium has two isotopes of masses 235 and 238 units. If both are present in Uranium hexafluoride gas which would have the larger average speed ? If atomic mass of fluorine is 19 units, estimate the percentage difference in speeds at any temperature.

Answer At a fixed temperature the average energy = ½ $m<v^2>$ is constant. So smaller the mass of the molecule, faster will be the speed. The ratio of speeds is inversely proportional to the square root of the ratio of the masses. The masses are 349 and 352 units. So

$$v_{349} / v_{352} = (352/349)^{1/2} = 1.0044 .$$

$$\text{Hence difference } \frac{V}{V} = 0.44\ \%.$$

[^{235}U is the isotope needed for nuclear fission. To separate it from the more abundant isotope ^{238}U, the mixture is surrounded by a porous cylinder. The porous cylinder must be thick and narrow, so that the molecule wanders through individually, colliding with the walls of the long pore. The faster molecule will leak out more than

the slower one and so there is more of the lighter molecule (enrichment) outside the porous cylinder (Fig. 13.5). The method is not very efficient and has to be repeated several times for sufficient enrichment.]. ◀

When gases diffuse, their rate of diffusion is inversely proportional to square root of the masses (see Exercise 13.12). Can you guess the explanation from the above answer?

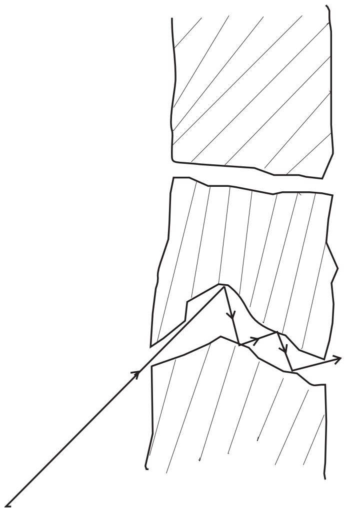


Fig. 13.5 *Molecules going through a porous wall.*

▶ ***Example 13.7*** (a) When a molecule (or an elastic ball) hits a (massive) wall, it rebounds with the same speed. When a ball hits a massive bat held firmly, the same thing happens. However, when the bat is moving towards the ball, the ball rebounds with a different speed. Does the ball move faster or slower? (Ch.6 will refresh your memory on elastic collisions.)

(b) When gas in a cylinder is compressed by pushing in a piston, its temperature rises. Guess at an explanation of this in terms of kinetic theory using (a) above.

(c) What happens when a compressed gas pushes a piston out and expands. What would you observe ?

(d) Sachin Tendulkar uses a heavy cricket bat while playing. Does it help him in anyway ?

Answer (a) Let the speed of the ball be u relative to the wicket behind the bat. If the bat is moving towards the ball with a speed V relative to the wicket, then the relative speed of the ball to bat is $V + u$ towards the bat. When the ball rebounds (after hitting the massive bat) its speed, relative to bat, is $V + u$ moving away from the bat. So relative to the wicket the speed of the rebounding ball is $V + (V + u) = 2V + u$, moving away from the wicket. So the ball speeds up after the collision with the bat. The rebound speed will be less than u if the bat is not massive. For a molecule this would imply an increase in temperature.

You should be able to answer (b) (c) and (d) based on the answer to (a).

(Hint: Note the correspondence, piston → bat, cylinder → wicket, molecule → ball.) ◀

13.5 LAW OF EQUIPARTITION OF ENERGY

The kinetic energy of a single molecule is

$$_t \quad \frac{1}{2}mv_x^2 \quad \frac{1}{2}mv_y^2 \quad \frac{1}{2}mv_z^2 \qquad (13.22)$$

For a gas in thermal equilibrium at temperature T the average value of energy denoted by $< _t >$ is

$$\langle \ _t \rangle \quad \left\langle \frac{1}{2}mv_x^2 \right\rangle \quad \left\langle \frac{1}{2}mv_y^2 \right\rangle \quad \left\langle \frac{1}{2}mv_z^2 \right\rangle \quad \frac{3}{2}k_BT \qquad (13.23)$$

Since there is no preferred direction, Eq. (13.23) implies

$$\left\langle \frac{1}{2}\ mv_x^2 \right\rangle \quad \frac{1}{2}\ k_BT\ , \left\langle \frac{1}{2}\ mv_y^2 \right\rangle \quad \frac{1}{2}\ k_BT\ ,$$

$$\left\langle \frac{1}{2}\ mv_z^2 \right\rangle \quad \frac{1}{2}\ k_BT \qquad (13.24)$$

A molecule free to move in space needs three coordinates to specify its location. If it is constrained to move in a plane it needs two;and if constrained to move along a line, it needs just one coordinate to locate it. This can also be expressed in another way. We say that it has one degree of freedom for motion in a line, two for motion in a plane and three for motion in space. Motion of a body as a whole from one point to another is called translation. Thus, a molecule free to move in space has three translational degrees of freedom. Each translational degree of freedom contributes a term that contains square of some variable of motion, e.g., ½ mv_x^2 and similar terms in v_y and v_z. In, Eq. (13.24) we see that in thermal equilibrium, the average of each such term is ½ k_BT .

Molecules of a monatomic gas like argon have only translational degrees of freedom. But what about a diatomic gas such as O_2 or N_2? A molecule of O_2 has three translational degrees of freedom. But in addition it can also rotate about its centre of mass. Figure 13.6 shows the two independent axes of rotation 1 and 2, normal to the axis joining the two oxygen atoms about which the molecule can rotate*. The molecule thus has two rotational degrees of freedom, each of which contributes a term to the total energy consisting of translational energy $_t$ and rotational energy $_r$.

$$ {}_t \quad {}_r \quad \frac{1}{2}mv_x^2 \quad \frac{1}{2}mv_y^2 \quad \frac{1}{2}mv_z^2 \quad \frac{1}{2}I_1 \;{}^2_1 \quad \frac{1}{2}I_2 \;{}^2_2 \qquad (13.25)$$

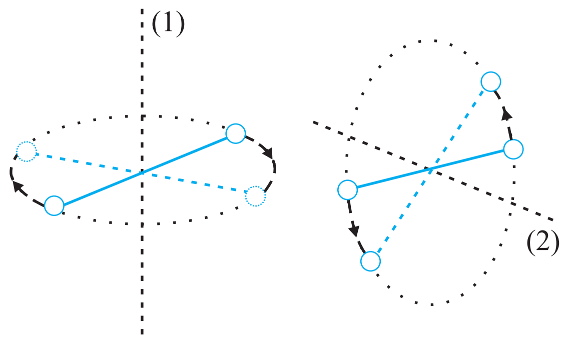


Fig. 13.6 *The two independent axes of rotation of a diatomic molecule*

where ω_1 and ω_2 are the angular speeds about the axes 1 and 2 and I_1, I_2 are the corresponding moments of inertia. Note that each rotational degree of freedom contributes a term to the energy that contains square of a rotational variable of motion.

We have assumed above that the O_2 molecule is a 'rigid rotator', i.e. the molecule does not vibrate. This assumption, though found to be true (at moderate temperatures) for O_2, is not always valid. Molecules like CO even at moderate temperatures have a mode of vibration, i.e. its atoms oscillate along the interatomic axis like a one-dimensional oscillator, and contribute a vibrational energy term ε_v to the total energy:

$$ {}_v \quad \frac{1}{2}m \; \frac{dy}{dt}^{\;2} \quad \frac{1}{2}ky^2 $$

$$ {}_t \quad {}_r \quad {}_v \qquad (13.26)$$

where k is the force constant of the oscillator and y the vibrational co-ordinate.

Once again the vibrational energy terms in Eq. (13.26) contain squared terms of vibrational variables of motion y and dy/dt .

At this point, notice an important feature in Eq.(13.26). While each translational and rotational degree of freedom has contributed only one 'squared term' in Eq.(13.26), one vibrational mode contributes two 'squared terms' : kinetic and potential energies.

Each quadratic term occurring in the expression for energy is a mode of absorption of energy by the molecule. We have seen that in thermal equilibrium at absolute temperature T, for each translational mode of motion, the average energy is ½ k_BT. A most elegant principle of classical statistical mechanics (first proved by Maxwell) states that this is so for each mode of energy: translational, rotational and vibrational. That is, in equilibrium, the total energy is equally distributed in all possible energy modes, with each mode having an average energy equal to ½ k_BT. This is known as the **law of equipartition of energy**. Accordingly, each translational and rotational degree of freedom of a molecule contributes ½ k_BT to the energy while each vibrational frequency contributes $2 \times ½\, k_BT = k_BT$, since a vibrational mode has both kinetic and potential energy modes.

The proof of the law of equipartition of energy is beyond the scope of this book. Here we shall apply the law to predict the specific heats of gases theoretically. Later we shall also discuss briefly, the application to specific heat of solids.

13.6 SPECIFIC HEAT CAPACITY

13.6.1 Monatomic Gases

The molecule of a monatomic gas has only three translational degrees of freedom. Thus, the average energy of a molecule at temperature T is $(3/2)k_BT$. The total internal energy of a mole of such a gas is

* *Rotation along the line joining the atoms has very small moment of inertia and does not come into play for quantum mechanical reasons. See end of section 13.6.*

$$U \quad \frac{3}{2}k_BT \quad N_A \quad \frac{3}{2}RT \qquad (13.27)$$

The molar specific heat at constant volume, C_v, is

$$C_v \text{ (monatomic gas)} = \frac{dU}{dT} = \frac{3}{2}RT \qquad (13.28)$$

For an ideal gas,

$$C_p - C_v = R \qquad (13.29)$$

where C_p is the molar specific heat at constant pressure. Thus,

$$C_p = \frac{5}{2}R \qquad (13.30)$$

The ratio of specific heats $\quad \frac{C_p}{C_v} \quad \frac{5}{3} \qquad (13.31)$

13.6.2 Diatomic Gases

As explained earlier, a diatomic molecule treated as a rigid rotator like a dumbbell has 5 degrees of freedom : 3 translational and 2 rotational. Using the law of equipartition of energy, the total internal energy of a mole of such a gas is

$$U \quad \frac{5}{2}k_BT \quad N_A \quad \frac{5}{2}RT \qquad (13.32)$$

The molar specific heats are then given by

$$C_v \text{ (rigid diatomic)} = \frac{5}{2}R,\ C_p = \frac{7}{2}R \qquad (13.33)$$

$$\gamma \text{ (rigid diatomic)} = \frac{7}{5} \qquad (13.34)$$

If the diatomic molecule is not rigid but has in addition a vibrational mode

$$U = \left(\frac{5}{2}k_BT + k_BT\right)N_A = \frac{7}{2}RT$$

$$C_v \quad \frac{7}{2}R,\ C_p \quad \frac{9}{2}R, \quad \frac{9}{7}R \qquad (13.35)$$

13.6.3 Polyatomic Gases

In general a polyatomic molecule has 3 translational, 3 rotational degrees of freedom and a certain number (f) of vibrational modes. According to the law of equipartition of energy, it is easily seen that one mole of such a gas has

$$U = (\frac{3}{2}\ k_BT + \frac{3}{2}\ k_BT + f\,k_BT)\ N_A$$

i.e. $C_v = (3 + f)\ R,\ \ C_p = (4 + f)\ R,$

$$\frac{f}{f} \qquad (13.36)$$

Note that $C_p - C_v = R$ is true for any ideal gas, whether mono, di or polyatomic.

Table 13.1 summarises the theoretical predictions for specific heats of gases ignoring any vibrational modes of motion. The values are in good agreement with experimental values of specific heats of several gases given in Table 13.2. Of course, there are discrepancies between predicted and actual values of specific heats of several other gases (not shown in the table), such as Cl_2, C_2H_6 and many other polyatomic gases. Usually, the experimental values for specific heats of these gases are greater than the predicted values given in Table13.1 suggesting that the agreement can be improved by including vibrational modes of motion in the calculation. The law of equipartition of energy is thus well

Table 13.1 **Predicted values of specific heat capacities of gases (ignoring vibrational modes),**

Nature of Gas	C_v (J mol^{-1} K^{-1})	C_p (J mol^{-1} K^{-1})	$C_p - C_v$ (J mol^{-1} K^{-1})	γ
Monatomic	12.5	20.8	8.31	1.67
Diatomic	20.8	29.1	8.31	1.40
Triatomic	24.93	33.24	8.31	1.33

Table13.2 **Measured values of specific heat capacities of some gases**

Nature of gas	Gas	C_v (J mol^{-1} K^{-1})	C_p (J mol^{-1} K^{-1})	$C_p - C_v$ (J mol^{-1} K^{-1})	γ
Monatomic	He	12.5	20.8	8.30	1.66
Monatomic	Ne	12.7	20.8	8.12	1.64
Monatomic	Ar	12.5	20.8	8.30	1.67
Diatomic	H_2	20.4	28.8	8.45	1.41
Diatomic	O_2	21.0	29.3	8.32	1.40
Diatomic	N_2	20.8	29.1	8.32	1.40
Triatomic	H_2O	27.0	35.4	8.35	1.31
Polyatomic	CH_4	27.1	35.4	8.36	1.31

verified experimentally at ordinary temperatures.

▶ ***Example 13.8*** A cylinder of fixed capacity 44.8 litres contains helium gas at standard temperature and pressure. What is the amount of heat needed to raise the temperature of the gas in the cylinder by 15.0 °C ? (R = 8.31 J mo1^{-1} K^{-1}).

Answer Using the gas law $PV = \mu RT$, you can easily show that 1 mol of any (ideal) gas at standard temperature (273 K) and pressure (1 atm = 1.01 × 10^5 Pa) occupies a volume of 22.4 litres. This universal volume is called molar volume. Thus the cylinder in this example contains 2 mol of helium. Further, since helium is monatomic, its predicted (and observed) molar specific heat at constant volume, C_v = (3/2) R, and molar specific heat at constant pressure, C_p = (3/2) $R + R$ = (5/2) R. Since the volume of the cylinder is fixed, the heat required is determined by C_v. Therefore,
Heat required = no. of moles × molar specific heat × rise in temperature

$= 2 \times 1.5\, R \times 15.0 = 45\, R$

$= 45 \times 8.31 = 374$ J. ◀

13.6.4 Specific Heat Capacity of Solids

We can use the law of equipartition of energy to determine specific heats of solids. Consider a solid of N atoms, each vibrating about its mean position. An oscillation in one dimension has average energy of $2 \times ½\, k_BT = k_BT$. In three dimensions, the average energy is 3 k_BT. For a mole of solid, $N = N_A$, and the total energy is

$U = 3\ k_BT \times N_A = 3\ RT$

Now at constant pressure $\Delta Q = \Delta U + P\Delta V = \Delta U$, since for a solid ΔV is negligible. Hence,

$$C \quad \frac{Q}{T} \quad \frac{U}{T} \quad 3R \qquad (13.37)$$

Table 13.3 **Specific Heat Capacity of some solids at room temperature and atmospheric pressure**

Substance	Specific heat (J kg^{-1} K^{-1})	Molar specific Heat(J mol^{-1} K^{-1})
Aluminium	900.0	24.4
Carbon	506.5	6.1
Copper	386.4	24.5
Lead	127.7	26.5
Silver	236.1	25.5
Tungsten	134.4	24.9

As Table 13.3 shows the prediction generally agrees with experimental values at ordinary temperature (Carbon is an exception).

13.6.5 Specific Heat Capacity of Water

We treat water like a solid. For each atom average energy is $3k_BT$. Water molecule has three atoms, two hydrogen and one oxygen. So it has

$U = 3 \times 3\ k_BT \times N_A = 9\ RT$

and $C = \Delta Q/\ \Delta T = \Delta\ U\ /\ \Delta T = 9R$.

This is the value observed and the agreement is very good. In the calorie, gram, degree units, water is defined to have unit specific heat. As 1 calorie = 4.179 joules and one mole of water is 18 grams, the heat capacity per mole is ~ 75 J mol^{-1} K^{-1} ~ 9R. However with more complex molecules like alcohol or acetone the arguments, based on degrees of freedom, become more complicated.

Lastly, we should note an important aspect of the predictions of specific heats, based on the classical law of equipartition of energy. The predicted specific heats are independent of temperature. As we go to low temperatures, however, there is a marked departure from this prediction. Specific heats of all substances approach zero as $T \rightarrow 0$. This is related to the fact that degrees of freedom get frozen and ineffective at low temperatures. According to classical physics degrees of freedom must remain unchanged at all times. The behaviour of specific heats at low temperatures shows the inadequacy of classical physics and can be explained only by invoking quantum considerations, as was first shown by Einstein. Quantum mechanics requires a minimum, nonzero amount of energy before a degree of freedom comes into play. This is also the reason why vibrational degrees of freedom come into play only in some cases.

13.7 MEAN FREE PATH

Molecules in a gas have rather large speeds of the order of the speed of sound. Yet a gas leaking from a cylinder in a kitchen takes considerable time to diffuse to the other corners of the room. The top of a cloud of smoke holds together for hours. This happens because molecules in a gas have a finite though small size, so they are bound to undergo collisions. As a result, they cannot

Seeing is Believing

Can one see atoms rushing about. Almost but not quite. One can see pollen grains of a flower being pushed around by molecules of water. The size of the grain is ~ 10^{-5} m. In 1827, a Scottish botanist Robert Brown, while examining, under a microscope, pollen grains of a flower suspended in water noticed that they continuously moved about in a zigzag, random fashion.

Kinetic theory provides a simple explanation of the phenomenon. Any object suspended in water is continuously bombarded from all sides by the water molecules. Since the motion of molecules is random, the number of molecules hitting the object in any direction is about the same as the number hitting in the opposite direction. The small difference between these molecular hits is negligible compared to the total number of hits for an object of ordinary size, and we do not notice any movement of the object.

When the object is sufficiently small but still visible under a microscope, the difference in molecular hits from different directions is not altogether negligible, i.e. the impulses and the torques given to the suspended object through continuous bombardment by the molecules of the medium (water or some other fluid) do not exactly sum to zero. There is a net impulse and torque in this or that direction. The suspended object thus, moves about in a zigzag manner and tumbles about randomly. This motion called now 'Brownian motion' is a visible proof of molecular activity. In the last 50 years or so molecules have been seen by scanning tunneling and other special microscopes.

In 1987 Ahmed Zewail, an Egyptian scientist working in USA was able to observe not only the molecules but also their detailed interactions. He did this by illuminating them with flashes of laser light for very short durations, of the order of tens of femtoseconds and photographing them. (1 femto-second = 10^{-15} s). One could study even the formation and breaking of chemical bonds. That is really seeing !

move straight unhindered; their paths keep getting incessantly deflected.

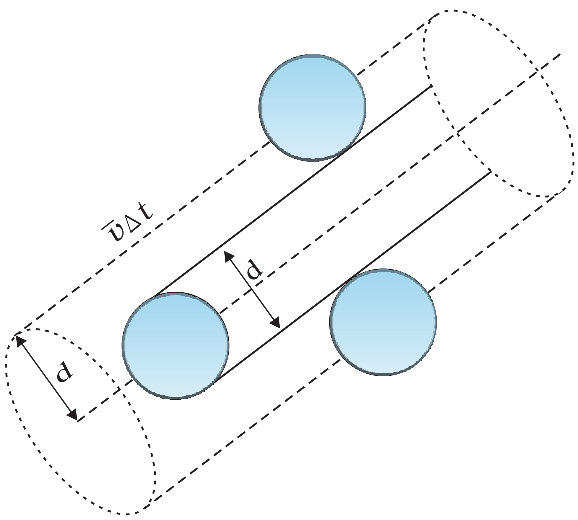


Fig. 13.7 *The volume swept by a molecule in time Δt in which any molecule will collide with it.*

Suppose the molecules of a gas are spheres of diameter d. Focus on a single molecule with the average speed $<v>$. It will suffer collision with any molecule that comes within a distance d between the centres. In time Δt, it sweeps a volume $\pi d^2 <v> \Delta t$ wherein any other molecule will collide with it (see Fig. 13.7). If n is the number of molecules per unit volume, the molecule suffers $n\pi d^2 <v> \Delta t$ collisions in time Δt. Thus the rate of collisions is $n\pi d^2 <v>$ or the time between two successive collisions is on the average,

$$\tau = 1/(n\pi <v> d^2) \qquad (13.38)$$

The average distance between two successive collisions, called the mean free path l, is :

$$l = <v> \tau = 1/(n\pi d^2) \qquad (13.39)$$

In this derivation, we imagined the other molecules to be at rest. But actually all molecules are moving and the collision rate is determined by the average relative velocity of the molecules. Thus we need to replace $<v>$ by $<v_r>$ in Eq. (13.38). A more exact treatment gives

$$l \quad 1/\sqrt{2}\; n \; d^2 \qquad (13.40)$$

Let us estimate l and τ for air molecules with average speeds $<v> = (485 \text{m/s})$. At STP

$$n = \frac{0.02 \quad 10^{23}}{22.4 \quad 10^{-3}}$$

$= 2.7 \times 10^{25}\ \text{m}^{-3.}$

Taking, $d = 2 \times 10^{-10}$ m,

$\tau = 6.1 \times 10^{-10}$ s

and $l = 2.9 \times 10^{-7}\ \text{m} \approx 1500d$ (13.41)

As expected, the mean free path given by Eq. (13.40) depends inversely on the number density and the size of the molecules. In a highly evacuated tube n is rather small and the mean free path can be as large as the length of the tube.

Example 13.9 Estimate the mean free path for a water molecule in water vapour at 373 K. Use information from Exercises 13.1 and Eq. (13.41) above.

Answer The d for water vapour is same as that of air. The number density is inversely proportional to absolute temperature.

$$\text{So } n \quad 2.7 \quad 10^{25} \frac{273}{373} \quad 2 \quad 10^{25}\ \text{m}^{-3}$$

Hence, mean free path $l \quad 4 \quad 10^{-7}$ m

Note that the mean free path is 100 times the interatomic distance ~ 40 Å = 4×10^{-9} m calculated earlier. It is this large value of mean free path that leads to the typical gaseous behaviour. Gases can not be confined without a container.

Using, the kinetic theory of gases, the bulk measurable properties like viscosity, heat conductivity and diffusion can be related to the microscopic parameters like molecular size. It is through such relations that the molecular sizes were first estimated.

SUMMARY

1. The ideal gas equation connecting pressure (P), volume (V) and absolute temperature (T) is

$$PV = \mu RT \quad = k_B NT$$

where μ is the number of moles and N is the number of molecules. R and k_B are universal constants.

$$R = 8.314 \text{ J mol}^{-1} \text{ K}^{-1}, \qquad k_B = \frac{R}{N_A} \quad = 1.38 \times 10^{-23} \text{ J K}^{-1}$$

Real gases satisfy the ideal gas equation only approximately, more so at low pressures and high temperatures.

2. Kinetic theory of an ideal gas gives the relation

$$P \quad \frac{1}{3} n m \overline{v^2}$$

where n is number density of molecules, m the mass of the molecule and $\overline{v^2}$ is the mean of squared speed. Combined with the ideal gas equation it yields a kinetic interpretation of temperature.

$$\frac{1}{2} m \overline{v^2} \quad \frac{3}{2} k_B T, \quad v_{rms} \quad \overline{v^2}^{\,1/2} \quad \sqrt{\frac{3k_BT}{m}}$$

This tells us that the temperature of a gas is a measure of the average kinetic energy of a molecule, *independent of the nature of the gas or molecule.* In a mixture of gases at a fixed temperature the heavier molecule has the lower average speed.

3. The translational kinetic energy

$$E = \frac{3}{2} k_B NT.$$

This leads to a relation

$$PV = \frac{2}{3} E$$

4. The law of equipartition of energy states that if a system is in equilibrium at absolute temperature T, the total energy is distributed equally in different energy modes of

absorption, the energy in each mode being equal to ½ k_B T. Each translational and rotational degree of freedom corresponds to one energy mode of absorption and has energy ½ $k_B T$. Each vibrational frequency has two modes of energy (kinetic and potential) with corresponding energy equal to
$2 \times ½\ k_B\ T = k_B\ T$.

5. Using the law of equipartition of energy, the molar specific heats of gases can be determined and the values are in agreement with the experimental values of specific heats of several gases. The agreement can be improved by including vibrational modes of motion.

6. The mean free path l is the average distance covered by a molecule between two successive collisions :

$$\bar{l} \quad \frac{1}{\sqrt{2}\, n \quad d^2}$$

where n is the number density and d the diameter of the molecule.

POINTS TO PONDER

1. Pressure of a fluid is not only exerted on the wall. Pressure exists everywhere in a fluid. Any layer of gas inside the volume of a container is in equilibrium because the pressure is the same on both sides of the layer.

2. We should not have an exaggerated idea of the intermolecular distance in a gas. At ordinary pressures and temperatures, this is only 10 times or so the interatomic distance in solids and liquids. What is different is the mean free path which in a gas is 100 times the interatomic distance and 1000 times the size of the molecule.

3. The law of equipartition of energy is stated thus: the energy for each degree of freedom in thermal equilibrium is ½ k_B T. Each quadratic term in the total energy expression of a molecule is to be counted as a degree of freedom. Thus, each vibrational mode gives 2 (not 1) degrees of freedom (kinetic and potential energy modes), corresponding to the energy $2 \times ½\ k_B\ T = k_B\ T$.

4. Molecules of air in a room do not all fall and settle on the ground (due to gravity) because of their high speeds and incessant collisions. In equilibrium, there is a very slight increase in density at lower heights (like in the atmosphere). The effect is small since the potential energy (mgh) for ordinary heights is much less than the average kinetic energy ½ mv^2 of the molecules.

5. $< v^2 >$ is not always equal to $(< v >)^2$. The average of a squared quantity is not necessarily the square of the average. Can you find examples for this statement.

EXERCISES

13.1 Estimate the fraction of molecular volume to the actual volume occupied by oxygen gas at STP. Take the diameter of an oxygen molecule to be 3 Å.

13.2 Molar volume is the volume occupied by 1 mol of any (ideal) gas at standard temperature and pressure (STP : 1 atmospheric pressure, 0 °C). Show that it is 22.4 litres.

13.3 Figure 13.8 shows plot of PV/T versus P for 1.00×10^{-3} kg of oxygen gas at two different temperatures.

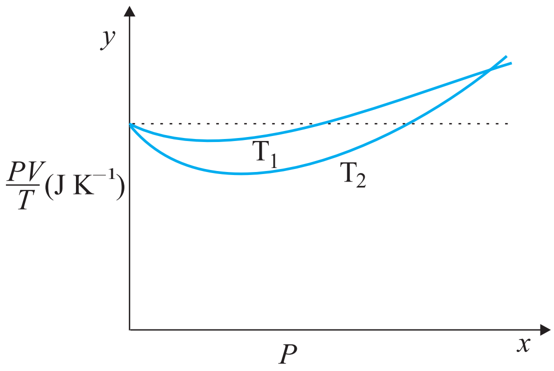


Fig. 13.8

(a) What does the dotted plot signify?
(b) Which is true: $T_1 > T_2$ or $T_1 < T_2$?
(c) What is the value of PV/T where the curves meet on the y-axis?
(d) If we obtained similar plots for 1.00×10^{-3} kg of hydrogen, would we get the same value of PV/T at the point where the curves meet on the y-axis? If not, what mass of hydrogen yields the same value of PV/T (for low pressurehigh temperature region of the plot) ? (Molecular mass of H_2 = 2.02 u, of O_2 = 32.0 u, R = 8.31 J mo1^{-1} K^{-1}.)

13.4 An oxygen cylinder of volume 30 litres has an initial gauge pressure of 15 atm and a temperature of 27 °C. After some oxygen is withdrawn from the cylinder, the gauge pressure drops to 11 atm and its temperature drops to 17 °C. Estimate the mass of oxygen taken out of the cylinder (R = 8.31 J mol^{-1} K^{-1}, molecular mass of O_2 = 32 u).

13.5 An air bubble of volume 1.0 cm^3 rises from the bottom of a lake 40 m deep at a temperature of 12 °C. To what volume does it grow when it reaches the surface, which is at a temperature of 35 °C ?

13.6 Estimate the total number of air molecules (inclusive of oxygen, nitrogen, water vapour and other constituents) in a room of capacity 25.0 m^3 at a temperature of 27 °C and 1 atm pressure.

13.7 Estimate the average thermal energy of a helium atom at (i) room temperature (27 °C), (ii) the temperature on the surface of the Sun (6000 K), (iii) the temperature of 10 million kelvin (the typical core temperature in the case of a star).

13.8 Three vessels of equal capacity have gases at the same temperature and pressure. The first vessel contains neon (monatomic), the second contains chlorine (diatomic), and the third contains uranium hexafluoride (polyatomic). Do the vessels contain equal number of respective molecules ? Is the root mean square speed of molecules the same in the three cases? If not, in which case is v_{rms} the largest ?

13.9 At what temperature is the root mean square speed of an atom in an argon gas cylinder equal to the rms speed of a helium gas atom at – 20 °C ? (atomic mass of Ar = 39.9 u, of He = 4.0 u).

13.10 Estimate the mean free path and collision frequency of a nitrogen molecule in a cylinder containing nitrogen at 2.0 atm and temperature 17 ^{0}C. Take the radius of a nitrogen molecule to be roughly 1.0 Å. Compare the collision time with the time the molecule moves freely between two successive collisions (Molecular mass of N_2 = 28.0 u).

Additional Exercises

13.11 A metre long narrow bore held horizontally (and closed at one end) contains a 76 cm long mercury thread, which traps a 15 cm column of air. What happens if the tube is held vertically with the open end at the bottom ?

13.12 From a certain apparatus, the diffusion rate of hydrogen has an average value of 28.7 $cm^3 s^{-1}$. The diffusion of another gas under the same conditions is measured to have an average rate of 7.2 $cm^3 s^{-1}$. Identify the gas.
[**Hint** : Use Graham's law of diffusion: $R_1/R_2 = (M_2/M_1)^{1/2}$, where R_1, R_2 are diffusion rates of gases 1 and 2, and M_1 and M_2 their respective molecular masses. The law is a simple consequence of kinetic theory.]

13.13 A gas in equilibrium has uniform density and pressure throughout its volume. This is strictly true only if there are no external influences. A gas column under gravity, for example, does not have uniform density (and pressure). As you might expect, its density decreases with height. The precise dependence is given by the so-called law of atmospheres

$$n_2 = n_1 \exp [-mg (h_2 - h_1)/ k_B T]$$

where n_2, n_1 refer to number density at heights h_2 and h_1 respectively. Use this relation to derive the equation for sedimentation equilibrium of a suspension in a liquid column:

$$n_2 = n_1 \exp [-mg N_A (\rho - P') (h_2 - h_1)/ (\rho RT)]$$

where ρ is the density of the suspended particle, and ρ' that of surrounding medium. [N_A is Avogadro's number, and R the universal gas constant.] [**Hint** : Use Archimedes principle to find the apparent weight of the suspended particle.]

13.14 Given below are densities of some solids and liquids. Give rough estimates of the size of their atoms :

Substance	Atomic Mass (u)	Density (10^3 Kg m^{-3})
Carbon (diamond)	12.01	2.22
Gold	197.00	19.32
Nitrogen (liquid)	14.01	1.00
Lithium	6.94	0.53
Fluorine (liquid)	19.00	1.14

[**Hint** : Assume the atoms to be 'tightly packed' in a solid or liquid phase, and use the known value of Avogadro's number. You should, however, not take the actual numbers you obtain for various atomic sizes too literally. Because of the crudeness of the tight packing approximation, the results only indicate that atomic sizes are in the range of a few Å].

CHAPTER FOURTEEN

OSCILLATIONS

14.1 INTRODUCTION

In our daily life we come across various kinds of motions. You have already learnt about some of them, e.g. rectilinear motion and motion of a projectile. Both these motions are non-repetitive. We have also learnt about uniform circular motion and orbital motion of planets in the solar system. In these cases, the motion is repeated after a certain interval of time, that is, it is periodic. In your childhood you must have enjoyed rocking in a cradle or swinging on a swing. Both these motions are repetitive in nature but different from the periodic motion of a planet. Here, the object moves to and fro about a mean position. The pendulum of a wall clock executes a similar motion. There are leaves and branches of a tree oscillating in breeze, boats bobbing at anchor and the surging pistons in the engines of cars. All these objects execute a periodic to and fro motion. Such a motion is termed as oscillatory motion. In this chapter we study this motion.

The study of oscillatory motion is basic to physics; its concepts are required for the understanding of many physical phenomena. In musical instruments like the sitar, the guitar or the violin, we come across vibrating strings that produce pleasing sounds. The membranes in drums and diaphragms in telephone and speaker systems vibrate to and fro about their mean positions. The vibrations of air molecules make the propagation of sound possible. Similarly, the atoms in a solid oscillate about their mean positions and convey the sensation of temperature. The oscillations of electrons in the antennas of radio, TV and satellite transmitters convey information.

The description of a periodic motion in general, and oscillatory motion in particular, requires some fundamental concepts like period, frequency, displacement, amplitude and phase. These concepts are developed in the next section.

14.2 PERIODIC AND OSCILLATORY MOTIONS

Fig 14.1 shows some periodic motions. Suppose an insect climbs up a ramp and falls down it comes back to the initial point and repeats the process identically. If you draw a graph of its height above the ground versus time, it would look something like Fig. 14.1 (a). If a child climbs up a step, comes down, and repeats the process, its height above the ground would look like that in Fig 14.1 (b). When you play the game of bouncing a ball off the ground, between your palm and the ground, its height versus time graph would look like the one in Fig 14.1 (c). Note that both the curved parts in Fig 14.1 (c) are sections of a parabola given by the Newton's equation of motion (see section 3.6),

$$h = ut + \frac{1}{2}gt^2 \text{ for downward motion, and}$$

$$h = ut - \frac{1}{2}gt^2 \text{ for upward motion,}$$

with different values of u in each case. These are examples of periodic motion. Thus, a motion that repeats itself at regular intervals of time is called **periodic motion.**

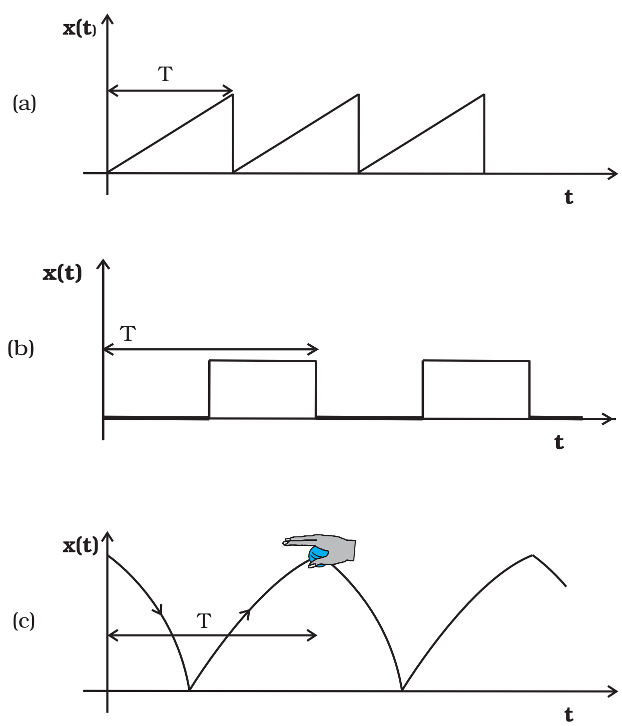


Fig 14.1 Examples of periodic motion. The period T is shown in each case.

Very often the body undergoing periodic motion has an equilibrium position somewhere inside its path. When the body is at this position no net external force acts on it. Therefore, if it is left there at rest, it remains there forever. If the body is given a small displacement from the position, a force comes into play which tries to bring the body back to the equilibrium point, giving rise to **oscillations** or **vibrations**. For example, a ball placed in a bowl will be in equilibrium at the bottom. If displaced a little from the point, it will perform oscillations in the bowl. Every oscillatory motion is periodic, but every periodic motion need not be oscillatory. Circular motion is a periodic motion, but it is not oscillatory.

There is no significant difference between oscillations and vibrations. It seems that when the frequency is small, we call it oscillation (like the oscillation of a branch of a tree), while when the frequency is high, we call it vibration (like the vibration of a string of a musical instrument).

Simple harmonic motion is the simplest form of oscillatory motion. This motion arises when the force on the oscillating body is directly proportional to its displacement from the mean position, which is also the equilibrium position. Further, at any point in its oscillation, this force is directed towards the mean position.

In practice, oscillating bodies eventually come to rest at their equilibrium positions, because of the damping due to friction and other dissipative causes. However, they can be forced to remain oscillating by means of some external periodic agency. We discuss the phenomena of damped and forced oscillations later in the chapter.

Any material medium can be pictured as a collection of a large number of coupled oscillators. The collective oscillations of the constituents of a medium manifest themselves as waves. Examples of waves include water waves, seismic waves, electromagnetic waves. We shall study the wave phenomenon in the next chapter.

14.2.1 Period and frequency

We have seen that any motion that repeats itself at regular intervals of time is called **periodic motion**. **The smallest interval of time after which the motion is repeated is called its period**. Let us denote the period by the symbol T. Its SI unit is second. For periodic motions,

which are either too fast or too slow on the scale of seconds, other convenient units of time are used. The period of vibrations of a quartz crystal is expressed in units of microseconds (10^{-6} s) abbreviated as μs. On the other hand, the orbital period of the planet Mercury is 88 earth days. The Halley's comet appears after every 76 years.

The reciprocal of T gives the number of repetitions that occur per unit time. This quantity is called the **frequency of the periodic motion**. It is represented by the symbol ν . The relation between ν and T is

$$\nu = 1/T \quad (14.1)$$

The unit of ν is thus s^{-1}. After the discoverer of radio waves, Heinrich Rudolph Hertz (1857-1894), a special name has been given to the unit of frequency. It is called hertz (abbreviated as Hz). Thus,

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1 s^{-1} \quad (14.2)$$

Note, that the frequency, ν, is not necessarily an integer.

Example 14.1 On an average a human heart is found to beat 75 times in a minute. Calculate its frequency and period.

Answer The beat frequency of heart = 75/(1 min)
= 75/(60 s)
= 1.25 s^{-1}
= 1.25 Hz
The time period T = 1/(1.25 s^{-1})
= 0.8 s

14.2.2 Displacement

In section 4.2, we defined displacement of a particle as the change in its position vector. In this chapter, we use the term displacement in a more general sense. It refers to change with time of any physical property under consideration. For example, in case of rectilinear motion of a steel ball on a surface, the distance from the starting point as a function of time is its position displacement. The choice of origin is a matter of convenience. Consider a block attached to a spring, the other end of which is fixed to a rigid wall [see Fig.14.2(a)]. Generally it is convenient to measure displacement of the body from its equilibrium position. For an oscillating simple pendulum, the angle from the vertical as a function of time may be regarded as a displacement variable [see Fig.14.2(b)]. The term displacement is not always to be referred in the context of position only. There can be many other kinds of displacement variables. The voltage across a capacitor, changing with time in an a.c. circuit, is also a displacement variable. In the same way, pressure variations in time in the propagation of sound wave, the changing electric and magnetic fields in a light wave are examples of displacement in different contexts. The displacement variable may take both positive and negative values. In experiments on oscillations, the displacement is measured for different times.

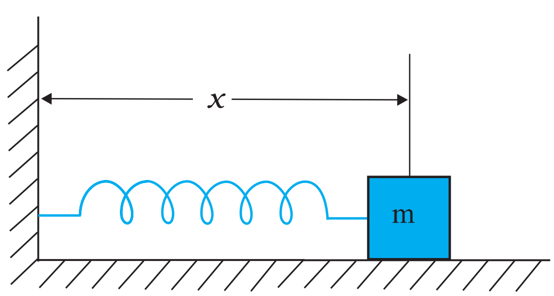


Fig. 14.2(a) *A block attached to a spring, the other end of which is fixed to a rigid wall. The block moves on a frictionless surface. The motion of the block can be described in terms of its distance or displacement x from the wall.*

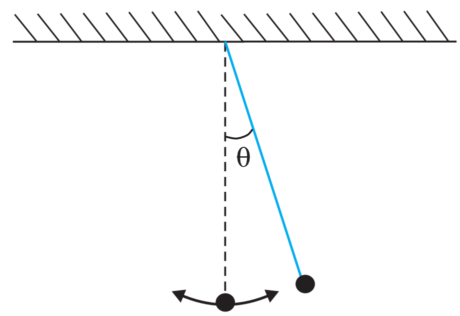


Fig.14.2(b) *An oscillating simple pendulum; its motion can be described in terms of angular displacement θ from the vertical.*

The displacement can be represented by a mathematical function of time. In case of periodic motion, this function is periodic in time. One of the simplest periodic functions is given by

$$f(t) = A \cos \omega t \quad (14.3a)$$

If the argument of this function, ωt, is increased by an integral multiple of 2π radians,

the value of the function remains the same. The function $f(t)$ is then periodic and its period, T, is given by

$$T = \frac{2\pi}{\omega} \qquad (14.3b)$$

Thus, the function $f(t)$ is periodic with period T,

$$f(t) = f(t+T)$$

The same result is obviously correct if we consider a sine function, $f(t) = A \sin \omega t$. Further, a linear combination of sine and cosine functions like,

$$f(t) = A \sin \omega t + B \cos \omega t \qquad (14.3c)$$

is also a periodic function with the same period T. Taking,

$$A = D \cos \phi \ \text{and} \ B = D \sin \phi$$

Eq. (14.3c) can be written as,

$$f(t) = D \sin (\omega t + \phi)\ , \qquad (14.3d)$$

Here D and ϕ are constant given by

$$D = \sqrt{A^2 + B^2} \ \text{and} \qquad \tan^{-1} \frac{B}{A}$$

The great importance of periodic sine and cosine functions is due to a remarkable result proved by the French mathematician, Jean Baptiste Joseph Fourier (1768-1830): **Any periodic function can be expressed as a superposition of sine and cosine functions of different time periods with suitable coefficients**.

Example 14.2 Which of the following functions of time represent (a) periodic and (b) non-periodic motion? Give the period for each case of periodic motion [ω is any positive constant].

(i) $\sin \omega t + \cos \omega t$
(ii) $\sin \omega t + \cos 2\omega t + \sin 4\omega t$
(iii) $e^{-\omega t}$
(iv) $\log (\omega t)$

Answer

(i) $\sin \omega t + \cos \omega t$ is a periodic function, it can also be written as $\sqrt{2} \sin (\omega t + \pi/4)$.

Now $\sqrt{2} \sin (\omega t + \pi/4) = \sqrt{2} \sin (\omega t + \pi/4 + 2\pi)$

$= \sqrt{2} \sin [\omega(t + 2\pi/\omega) + \pi/4]$

The periodic time of the function is $2\pi/\omega$

(ii) This is an example of a periodic motion. It can be noted that each term represents a periodic function with a different angular frequency. Since period is the least interval of time after which a function repeats its value, $\sin \omega t$ has a period $T_0 = 2\pi/\omega$; $\cos 2\omega t$ has a period $\pi/\omega = T_0/2$; and $\sin 4\omega t$ has a period $2\pi/4\omega = T_0/4$. The period of the first term is a multiple of the periods of the last two terms. Therefore, the smallest interval of time after which the sum of the three terms repeats is T_0, and thus the sum is a periodic function with a period $2\pi/\omega$

(iii) The function $e^{-\omega t}$ is not periodic, it decreases monotonically with increasing time and tends to zero as $t \to \infty$ and thus, never repeats its value.

(iv) The function $\log(\omega t)$ increases monotonically with time t. It, therefore, never repeats its value and is a non-periodic function. It may be noted that as $t \to \infty$, $\log(\omega t)$ diverges to ∞. It, therefore, cannot represent any kind of physical displacement. ◀

14.3 SIMPLE HARMONIC MOTION

Let us consider a particle vibrating back and forth about the origin of an x-axis between the limits $+A$ and $-A$ as shown in Fig. 14.3. In between these extreme positions the particle

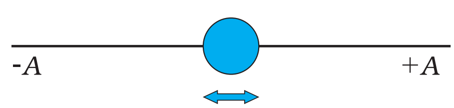


Fig. 14.3 *A particle vibrating back and forth about the origin of x-axis, between the limits +A and –A.*

moves in such a manner that its speed is maximum when it is at the origin and zero when it is at $\pm A$. The time t is chosen to be zero when the particle is at $+A$ and it returns to $+A$ at $t = T$. In this section we will describe this motion. Later, we shall discuss how to achieve it. To study the motion of this particle, we record its positions as a function of time

by taking 'snapshots' at regular intervals of time. A set of such snapshots is shown in Fig. 14.4. The position of the particle with reference to the origin gives its displacement at any instant of time. For such a motion the displacement $x(t)$ of the particle from a certain chosen origin is found to vary with time as,

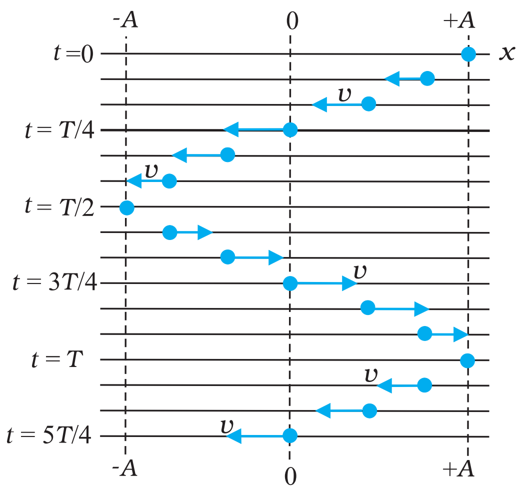


Fig. 14.4 *A sequence of 'snapshots' (taken at equal intervals of time) showing the position of a particle as it oscillates back and forth about the origin along an x-axis, between the limits +A and –A. The length of the vector arrows is scaled to indicate the speed of the particle. The speed is maximum when the particle is at the origin and zero when it is at ± A. If the time t is chosen to be zero when the particle is at +A, then the particle returns to +A at t = T, where T is the period of the motion. The motion is then repeated. It is represented by Eq. (14.4) for ϕ = 0.*

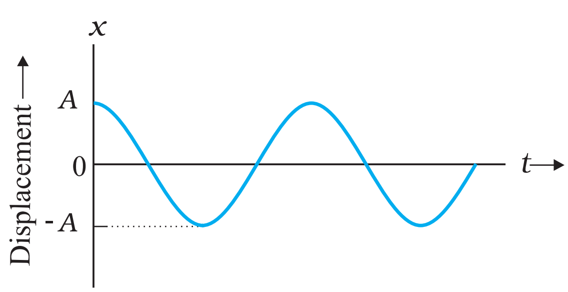


Fig. 14.5 *A graph of x as a function of time for the motion represented by Eq. (14.4).*

$$x(t) = A\cos(\omega t + \phi) \qquad (14.4)$$

in which A, ω and ϕ are constants.

$$\underset{\text{Displacement}}{x(t)} = \underset{\text{Amplitude}}{A} \cos(\overbrace{\underset{\substack{\text{Angular}\\\text{frequency}}}{\omega t} + \underset{\substack{\text{Phase}\\\text{constant}}}{\phi}}^{\text{Phase}})$$

Fig. 14.6 *A reference of the quantities in Eq. (14.4).*

The motion represented by Eq. (14.4) is called **simple harmonic motion** (SHM); a term that means the periodic motion is a sinusoidal function of time. Equation (14.4), in which the sinusoidal function is a cosine function, is plotted in Fig. 14.5. The quantities that determine

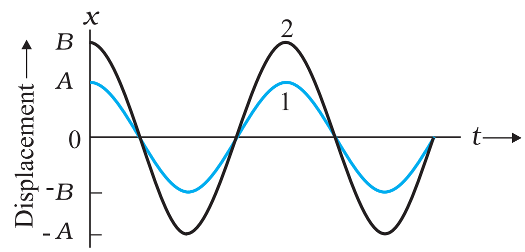


Fig. 14.7 (a) *A plot of displacement as a function of time as obtained from Eq. (14.4) with ϕ = 0. The curves 1 and 2 are for two different amplitudes A and B.*

the shape of the graph are displayed in Fig. 14.6 along with their names.

We shall now define these quantities.

The quantity A is called the **amplitude** of the motion. It is a positive constant which represents the magnitude of the maximum displacement of the particle in either direction. The cosine function in Eq. (14.4) varies between the limits ±1, so the displacement $x(t)$ varies between the limits ± A. In Fig. 14.7 (a), the curves 1 and 2 are plots of Eq. (14.4) for two different amplitudes A and B. The difference between these curves illustrates the significance of amplitude.

The time varying quantity, $(\omega t + \phi)$, in Eq. (14.4) is called the **phase** of the motion. It describes the state of motion at a given time. The constant ϕ is called the **phase constant** (or **phase angle**). The value of ϕ depends on the displacement and velocity of the particle at t = 0. This can be understood better by considering Fig. 14.7(b). In this figure, the curves 3 and 4 represent plots of Eq. (14.4) for two values of the phase constant ϕ.

It can be seen that the phase constant signifies the initial conditions.

The constant ω, called the angular frequency of the motion, is related to the period T. To get

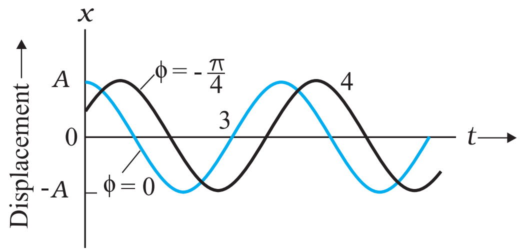


Fig. 14.7 (b) *A plot obtained from Eq. 14.4. The curves 3 and 4 are for ϕ = 0 and -π/4 respectively. The amplitude A is same for both the plots.*

their relationship, let us consider Eq. (14.4) with ϕ = 0; it then reduces to,

$$x(t) = A \cos \omega t \qquad (14.5)$$

Now since the motion is periodic with a period T, the displacement $x(t)$ must return to its initial value after one period of the motion; that is, **$x(t)$ must be equal to $x(t + T)$** for all t. Applying this condition to Eq. (14.5) leads to,

$$A \cos \omega t = A \cos \omega(t + T) \qquad (14.6)$$

As the cosine function first repeats itself when its argument (the phase) has increased by 2π, Eq. (14.6) gives,

$$\omega(t + T) = \omega t + 2\pi$$

or $\qquad \omega T = 2\pi$

Thus, the angular frequency is,

$$\omega = 2\pi / T \qquad (14.7)$$

The SI unit of angular frequency is radians per second. To illustrate the significance of period T, sinusoidal functions with two different periods are plotted in Fig. 14.8.

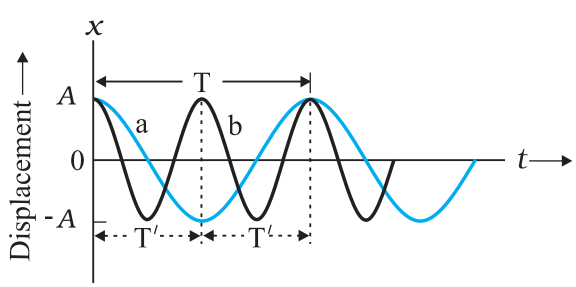


Fig. 14.8 *Plots of Eq. (14.4) for ϕ = 0 for two different periods.*

In this plot the SHM represented by curve *a*, has a period T and that represented by curve *b*, has a period $T' = T/2$.

We have had an introduction to simple harmonic motion. In the next section we will discuss the simplest example of simple harmonic motion. It will be shown that the projection of uniform circular motion on a diameter of the circle executes simple harmonic motion.

Example 14.3 Which of the following functions of time represent (a) simple harmonic motion and (b) periodic but not simple harmonic? Give the period for each case.

(1) $\sin \omega t - \cos \omega t$

(2) $\sin^2 \omega t$

Answer

(a) $\sin \omega t - \cos \omega t$

$$= \sin \omega t - \sin (\pi/2 - \omega t)$$

$$= 2 \cos (\pi/4) \sin (\omega t - \pi/4)$$

$$= \sqrt{2} \sin (\omega t - \pi/4)$$

This function represents a simple harmonic motion having a period $T = 2\pi/\omega$ and a phase angle $(-\pi/4)$ or $(7\pi/4)$

(b) $\sin^2 \omega t$

$$= \tfrac{1}{2} - \tfrac{1}{2} \cos 2\omega t$$

The function is periodic having a period $T = \pi/\omega$ It also represents a harmonic motion with the point of equilibrium occurring at ½ instead of zero.

14.4 SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

In 1610, Galileo discovered four principal moons of the planet Jupiter. To him, each moon seemed to move back and forth relative to the planet in a simple harmonic motion; the disc of the planet forming the mid point of the motion. The record of his observations, written in his own hand, is still available. Based on his data, the position of the moon Callisto relative to Jupiter is plotted in Fig. 14.9. In this figure, the circles represent Galileo's data points and the curve drawn is a best fit to the data. The curve obeys Eq. (14.4), which is the displacement function for SHM. It gives a period of about 16.8 days.

It is now well known that Callisto moves with essentially a constant speed in an almost circular orbit around Jupiter. Its true motion is uniform circular motion. What Galileo saw and what we can also see, with a good pair of binoculars, is the projection of this uniform circular motion on a line in the plane of motion. This can easily be visualised by performing a

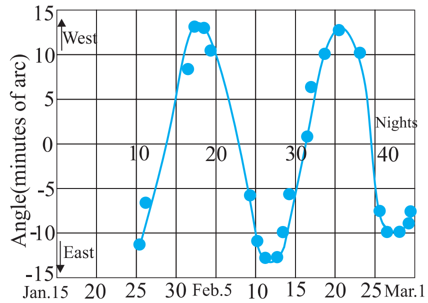


Fig. 14.9 *The angle between Jupiter and its moon Callisto as seen from earth. The circles are based on Galileo's measurements of 1610. The curve is a best fit suggesting a simple harmonic motion. At Jupiter's mean distance, 10 minutes of arc corresponds to about 2×10^6 km.*

simple experiment. Tie a ball to the end of a string and make it move in a horizontal plane about a fixed point with a constant angular speed. The ball would then perform a uniform circular motion in the horizontal plane. Observe the ball sideways or from the front, fixing your attention in the plane of motion. The ball will appear to execute to and fro motion along a horizontal line with the point of rotation as the midpoint. You could alternatively observe the shadow of the ball on a wall which is perpendicular to the plane of the circle. In this process what we are observing is the motion of the ball on a diameter of the circle normal to the direction of viewing. This experiment provides an analogy to Galileo's observation.

In Fig. 14.10, we show the motion of a **reference particle** P executing a uniform circular motion with (constant) angular speed ω in a **reference circle**. The radius A of the circle is the magnitude of the particle's position vector. At any time t, the angular position of the particle

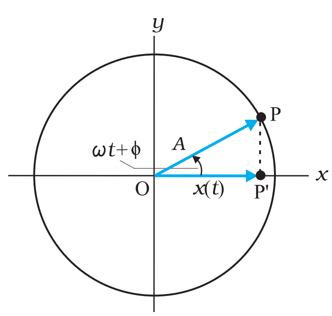


Fig. 14.10 *The motion of a reference particle P executing a uniform circular motion with (constant) angular speed ω in a reference circle of radius A.*

is $\omega t + \phi$, where ϕ is its angular position at $t = 0$. The projection of particle P on the x-axis is a point P′, which we can take as a second particle. The projection of the position vector of particle P on the x-axis gives the location $x(t)$ of P′. Thus we have,

$$x(t) = A \cos (\omega t + \phi)$$

which is the same as Eq. (14.4). This shows that if the reference particle P moves in a uniform circular motion, its projection particle P′ executes a simple harmonic motion along a diameter of the circle.

From Galileo's observation and the above considerations, we are led to the conclusion that circular motion viewed edge-on is simple harmonic motion. In a more formal language we can say that : **Simple harmonic motion is the projection of uniform circular motion on a diameter of the circle in which the latter motion takes place**.

Example 14.4 Fig. 14.11 depicts two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated on the figures. Obtain the simple harmonic motions of the x-projection of the radius vector of the rotating particle P in each case.

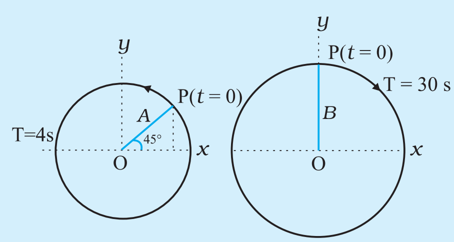


Fig. 14.11

Answer

(a) At $t = 0$, OP makes an angle of $45° = \pi/4$ rad with the (positive direction of) x-axis. After time t, it covers an angle $\frac{2\pi}{T}t$ in the anticlockwise sense, and makes an angle of $\frac{2\pi}{T}t + \frac{\pi}{4}$ with the x-axis.

The projection of OP on the x-axis at time t is given by,

$x(t) = A \cos\left(\frac{2}{T}t + \frac{}{4}\right)$

For $T = 4$ s,

$x(t) = A \cos \quad \frac{2}{4}t + \frac{}{4}$

which is a SHM of amplitude A, period 4 s,

and an initial phase* = $\frac{}{4}$.

(b) In this case at $t = 0$, OP makes an angle of

$90^o = \frac{}{2}$ with the x-axis. After a time t, it

covers an angle of $\frac{2}{T}t$ in the clockwise

sense and makes an angle of $\frac{}{2} - \frac{2}{T}t$

with the x-axis. The projection of OP on the x-axis at time t is given by

$$x(t) = B \cos \quad \frac{}{2} - \frac{2}{T}t$$

$$= B \sin\left(\frac{2}{T}t\right)$$

For $T = 30$ s,

$$x(t) = B \sin \quad \frac{}{15}t$$

Writing this as $x(t) = B \cos \quad \frac{}{15}t - \frac{}{2}$, and comparing with Eq. (14.4). We find that this represents a SHM of amplitude B, period 30 s,

and an initial phase of $\frac{}{2}$. ◀

14.5 VELOCITY AND ACCELERATION IN SIMPLE HARMONIC MOTION

It can be seen easily that the magnitude of velocity, **v**, with which the reference particle P (Fig. 14.10) is moving in a circle is related to its angular speed, ω as

$$v = \omega A \qquad (14.8)$$

where A is the radius of the circle described by the particle P. The magnitude of the velocity vector **v** of the projection particle is ωA ; its projection on the x-axis at any time t, as shown in Fig. 14.12, is

$$v(t) = -\omega A \sin(\omega t + \phi) \qquad (14.9)$$

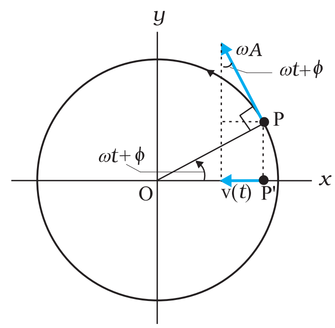


Fig. 14.12 *The velocity, $v(t)$, of the particle P′ is the projection of the velocity v of the reference particle, P.*

The negative sign appears because the velocity component of P is directed towards the left, in the negative direction of x. Equation (14.9) expresses the instantaneous velocity of the particle P′ (projection of P). Therefore, **it expresses the instantaneous velocity of a particle executing SHM.** Equation (14.9) can also be obtained by differentiating Eq. (14.4) with respect to time as,

$$v(t) = \frac{d}{dt}x(t) \qquad (14.10)$$

* *The natural unit of angle is radian, defined through the ratio of arc to radius. Angle is a dimensionless quantity. Therefore it is not always necessary to mention the unit 'radian' when we use π, its multiples or submultiples. The conversion between radian and degree is not similar to that between metre and centimetre or mile. If the argument of a trigonometric function is stated without units, it is understood that the unit is radian. On the other hand, if degree is to be used as the unit of angle, then it must be shown explicitly. For example, sin(15^0) means sine of 15 degree, but sin(15) means sine of 15 radians. Hereafter, we will often drop 'rad' as the unit, and it should be understood that whenever angle is mentioned as a numerical value, without units, it is to be taken as radians.*

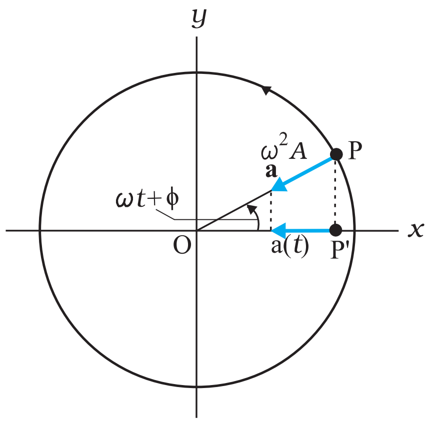


Fig. 14.13 *The acceleration, a(t), of the particle P′ is the projection of the acceleration* **a** *of the reference particle P.*

We have seen that a particle executing a uniform circular motion is subjected to a radial acceleration **a** directed towards the centre. Figure 14.13 shows such a radial acceleration, **a**, of the reference particle P executing uniform circular motion. The magnitude of the radial acceleration of P is $\omega^2 A$. Its projection on the x-axis at any time t is,

$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$
$$= -\omega^2 x(t) \quad (14.11)$$

which is the acceleration of the particle P′ (the projection of particle P). Equation (14.11), therefore, represents the instantaneous acceleration of the particle P′, which is executing **SHM**. Thus Eq. (14.11) **expresses the acceleration of a particle executing SHM**. It is an important result for *SHM*. It shows that in **SHM**, **the acceleration is proportional to the displacement and is always directed towards the mean position**. Eq. (14.11) can also be obtained by differentiating Eq. (14.9) with respect to time as,

$$a(t) = \frac{\mathrm{d}}{\mathrm{d}t} v(t) \quad (14.12)$$

The inter-relationship between the displacement of a particle executing simple harmonic motion, its velocity and acceleration can be seen in Fig. 14.14. In this figure (a) is a plot of Eq. (14.4) with $\phi = 0$ and (b) depicts Eq. (14.9) also with $\phi = 0$. Similar to the amplitude A in Eq. (14.4), the positive quantity ωA in Eq. (14.9) is called the **velocity amplitude** v_m. In Fig. 14.14(b), it can be seen that the velocity of the oscillating particle varies between the limits $\pm v_m = \pm \omega A$. Note that the curve of $v(t)$ is shifted (to the left) from the curve of $x(t)$ by one quarter period and thus the particle velocity lags behind the displacement by a phase angle of $\pi/2$; when the magnitude of displacement is the greatest, the magnitude of the velocity is the least. When the magnitude of displacement is the least, the velocity is the greatest. Figure14.14(c) depicts the variation of the particle acceleration $a(t)$. It is seen that when the displacement has its greatest positive value, the acceleration has its greatest negative value and vice versa. When the displacement is zero, the acceleration is also zero.

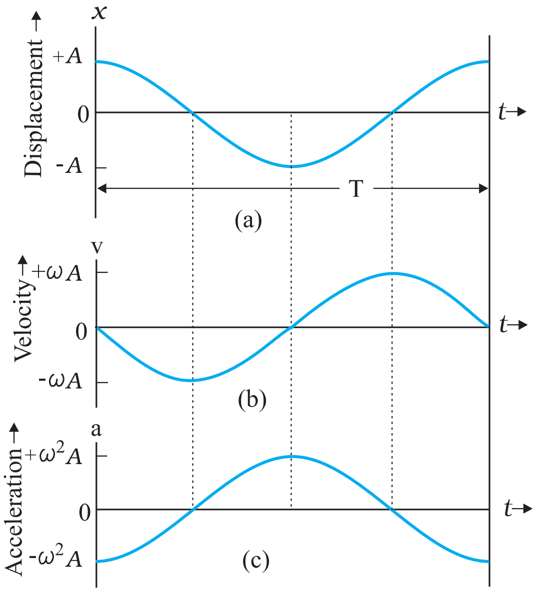


Fig. 14.14 *The particle displacement, velocity and acceleration in a simple harmonic motion. (a) The displacement x (t) of a particle executing SHM with phase angle ϕ equal to zero. (b) The velocity v (t) of the particle. (c) The acceleration a (t) of the particle.*

▶ ***Example 14.5*** A body oscillates with SHM according to the equation (in SI units),

$$x = 5 \cos [2\pi\, t + \pi/4].$$

At $t = 1.5$ s, calculate the (a) displacement, (b) speed and (c) acceleration of the body.

Answer The angular frequency ω of the body $= 2\pi\ \mathrm{s}^{-1}$ and its time period $T = 1$ s.

At $t = 1.5$ s

(a) displacement = $(5.0\ \mathrm{m}) \cos [(2\pi\ \mathrm{s}^{-1}) \times 1.5\ \mathrm{s} + \pi/4]$

$= (5.0 \text{ m}) \cos [(3\pi + \pi/4)]$
$= -5.0 \times 0.707 \text{ m}$
$= -3.535 \text{ m}$

(b) Using Eq. (14.9), the speed of the body
$= -(5.0 \text{ m})(2\pi \text{ s}^{-1}) \sin [(2\pi \text{ s}^{-1}) \times 1.5 \text{ s} + \pi/4]$
$= -(5.0 \text{ m})(2\pi \text{ s}^{-1}) \sin [(3\pi + \pi/4)]$
$= 10\pi \times 0.707 \text{ m s}^{-1}$
$= 22 \text{ m s}^{-1}$

(c) Using Eq.(14.10), the acceleration of the body
$= -(2\pi \text{ s}^{-1})^2 \times$ displacement
$= -(2\pi \text{ s}^{-1})^2 \times (-3.535 \text{ m})$
$= 140 \text{ m s}^{-2}$ ◀

14.6 FORCE LAW FOR SIMPLE HARMONIC MOTION

In Section14.3, we described the simple harmonic motion. Now we discuss how it can be generated. Newton's second law of motion relates the force acting on a system and the corresponding acceleration produced. Therefore, if we know how the acceleration of a particle varies with time, this law can be used to learn about the force, which must act on the particle to give it that acceleration. If we combine Newton's second law and Eq. (14.11), we find that for simple harmonic motion,

$$F(t) = ma$$
$$= -m\omega^2 x(t)$$

or $$F(t) = -k\, x(t) \quad (14.13)$$

where $$k = m\omega^2 \quad (14.14a)$$

or

$$\omega = \sqrt{\frac{k}{m}} \quad (14.14b)$$

Equation (14.13) gives the force acting on the particle. It is proportional to the displacement and directed in an opposite direction. Therefore, it is a restoring force. Note that unlike the centripetal force for uniform circular motion that is constant in magnitude, the restoring force for SHM is time dependent. The force law expressed by Eq. (14.13) can be taken as an alternative definition of simple harmonic motion. It states : **Simple harmonic motion is the motion executed by a particle subject to a force, which is proportional to the displacement of the particle and is directed towards the mean position.**

Since the force F is proportional to x rather than to some other power of x, such a system is also referred to as a **linear harmonic oscillator**. Systems in which the restoring force is a non-linear function of x are termed as non-linear harmonic or anharmonic oscillators.

▶ ***Example 14.6*** Two identical springs of spring constant k are attached to a block of mass m and to fixed supports as shown in Fig. 14.15. Show that when the mass is displaced from its equilibrium position on either side, it executes a simple harmonic motion. Find the period of oscillations.

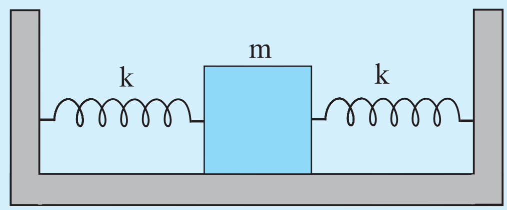


Fig. 14.15

Answer Let the mass be displaced by a small distance x to the right side of the equilibrium position, as shown in Fig. 14.16. Under this situation the spring on the left side gets

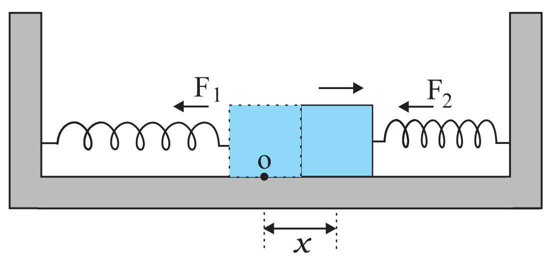


Fig. 14.16

elongated by a length equal to x and that on the right side gets compressed by the same length. The forces acting on the mass are then,

$F_1 = -k\,x$ (force exerted by the spring on the left side, trying to pull the mass towards the mean position)

$F_2 = -k\,x$ (force exerted by the spring on the right side, trying to push the mass towards the mean position)

The net force, F, acting on the mass is then given by,

$$F = -2kx$$

Hence the force acting on the mass is proportional to the displacement and is directed towards the mean position; therefore, the motion executed by the mass is simple harmonic. The time period of oscillations is,

$$T = 2\pi\sqrt{\frac{m}{2k}}$$ ◀

14.7 ENERGY IN SIMPLE HARMONIC MOTION

A particle executing simple harmonic motion has kinetic and potential energies, both varying between the limits, zero and maximum.

In section14.5 we have seen that the velocity of a particle executing SHM, is a periodic function of time. It is zero at the extreme positions of displacement. Therefore, the kinetic energy (K) of such a particle, which is defined as

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\,\omega^2 A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}k\,A^2 \sin^2(\omega t + \phi) \qquad (14.15)$$

is also a periodic function of time, being zero when the displacement is maximum and maximum when the particle is at the mean position. Note, since the sign of v is immaterial in K, the period of K is $T/2$.

What is the potential energy (PE) of a particle executing simple harmonic motion ? In Chapter 6, we have seen that the concept of potential energy is possible only for conservative forces. The spring force $F = -kx$ is a conservative force, with associated potential energy

$$U = \frac{1}{2}k\,x^2 \qquad 14.16)$$

Hence the potential energy of a particle executing simple harmonic motion is,

$$U(x) = \frac{1}{2}k\,x^2$$

$$= \frac{1}{2}k\,A^2 \cos^2(\omega t + \phi) \qquad (14.17)$$

Thus the potential energy of a particle executing simple harmonic motion is also periodic, with period $T/2$, being zero at the mean position and maximum at the extreme displacements.

It follows from Eqs. (14.15) and (14.17) that the total energy, E, of the system is,

$$E = U + K$$

$$= \frac{1}{2}k\,A^2 \cos^2(\omega t + \phi) + \frac{1}{2}k\,A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}k\,A^2 \left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right]$$

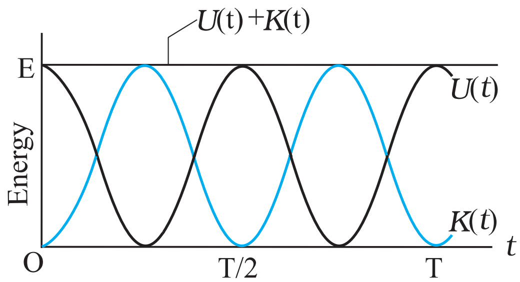
The quantity within the square brackets above is unity and we have,

$$E = \frac{1}{2}k\,A^2 \qquad (14.18)$$

The total mechanical energy of a harmonic oscillator is thus independent of time as expected for motion under any conservative force. The time and displacement dependence of the potential and kinetic energies of a linear simple harmonic oscillator are shown in Fig. 14.17.

It is observed that in a linear harmonic oscillator, all energies are positive and peak twice during every period. For $x = 0$, the energy is all kinetic and for $x = \pm A$ it is all potential.

In between these extreme positions, the potential energy increases at the expense of kinetic energy. This behaviour of a linear harmonic oscillator suggests that it possesses an element of springiness and an element of inertia. The former stores its potential energy and the latter stores its kinetic energy.



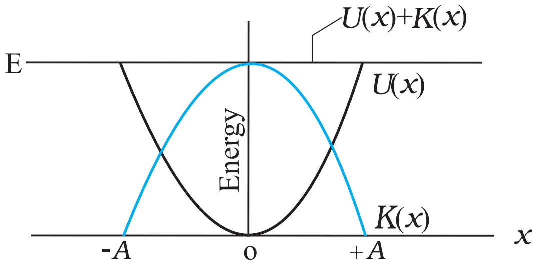


Fig. 14.17 *(a) Potential energy U(t), kinetic energy K(t) and the total energy E as functions of time t for a linear harmonic oscillator. All energies are positive and the potential and kinetic energies peak twice in every period of the oscillator. (b) Potential energy U(x), kinetic energy K(x) and the total energy E as functions of position x for a linear harmonic oscillator with amplitude A. For x = 0, the energy is all kinetic and for x = ± A it is all potential.*

Example 14.7 A block whose mass is 1 kg is fastened to a spring. The spring has a spring constant of 50 N m^{-1}. The block is pulled to a distance x = 10 cm from its equilibrium position at x = 0 on a frictionless surface from rest at t = 0. Calculate the kinetic, potential and total energies of the block when it is 5 cm away from the mean position.

Answer The block executes SHM, its angular frequency, as given by Eq. (14.14b), is

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{50 \text{ N m}^{-1}}{1\text{kg}}}$$

$$= 7.07 \text{ rad s}^{-1}$$

Its displacement at any time t is then given by,

$$x(t) = 0.1 \cos (7.07t)$$

Therefore, when the particle is 5 cm away from the mean position, we have

$$0.05 = 0.1 \cos (7.07t)$$

Or $\cos (7.07t) = 0.5$ and hence

$$\sin (7.07t) = \frac{\sqrt{3}}{2} = 0.866,$$

Then the velocity of the block at x = 5 cm is

$= 0.1 \times 7.07 \times 0.866 \text{ m s}^{-1}$

$= 0.61 \text{ m s}^{-1}$

Hence the K.E. of the block,

$$= \frac{1}{2} m v^2$$

$= ½[1\text{kg} \times (0.6123 \text{ m s}^{-1})^2]$

$= 0.19$ J

The P.E. of the block,

$$= \frac{1}{2} k x^2$$

$= ½(50 \text{ N m}^{-1} \times 0.05 \text{ m} \times 0.05 \text{ m})$

$= 0.0625$ J

The total energy of the block at x = 5 cm,

= K.E. + P.E.

= 0.25 J

we also know that at maximum displacement, K.E. is zero and hence the total energy of the system is equal to the P.E. Therefore, the total energy of the system,

$= ½(50 \text{ N m}^{-1} \times 0.1 \text{ m} \times 0.1 \text{ m})$

$= 0.25$ J

which is same as the sum of the two energies at a displacement of 5 cm. This is in conformity with the principle of conservation of energy. ◀

14.8 SOME SYSTEMS EXECUTING SIMPLE HARMONIC MOTION

There are no physical examples of absolutely pure **simple harmonic motion**. In practice we come across systems that execute simple harmonic motion approximately under certain conditions. In the subsequent part of this section, we discuss the motion executed by some such systems.

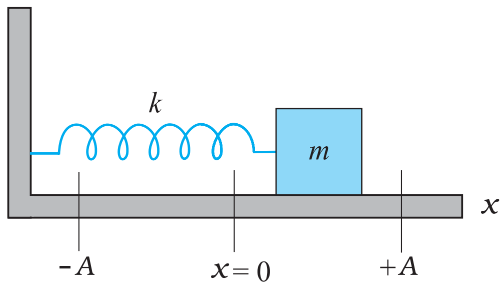


Fig. 14.18 *A linear simple harmonic oscillator consisting of a block of mass m attached to a spring. The block moves over a frictionless surface. Once pulled to the side and released, it executes simple harmonic motion.*

14.8.1 Oscillations due to a Spring

The simplest observable example of simple harmonic motion is the small oscillations of a block of mass m fixed to a spring, which in turn is fixed to a rigid wall as shown in Fig. 14.18. The block is placed on a frictionless horizontal surface. If the block is pulled on one side and is released, it then executes a to and fro motion about a mean position. Let $x = 0$, indicate the position of the centre of the block when the spring is in equilibrium. The positions marked as $-A$ and $+A$ indicate the maximum displacements to the left and the right of the mean position. We have already learnt that springs have special properties, which were first discovered by the English physicist Robert Hooke. He had shown that such a system when deformed, is subject to a restoring force, the magnitude of which is proportional to the deformation or the displacement and acts in opposite direction. This is known as Hooke's law (Chapter 9). It holds good for displacements small in comparison to the length of the spring. At any time t, if the displacement of the block from its mean position is x, the restoring force F acting on the block is,

$$F(x) = -k\,x \qquad (14.19)$$

The constant of proportionality, k, is called the spring constant, its value is governed by the elastic properties of the spring. A stiff spring has large k and a soft spring has small k. Equation (14.19) is same as the force law for SHM and therefore the system executes a simple harmonic motion. From Eq. (14.14) we have,

$$\omega = \sqrt{\frac{k}{m}} \qquad (14.20)$$

and the period, T, of the oscillator is given by,

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad (14.21)$$

Equations (14.20) and (14.21) tell us that a large angular frequency and hence a small period is associated with a stiff spring (high k) and a light block (small m).

Example 14.8 A 5 kg collar is attached to a spring of spring constant 500 N m^{-1}. It slides without friction over a horizontal rod. The collar is displaced from its equilibrium position by 10.0 cm and released. Calculate
(a) the period of oscillation,
(b) the maximum speed and
(c) maximum acceleration of the collar.

Answer (a) The period of oscillation as given by Eq. (14.21) is,

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi\sqrt{\frac{5.0\text{ kg}}{500\text{ N m}^{-1}}}$$

$$= (2\pi/10)\text{ s}$$

$$= 0.63\text{ s}$$

(b) The velocity of the collar executing SHM is given by,

$$v(t) = -A\omega\sin(\omega t + \phi)$$

The maximum speed is given by,

$$v_m = A\omega$$

$$= 0.1 \times \sqrt{\frac{k}{m}}$$

$$= 0.1 \times \sqrt{\frac{500\text{ N m}^{-1}}{5\text{ kg}}}$$

$$= 1\text{ m s}^{-1}$$

and it occurs at $x = 0$

(c) The acceleration of the collar at the displacement $x(t)$ from the equilibrium is given by,

$$a(t) = -\omega^2 x(t)$$

$$= -\frac{k}{m}\,x(t)$$

Therefore the maximum acceleration is,

$$a_{max} = \omega^2 A$$

$$= \frac{500 \text{ N m}^{-1}}{5 \text{ kg}} \text{ x } 0.1 \text{ m}$$

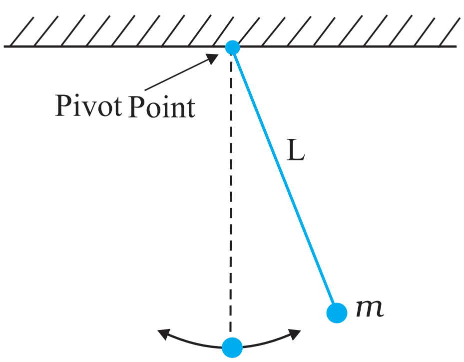
$$= 10 \text{ m s}^{-2}$$

and it occurs at the extremities. ◀

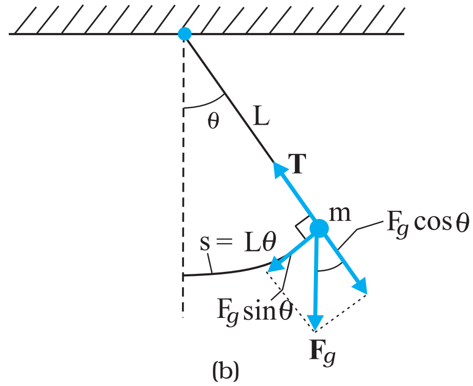
14.8.2 The Simple Pendulum

It is said that Galileo measured the periods of a swinging chandelier in a church by his pulse beats. He observed that the motion of the chandelier was periodic. The system is a kind of pendulum. You can also make your own pendulum by tying a piece of stone to a long unstretchable thread, approximately 100 cm long. Suspend your pendulum from a suitable support so that it is free to oscillate. Displace the stone to one side by a small distance and let it go. The stone executes a to and fro motion, it is periodic with a period of about two seconds. Is this motion simple harmonic ? To answer this question, we consider a **simple pendulum**, which consists of a particle of mass *m* (called the bob of the pendulum) suspended from one end of an unstretchable, massless string of length *L* fixed at the other end as shown in Fig. 14.19(a). The bob is free to swing to and fro in the plane of the page, to the left and right of a vertical line through the pivot point.

The forces acting on the bob are the force **T**, tension in the string and the gravitational force $\mathbf{F_g}$ (= $m\,\mathbf{g}$), as shown in Fig. 14.19(b). The string makes an angle θ with the vertical. We resolve the force $\mathbf{F_g}$ into a radial component $F_g \cos\theta$ and a tangential component $F_g \sin\theta$. The radial component is cancelled by the tension, since there is no motion along the length of the string. The tangential component produces a restoring torque about the pendulum's pivot point. This



(a)



(b)

Fig. 14.19 *(a) A simple pendulum. (b) The forces acting on the bob are the force due to gravity, F_g (= m g), and the tension T in the string. (b) The tangential component F_g of the gravitational force is a restoring force that tends to bring the pendulum back to the central position.*

torque always acts opposite to the displacement of the bob so as to bring it back towards its central location. The central location is called the **equilibrium position** (θ = 0), because at this position the pendulum would be at rest if it were not swinging.

The restoring torque τ is given by,

$$\tau = -L\,(F_g \sin\theta) \tag{14.22}$$

where the negative sign indicates that the torque acts to reduce θ, and L is the length of the moment arm of the force $F_g \sin\theta$ about the pivot point. For rotational motion we have,

$$\tau = I\,\alpha \tag{14.23}$$

where I is the pendulum's rotational inertia about the pivot point and α is its angular acceleration about that point. From Eqs. (14.22) and (14.23) we have,

$$-L\,(F_g \sin\theta) = I\,\alpha \tag{14.24}$$

Substituting the magnitude of F_g, i.e. mg, we have,

$$-L\,m\,g \sin\theta = I\,\alpha$$

Or,

$$\alpha = \frac{m\,g\,L}{I} \sin \tag{14.25}$$

We can simplify Eq. (14.25) if we assume that the displacement θ is small. We know that $\sin\theta$ can be expressed as,

$$\sin \qquad \frac{\ }{3!} \quad \frac{\ }{5!} \pm \ldots \tag{14.26}$$

where θ is in radians.

Now if θ is small, sin θ can be approximated by θ and Eq. (14.25) can then be written as,

$$\alpha = -\frac{mgL}{I}\theta \qquad (14.27)$$

In Table 14.1, we have listed the angle θ in degrees, its equivalent in radians, and the value of the function sin θ . From this table it can be seen that for θ as large as 20 degrees, sin θ is nearly the same as θ **expressed in radians**.

Table 14.1 **sin θ as a function of angle θ**

θ (degrees)	θ (radians)	sin θ
0	0	0
5	0.087	0.087
10	0.174	0.174
15	0.262	0.256
20	0.349	0.342

Equation (14.27) is the angular analogue of Eq. (14.11) and tells us that the angular acceleration of the pendulum is proportional to the angular displacement θ but opposite in sign. Thus as the pendulum moves to the right, its pull to the left increases until it stops and begins to return to the left. Similarly, when it moves towards left, its acceleration to the right tends to return it to the right and so on, as it swings to and fro in SHM. Thus the motion of a **simple pendulum swinging through small angles is approximately SHM**.

Comparing Eq. (14.27) with Eq. (14.11), we see that the angular frequency of the pendulum is,

$$\omega = \sqrt{\frac{mgL}{I}}$$

and the period of the pendulum, T, is given by,

$$T \qquad \sqrt{\frac{I}{mgL}} \qquad (14.28)$$

All the mass of a simple pendulum is centred in the mass m of the bob, which is at a radius of L from the pivot point. Therefore, for this system, we can write $I = m\,L^2$ and substituting this in Eq. (14.28) we get,

SHM - how small should the amplitude be?

When you perform the experiment to determine the time period of a simple pendulum, your teacher tells you to keep the amplitude small. But have you ever asked how small is small? Should the amplitude to 5^0, 2^0· 1^0, or 0.5^0? Or could it be 10^0, 20^0, or 30^0?

To appreciate this, it would be better to measure the time period for different amplitudes, up to large amplitudes. Of course, for large oscillations, you will have to take care that the pendulum oscillates in a vertical plane. Let us denote the time period for small-amplitude oscillations as $T(0)$ and write the time period for amplitude θ_0 as $T(\theta_0) = cT(0)$, where c is the multiplying factor. If you plot a graph of c versus θ_0, you will get values somewhat like this:

θ_0	:	20^0	45^0	50^0	70^0	90^0
c	:	1.02	1.04	1.05	1.10	1.18

This means that the error in the time period is about 2% at an amplitude of 20^0, 5% at an amplitude of 50^0, and 10% at an amplitude of 70^0 and 18% at an amplitude of 90^0.

In the experiment, you will never be able to measure $T(0)$ because this means there are no oscillations. Even theoretically, sin θ is exactly equal to θ only for $\theta = 0$. There will be some inaccuracy for all other values of θ . The difference increases with increasing θ . Therefore we have to decide how much error we can tolerate. No measurement is ever perfectly accurate. You must also consider questions like these: What is the accuracy of the stopwatch? What is your own accuracy in starting and stopping the stopwatch? You will realise that the accuracy in your measurements at this level is never better than 5% or 10%. Since the above table shows that the time period of the pendulum increases hardly by 5% at an amplitude of 50^0 over its low amplitude value, you could very well keep the amplitude to be 50° in your experiments.

$$T \qquad \sqrt{\frac{L}{g}} \qquad (14.29)$$

Equation (14.29) represents a simple expression for the time period of a simple pendulum.

Example 14.9 What is the length of a simple pendulum, which ticks seconds ?

Answer From Eq. (14.29), the time period of a simple pendulum is given by,

$$T \qquad \sqrt{\frac{L}{g}}$$

From this relation one gets,

$$L \quad \frac{gT^2}{4 \quad ^2}$$

The time period of a simple pendulum, which ticks seconds, is 2 s. Therefore, for $g = 9.8$ m s^{-2} and $T = 2$ s, L is

$$\frac{9.8(\text{m s}^{-2}) \quad 4(\text{s}^2)}{4 \quad ^2}$$

$= 1$ m

14.9 DAMPED SIMPLE HARMONIC MOTION

We know that the motion of a simple pendulum, swinging in air, dies out eventually. Why does it happen ? This is because the air drag and the friction at the support oppose the motion of the pendulum and dissipate its energy gradually. The pendulum is said to execute **damped oscillations**. In damped oscillations, although the energy of the system is continuously dissipated, the oscillations remain apparently periodic. The dissipating forces are generally the frictional forces. To understand the effect of such external forces on the motion of an oscillator, let us consider a system as shown in Fig. 14.20. Here a block of mass m oscillates vertically on a spring with spring constant k. The block is connected to a vane through a rod (the vane and the rod are considered to be massless). The vane is submerged in a liquid. As the block oscillates up and down, the vane also moves along with it in the liquid. The up and down motion of the vane displaces the liquid, which in turn, exerts an inhibiting drag force (viscous drag) on it and thus on the entire oscillating system. With time, the mechanical energy of the block-spring system decreases, as energy is transferred to the thermal energy of the liquid and vane.

Let the damping force exerted by the liquid on the system be* $\mathbf{F}_d$. Its magnitude is proportional to the velocity $\mathbf{v}$ of the vane or the block. The force acts in a direction opposite to the direction of $\mathbf{v}$. This assumption is valid only when the vane moves slowly. Then for the motion along the x-axis (vertical direction as shown in Fig. 14.20), we have

$$\mathbf{F}_d = -b\,\mathbf{v} \qquad (14.30)$$

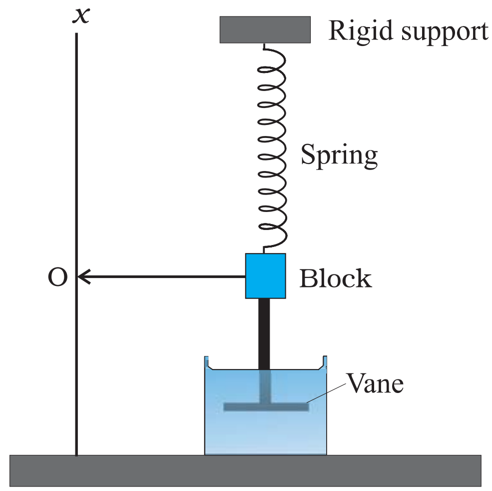


Fig. 14.20 *A damped simple harmonic oscillator. The vane immersed in a liquid exerts a damping force on the block as it oscillates up and down.*

where b is a **damping constant** that depends on the characteristics of the liquid and the vane. The negative sign makes it clear that the force is opposite to the velocity at every moment.

When the mass m is attached to the spring and released, the spring will elongate a little and the mass will settle at some height. This position, shown by O in Fig 14.20, is the equilibrium position of the mass. If the mass is pulled down or pushed up a little, the restoring force on the block due to the spring is $\mathbf{F}_S = -k\mathbf{x}$, where $\mathbf{x}$ is the displacement of the mass from its equilibrium position. Thus the total force acting

* *Under gravity, the block will be at a certain equilibrium position O on the spring; x here represents the displacement from that position.*

on the mass at any time t is $\mathbf{F} = -k\,\mathbf{x} - b\,\mathbf{v}$. If $\mathbf{a}(t)$ is the acceleration of the mass at time t, then by Newton's second law of motion for force components along the x-axis, we have

$$m\,a(t) = -k\,x(t) - b\,v(t) \qquad (14.31)$$

Here we have dropped the vector notation because we are discussing one-dimensional motion. Substituting dx/dt for $v(t)$ and d^2x/dt^2 for the acceleration $a(t)$ and rearranging gives us the differential equation,

$$m\frac{d^2x}{dt^2} \quad b\frac{dx}{dt} \quad k\,x \quad 0 \qquad (14.32)$$

The solution of Eq. (14.32) describes the motion of the block under the influence of a damping force which is proportional to velocity. The solution is found to be of the form

$$x(t) \;=\; A\,e^{-b\,t/2m}\cos(\omega' t + \phi) \qquad (14.33)$$

where a is the amplitude and ω' is the angular frequency of the damped oscillator given by,

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad (14.34)$$

In this function, the cosine function has a period $2\pi/\omega'$ but the function $x(t)$ is not strictly periodic because of the factor $e^{-b\,t/2m}$ which decreases continuously with time. However, if the decrease is small in one time period T, the motion represented by Eq. (14.33) is approximately periodic.

The solution, Eq. (14.33), can be graphically represented as shown in Fig. 14.21. We can

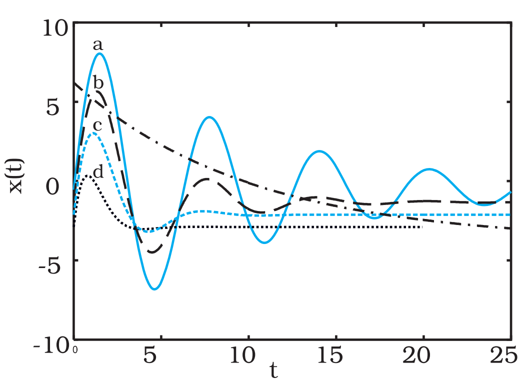


Fig. 14.21 *Displacement as a function of time in damped harmonic oscillations. Damping goes on increasing successively from curve* **a** *to* **d**.

regard it as a cosine function whose amplitude, which is $Ae^{-b\,t/2m}$, gradually decreases with time.

If $b = 0$ (there is no damping), then Eqs. (14.33) and (14.34) reduce to Eqs. (14.4) and (14.14b), expressions for the displacement and angular frequency of an undamped oscillator. We have seen that the mechanical energy of an undamped oscillator is constant and is given by Eq. (14.18) ($E = 1/2\;k\,A^2$). If the oscillator is damped, the mechanical energy is not constant but decreases with time. If the damping is small, we can find $E(t)$ by replacing A in Eq. (14.18) by $Ae^{-bt/2m}$, the amplitude of the damped oscillations. Thus we find,

$$E(t) = \frac{1}{2}k\,A^2\,e^{-b\,t/m} \qquad (14.35)$$

Equation (14.35) shows that the total energy of the system decreases exponentially with time. Note that small damping means that the dimensionless ratio $\left(\frac{b}{\sqrt{km}}\right)$ is much less than 1.

▶ *Example 14.10* For the damped oscillator shown in Fig. 14.20, the mass m of the block is 200 g, k = 90 N m^{-1} and the damping constant b is 40 g s^{-1}. Calculate (a) the period of oscillation, (b) time taken for its amplitude of vibrations to drop to half of its initial value and (c) the taken for its mechanical energy to drop to half its initial value.

Answer (a) We see that km = 90×0.2 = 18 kg N m^{-1} = kg^2 s^{-2}; therefore $\sqrt{km}$ = 4.243 kg s^{-1}, and b = 0.04 kg s^{-1}. Therefore b is much less than $\sqrt{km}$. Hence the time period T from Eq. (14.34) is given by

$$T \quad 2\sqrt{\frac{m}{k}}$$

$$2\sqrt{\frac{0.2\ kg}{90\ \text{N m}^{\pm 1}}}$$

$$= 0.3\ \text{s}$$

(b) Now, from Eq. (14.33), the time, $T_{1/2}$, for the amplitude to drop to half of its initial value is given by,

$$T_{1/2} = \frac{\ln(1/2)}{\boldsymbol{b/2m}}$$

$$\frac{0.693}{40} \quad 2 \quad 200 \text{ s}$$

$$= 6.93 \text{ s}$$

(c) For calculating the time, $t_{1/2}$, for its mechanical energy to drop to half its initial value we make use of Eq. (14.35). From this equation we have,

$$E\,(t_{1/2})/E\,(0) \;=\; \exp\,(-bt_{1/2}/m)$$

Or $$\tfrac{1}{2} \;=\; \exp\,(-bt_{1/2}/m)$$

$$\ln\,(1/2) \;=\; -(bt_{1/2}/m)$$

Or $$t_{1/2} \quad \frac{0.693}{40 \text{ g s}^{-1}} \quad 200 \text{ g}$$

$$= \; 3.46 \text{ s}$$

This is just half of the decay period for amplitude. This is not suprising, because, according to Eqs. (14.33) and (14.35), energy depends on the square of the amplitude. Notice that there is a factor of 2 in the exponents of the two exponentials. ◀

14.10 FORCED OSCILLATIONS AND RESONANCE

A person swinging in a swing without anyone pushing it or a simple pendulum, displaced and released, are examples of free oscillations. In both the cases, the amplitude of swing will gradually decrease and the system would, ultimately, come to a halt. Because of the ever-present dissipative forces, the free oscillations cannot be sustained in practice. They get damped as seen in section 14.9. However, while swinging in a swing if you apply a push periodically by pressing your feet against the ground, you find that not only the oscillations can now be maintained but the amplitude can also be increased. Under this condition the swing has **forced,** or **driven, oscillations**. In case of a system executing driven oscillations under the action of a harmonic force, two angular frequencies are important : (1) **the natural** angular frequency ω of the system, which is the angular frequency at which it will oscillate if it were displaced from equilibrium position and then left to oscillate freely, and (2) the angular frequency ω_d of the external force causing the driven oscillations.

Suppose an external force $F(t)$ of amplitude F_0 that varies periodically with time is applied to a damped oscillator. Such a force can be represented as,

$$F(t) \;=\; F_o \cos \omega_d t \qquad (14.36)$$

The motion of a particle under the combined action of a linear restoring force, damping force and a time dependent driving force represented by Eq. (14.36) is given by,

$$m\,a(t) = -k\,x(t) - bv(t) + F_o \cos \omega_d t \qquad (14.37a)$$

Substituting d^2x/dt^2 for acceleration in Eq. (14.37a) and rearranging it, we get

$$m\frac{d^2x}{dt^2} \quad b\frac{dx}{dt} \quad kx \quad F_o \cos \omega_d t \qquad (14.37b)$$

This is the equation of an oscillator of mass m on which a periodic force of (angular) frequency ω_d is applied. The oscillator initially oscillates with its natural frequency ω. When we apply the external periodic force, the oscillations with the natural frequency die out, and then the body oscillates with the (angular) frequency of the external periodic force. Its displacement, after the natural oscillations die out, is given by

$$x(t) \;=\; A\cos\,(\omega_d t \;+\; \phi) \qquad (14.38)$$

where t is the time measured from the moment when we apply the periodic force.

The amplitude A is a function of the forced frequency ω_d and the natural frequency ω. Analysis shows that it is given by

$$A \quad \frac{F}{m \qquad {}_d^{\;2} \qquad {}_d b}^{\,1/2} \qquad (14.39a)$$

and $$\tan\phi = \frac{-v_o}{\omega_d x_o} \qquad (14.39b)$$

where m is the mass of the particle and v_0 and x_0 are the velocity and the displacement of the particle at time $t = 0$, which is the moment when we apply the periodic force. Equation (14.39) shows that the amplitude of the forced oscillator depends on the (angular) frequency of the driving force. We can see a different behaviour of the oscillator when ω_d is far from ω and when it is close to ω. We consider these two cases.

(a) **Small Damping, Driving Frequency far from Natural Frequency :** In this case, $\omega_d b$ will be much smaller than $m(\omega^2 - \omega_d^2)$, and we can neglect that term. Then Eq. (14.39) reduces to

$$A \quad \frac{F}{m \qquad {}^{2}_{d}} \tag{14.40}$$

Figure 14.22 shows the dependence of the displacement amplitude of an oscillator on the angular frequency of the driving force for different amounts of damping present in the system. It may be noted that in all the cases the amplitude is greatest when $\omega_d / \omega = 1$. The curves in this figure show that smaller the damping, the taller and narrower is the resonance peak.

If we go on changing the driving frequency, the amplitude tends to infinity when it equals the natural frequency. But this is the ideal case of zero damping, a case which never arises in a real system as the damping is never perfectly zero. You must have experienced in a swing that when the timing of your push exactly matches with the time period of the swing, your swing gets the maximum amplitude. This amplitude is large, but not infinity, because there is always some damping in your swing. This will become clear in the (b).

(b) **Driving Frequency Close to Natural Frequency :** If ω_d is very close to ω, $m(\omega^2 - {}^{2}_{d})$

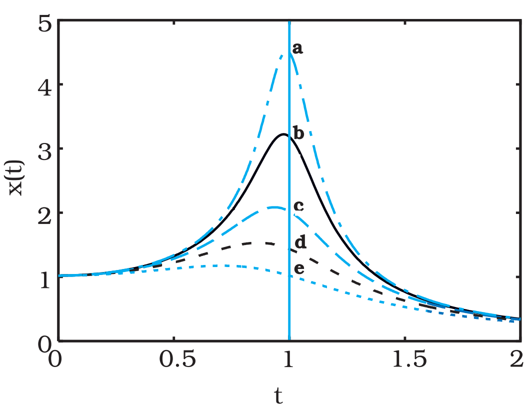


Fig. 14.22 *The amplitude of a forced oscillator as a function of the angular frequency of the driving force. The amplitude is greatest near $\omega_d / \omega = 1$. The five curves correspond to different extents of damping present in the system. Curve **a** corresponds to the least damping, and damping goes on increasing successively in curves **b, c, d, e**. Notice that the peak shifts to the left with increasing **b**.*

would be much less than $\omega_d b$, for any reasonable value of b, then Eq. (14.39) reduces to

$$A = \frac{F_o}{\omega_d b} \tag{14.41}$$

This makes it clear that the maximum possible amplitude for a given driving frequency is governed by the driving frequency and the damping, and is never infinity. The phenomenon of increase in amplitude when the driving force is close to the natural frequency of the oscillator is called **resonance**.

In our daily life we encounter phenomena which involve resonance. Your experience with swings is a good example of resonance. You might have realised that the skill in swinging to greater heights lies in the synchronisation of the rhythm of pushing against the ground with the natural frequency of the swing.

To illustrate this point further, let us consider a set of five simple pendulums of assorted lengths suspended from a common rope as shown in Fig. 14.23. The pendulums 1 and 4 have the same lengths and the others have different lengths. Now let us set pendulum 1 into motion. The energy from this pendulum gets transferred to other pendulums through the connecting rope and they start oscillating. The driving force is provided through the connecting rope. The frequency of this force is the frequency with which pendulum 1 oscillates. If we observe the response of pendulums 2, 3 and 5, they first start oscillating with their natural frequencies

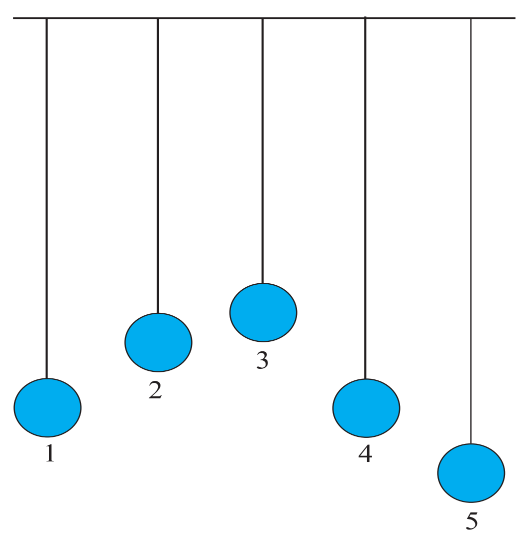


Fig. 14.23 *A system of five simple pendulums suspended from a common rope.*

of oscillations and different amplitudes, but this motion is gradually damped and not sustained. Their frequencies of oscillation gradually change and ultimately they oscillate with the frequency of pendulum 1, i.e. the frequency of the driving force but with different amplitudes. They oscillate with small amplitudes. The response of pendulum 4 is in contrast to this set of pendulums. It oscillates with the same frequency as that of pendulum 1 and its amplitude gradually picks up and becomes very large. A resonance-like response is seen. This happens because in this the condition for resonance is satisfied, i.e. the natural frequency of the system coincides with that of the driving force.

All mechanical structures have one or more natural frequencies, and if a structure is subjected to a strong external periodic driving force that matches one of these frequencies, the resulting oscillations of the structure may rupture it. The Tacoma Narrows Bridge at Puget Sound, Washington, USA was opened on July 1, 1940. Four months later winds produced a pulsating resultant force in resonance with the natural frequency of the structure. This caused a steady increase in the amplitude of oscillations until the bridge collapsed. It is for the same reason the marching soldiers break steps while crossing a bridge. Aircraft designers make sure that none of the natural frequencies at which a wing can oscillate match the frequency of the engines in flight. Earthquakes cause vast devastation. It is interesting to note that sometimes, in an earthquake, short and tall structures remain unaffected while the medium height structures fall down. This happens because the natural frequencies of the short structures happen to be higher and those of taller structures lower than the frequency of the seismic waves.

SUMMARY

1. The motions which repeat themselves are called *periodic motions.*
2. The *period T* is the time required for one complete oscillation, or cycle. It is related to the frequency ν by,

$$T = \frac{1}{\nu}$$

The *frequency* ν of periodic or oscillatory motion is the number of oscillations per unit time. In the SI, it is measured in hertz :

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1 \text{s}^{-1}$$

3. In *simple harmonic motion* (SHM), the displacement $x(t)$ of a particle from its equilibrium position is given by,

$$x(t) = A \cos(\omega t + \phi) \qquad \text{(displacement)},$$

in which A is the *amplitude* of the displacement, the quantity $(\omega t + \phi)$ is the phase of the motion, and ϕ is the *phase constant*. The *angular frequency* ω is related to the period and frequency of the motion by,

$$\frac{2}{T} \quad 2 \qquad \text{(angular frequency)}.$$

4. Simple harmonic motion is the projection of uniform circular motion on the diameter of the circle in which the latter motion occurs.
5. The particle velocity and acceleration during SHM as functions of time are given by,

$$v(t) = -\omega A \sin(\omega t + \phi) \qquad \text{(velocity)},$$

$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$

$= \quad -\omega^2 x\,(t)$ (acceleration),

Thus we see that both velocity and acceleration of a body executing simple harmonic motion are periodic functions, having the velocity *amplitude* $v_m = \omega A$ and *acceleration amplitude* $a_m = \omega^2 A$, respectively.

6. The force acting simple harmonic motion is proportional to the displacement and is always directed towards the centre of motion.

7. A particle executing simple harmonic motion has, at any time, kinetic energy $K = ½\, mv^2$ and potential energy $U = ½\, kx^2$. If no friction is present the mechanical energy of the system, $E = K + U$ always remains constant even though K and U change with time

8. A particle of mass m oscillating under the influence of a Hooke's law restoring force given by $F = -\,k\,x$ exhibits simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(angular frequency)}$$

$$T \quad 2\sqrt{\frac{m}{k}} \qquad \text{(period)}$$

Such a system is also called a linear oscillator.

9. The motion of a simple pendulum swinging through small angles is approximately simple harmonic. The period of oscillation is given by,

$$T \quad 2\sqrt{\frac{L}{g}}$$

10. The mechanical energy in a real oscillating system decreases during oscillations because external forces, such as drag, inhibit the oscillations and transfer mechanical energy to thermal energy. The real oscillator and its motion are then said to be *damped*. If the *damping force* is given by $F_d = -bv$, where v is the velocity of the oscillator and b is a *damping constant*, then the displacement of the oscillator is given by,

$$x\,(t) = A\,e^{-bt/2m} \cos\,(\omega' t + \phi)$$

where ω', the angular frequency of the damped oscillator, is given by

$$\sqrt{\frac{k}{m} \quad \frac{b^2}{4m^2}}$$

If the damping constant is small then $\omega' \approx \omega$, where ω is the angular frequency of the undamped oscillator. The mechanical energy E of the damped oscillator is given by

$$E(t) \quad \frac{1}{2}kA^2 e^{\quad bt/m}$$

11. If an external force with angular frequency ω_d acts on an oscillating system with natural angular frequency ω, the system oscillates with angular frequency ω_d. The amplitude of oscillations is the greatest when

$$\omega_d = \omega$$

a condition called *resonance*.

Physical quantity	Symbol	Dimensions	Unit	Remarks
Period	T	[T]	s	The least time for motion to repeat itself
Frequency	ν (or f)	$[T^{-1}]$	s^{-1}	$\nu = \frac{1}{T}$
Angular frequency	ω	$[T^{-1}]$	s^{-1}	$\omega = 2\,\pi\,\nu$
Phase constant	ϕ	Dimensionless	rad	Initial value of phase of displacement in SHM
Force constant	k	$[MT^{-2}]$	$N\ m^{-1}$	Simple harmonic motion $F = -k\,x$

POINTS TO PONDER

1. The period T is the *least time* after which motion repeats itself. Thus, motion repeats itself after nT where n is an integer.
2. Every periodic motion is not simple harmonic motion. Only that periodic motion governed by the force law $F = -\,k\,x$ is simple harmonic.
3. Circular motion can arise due to an inverse-square law force (as in planetary motion) as well as due to simple harmonic force in two dimensions equal to: $-m\omega^2 r$. In the latter case, the phases of motion, in two perpendicular directions (x and y) must differ by $\omega/2$. Thus, a particle subject to a force $-m\omega^2 r$ with initial position (o, A) and velocity (ωA, o) will move uniformly in a circle of radius A.
4. For linear simple harmonic motion with a given ω two arbitrary initial conditions are necessary and sufficient to determine the motion completely. The initial condition may be (i) initial position and initial velocity or (ii) amplitude and phase or (iii) energy and phase.
5. From point 4 above, given amplitude or energy, phase of motion is determined by the initial position or initial velocity.
6. A combination of two simple harmonic motions with arbitrary amplitudes and phases is not necessarily periodic. It is periodic only if frequency of one motion is an integral multiple of the other's frequency. However, a periodic motion can always be expressed as a sum of infinite number of harmonic motions with appropriate amplitudes.
7. The period of SHM does not depend on amplitude or energy or the phase constant. Contrast this with the periods of planetary orbits under gravitation (Kepler's third law).
8. The motion of a simple pendulum is simple harmonic for small angular displacement.
9. For motion of a particle to be simple harmonic, its displacement x must be expressible in either of the following forms :

   $$x = A\cos\omega t + B\sin\omega t$$

   $$x = A\cos(\omega t + \alpha),\ x = B\sin(\omega t + \beta)$$

   The three forms are completely equivalent (any one can be expressed in terms of any other two forms).

   Thus, damped simple harmonic motion [Eq. (14.31)] is not strictly simple harmonic. It is approximately so only for time intervals much less than $2m/b$ where b is the damping constant.
10. In forced oscillations, the steady state motion of the particle (after the force oscillations die out) is simple harmonic motion whose frequency is the frequency of the driving frequency ω_d, not the natural frequency ω of the particle.

11. In the ideal case of zero damping, the amplitude of simple harmonic motion at resonance is infinite. This is no problem since all real systems have some damping, however, small.
12. Under forced oscillation, the phase of harmonic motion of the particle differs from the phase of the driving force.

Exercises

14.1 Which of the following examples represent periodic motion?
(a) A swimmer completing one (return) trip from one bank of a river to the other and back.
(b) A freely suspended bar magnet displaced from its N-S direction and released.
(c) A hydrogen molecule rotating about its center of mass.
(d) An arrow released from a bow.

14.2 Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?
(a) the rotation of earth about its axis.
(b) motion of an oscillating mercury column in a U-tube.
(c) motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.
(d) general vibrations of a polyatomic molecule about its equilibrium position.

14.3 Figure 14.27 depicts four *x-t* plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion) ?

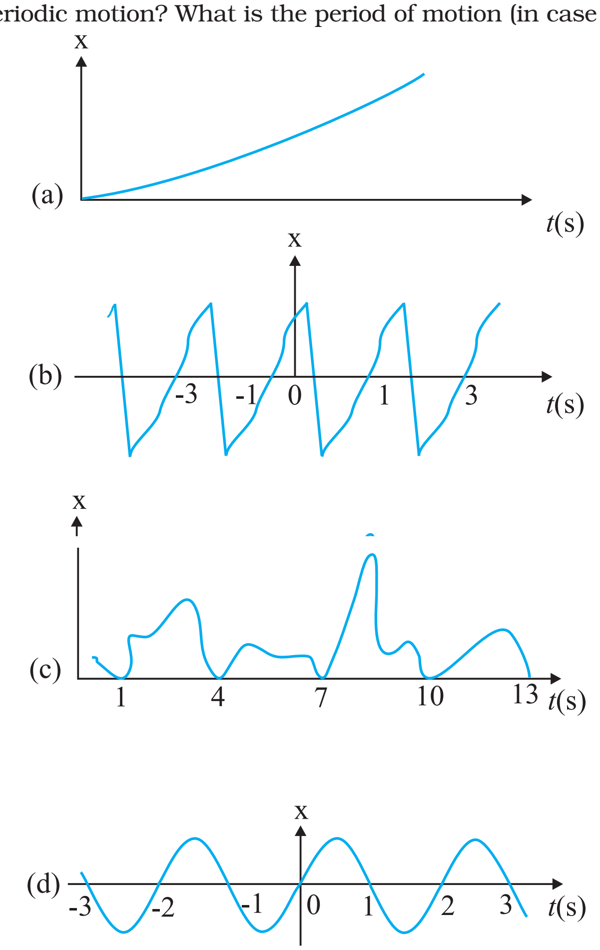


Fig. 14.27

14.4 Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant):
(a) $\sin \omega t - \cos \omega t$
(b) $\sin^3 \omega t$
(c) $3 \cos (\pi/4 - 2\omega t)$
(d) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$
(e) $\exp (-\omega^2 t^2)$
(f) $1 + \omega t + \omega^2 t^2$

14.5 A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is
(a) at the end A,
(b) at the end B,
(c) at the mid-point of AB going towards A,
(d) at 2 cm away from B going towards A,
(e) at 3 cm away from A going towards B, and
(f) at 4 cm away from B going towards A.

14.6 Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?
(a) $a = 0.7x$
(b) $a = -200x^2$
(c) $a = -10x$
(d) $a = 100x^3$

14.7 The motion of a particle executing simple harmonic motion is described by the displacement function,

$$x(t) = A \cos (\omega t + \phi).$$

If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle ? The angular frequency of the particle is $\pi \text{ s}^{-1}$. If instead of the cosine function, we choose the sine function to describe the SHM : $x = B \sin (\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions.

14.8 A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body ?

14.9 A spring having with a spring constant 1200 N m^{-1} is mounted on a horizontal table as shown in Fig. 14.28. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.

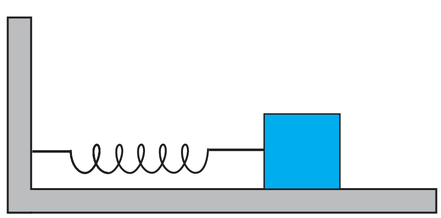


Fig. 14.28

Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass.

14.10 In Exercise 14.9, let us take the position of mass when the spring is unstreched as $x = 0$, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch ($t = 0$), the mass is

(a) at the mean position,
(b) at the maximum stretched position, and
(c) at the maximum compressed position.

In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

14.11 Figures 14.29 correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e. clockwise or anti-clockwise) are indicated on each figure.

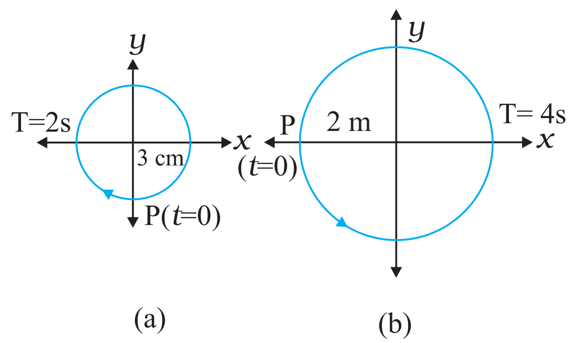


Fig. 14.29

Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P, in each case.

14.12 Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t =0) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s).

(a) $x = -2 \sin (3t + \pi/3)$

(b) $x = \cos (\pi/6 - t)$

(c) $x = 3 \sin (2\pi t + \pi/4)$

(d) $x = 2 \cos \pi t$

14.13 Figure 14.30 (a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force **F** applied at the free end stretches the spring. Figure 14.30 (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. 14.30(b) is stretched by the same force **F**.

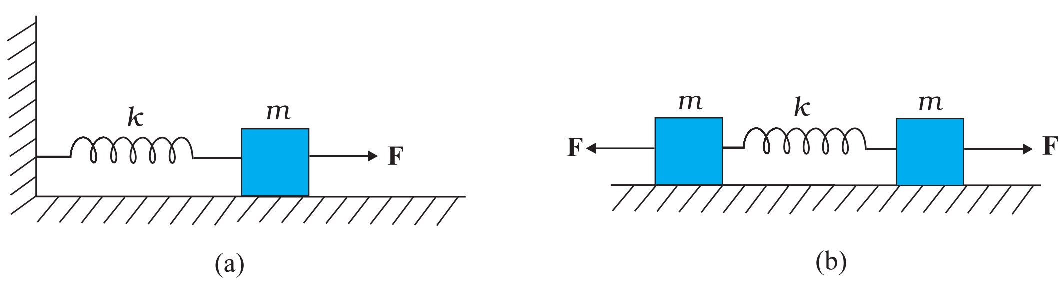


Fig. 14.30

(a) What is the maximum extension of the spring in the two cases ?
(b) If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case ?

14.14 The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed ?

14.15 The acceleration due to gravity on the surface of moon is 1.7 m s^{-2}. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s ? (g on the surface of earth is 9.8 m s^{-2})

14.16 Answer the following questions :

(a) Time period of a particle in SHM depends on the force constant k and mass m of the particle:

$T = 2\pi\sqrt{\frac{m}{k}}$. A simple pendulum executes SHM approximately. Why then is the time period of a pendulum independent of the mass of the pendulum?

(b) The motion of a simple pendulum is approximately simple harmonic for small angle oscillations. For larger angles of oscillation, a more involved analysis shows that T is greater than $2\pi\sqrt{\frac{l}{g}}$. Think of a qualitative argument to appreciate this result.

(c) A man with a wristwatch on his hand falls from the top of a tower. Does the watch give correct time during the free fall ?

(d) What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity ?

14.17 A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period ?

14.18 A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_l. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}$$

where ρ is the density of cork. (Ignore damping due to viscosity of the liquid).

14.19 One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.

Additional Exercises

14.20 An air chamber of volume V has a neck area of cross section a into which a ball of mass m just fits and can move up and down without any friction (Fig.14.33). Show that when the ball is pressed down a little and released , it executes SHM. Obtain an expression for the time period of oscillations assuming pressure-volume variations of air to be isothermal [see Fig. 14.33].

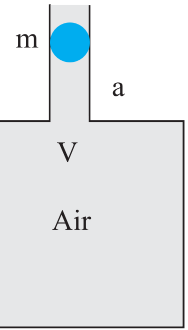


Fig.14.33

14.21 You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

14.22 Show that for a particle in linear SHM the average kinetic energy over a period of oscillation equals the average potential energy over the same period.

14.23 A circular disc of mass 10 kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations is found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire. (Torsional spring constant α is defined by the relation $J = -\alpha\,\theta$, where J is the restoring couple and θ the angle of twist).

14.24 A body describes simple harmonic motion with an amplitude of 5 cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (a) 5 cm, (b) 3 cm, (c) 0 cm.

14.25 A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time $t = 0$. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0 and v_0. [Hint : Start with the equation $x = a \cos(\omega t+\theta)$ and note that the initial velocity is negative.]

Chapter Fifteen

Waves

15.1 INTRODUCTION

In the previous Chapter, we studied the motion of objects oscillating in isolation. What happens in a system, which is a collection of such objects ? A material medium provides such an example. Here, elastic forces bind the constituents to each other and, therefore, the motion of one affects that of the other. If you drop a little pebble in a pond of still water, the water surface gets disturbed. The disturbance does not remain confined to one place, but propagates outward along a circle. If you continue dropping pebbles in the pond, you see circles rapidly moving outward from the point where the water surface is disturbed. It gives a feeling as if the water is moving outward from the point of disturbance. If you put some cork pieces on the disturbed surface, it is seen that the cork pieces move up and down but do not move away from the centre of disturbance. This shows that the water mass does not flow outward with the circles, but rather a moving disturbance is created. Similarly, when we speak, the sound moves outward from us, without any flow of air from one part of the medium to another. The disturbances produced in air are much less obvious and only our ears or a microphone can detect them. These patterns, which move without the actual physical transfer or flow of matter as a whole, are called **waves**. In this Chapter, we will study such waves.

In a wave, information and energy, in the form of signals, propagate from one point to another but no material object makes the journey. All our communications depend on the transmission of signals through waves. When we make a telephone call to a friend at a distant place, a sound wave carries the message from our vocal cords to the telephone. There, an electrical signal is generated which propagates along the copper wire. If the distance is too large, the electrical signal generated may be transformed into a light signal or

electromagnetic waves and transmitted through optical cables or the atmosphere, possibly by way of a communication satellite. At the receiving end, the electrical or light signal or the electromagnetic waves are transformed back into sound waves travelling from the telephone to the ear.

Not all waves require a medium for their propagation. We know that light waves can travel through vacuum. The light emitted by stars, which are hundreds of light years away, reaches us through inter-stellar space, which is practically a vacuum.

The waves we come across are mainly of three types: (a) mechanical waves, (b) electromagnetic waves and (c) matter waves. Mechanical waves are most familiar because we encounter them constantly; common examples include water waves, sound waves, seismic waves, etc. All these waves have certain central features : They are governed by Newton's laws, and can exist only within a material medium, such as water, air, and rock. The common examples of electromagnetic waves are visible and ultra-violet light, radio waves, microwaves, x-rays etc. All electromagnetic waves travel through vacuum at the same speed *c*, given by

$$c = 299,792,458 \text{ m s}^{-1} \quad \text{(speed of light)} \qquad (15.1)$$

Unlike the mechanical waves, the electro-magnetic waves do not require any medium for their propagation. You would learn more about these waves later.

Matter waves are associated with moving electrons, protons, neutrons and other fundamental particles, and even atoms and molecules. Because we commonly think of these as constituting matter, such waves are called **matter waves**. They arise in quantum mechanical description of nature that you will learn in your later studies. Though conceptually more abstract than mechanical or electro-magnetic waves, they have already found applications in several devices basic to modern technology; matter waves associated with electrons are employed in electron microscopes.

In this chapter we will study mechanical waves, which require a material medium for their propagation.

The aesthetic influence of waves on art and literature is seen from very early times; yet the first scientific analysis of wave motion dates back to the seventeenth century. Some of the famous scientists associated with the physics of wave motion are Christiaan Huygens (1629-1695), Robert Hooke and Isaac Newton. The understanding of physics of waves followed the physics of oscillations of masses tied to springs and physics of the simple pendulum. Waves in elastic media are intimately connected with harmonic oscillations. (Stretched strings, coiled springs, air, etc., are examples of elastic media.) We shall illustrate this connection through simple examples.

Consider a collection of springs connected to one another as shown in Fig. 15.1. If the spring at one end is pulled suddenly and released, the disturbance travels to the other end. What has happened ? The first spring is disturbed from its equilibrium length. Since the second spring is connected to the first, it is also stretched or compressed, and so on. The disturbance moves

Fig. 15.1 *A collection of springs connected to each other. The end A is pulled suddenly generating a disturbance, which then propagates to the other end.*

from one end to the other; but each spring only executes small oscillations about its equilibrium position. As a practical example of this situation, consider a stationary train at a railway station. Different bogies of the train are coupled to each other through a spring coupling. When an engine is attached at one end, it gives a push to the bogie next to it; this push is transmitted from one bogie to another without the entire train being bodily displaced.

Now let us consider the propagation of sound waves in air. As the wave passes through air, it compresses or expands a small region of air. This causes a change in the density of that region, say $\delta\rho$, this change induces a change in pressure, δp, in that region. Pressure is force per unit area, so there is a **restoring force proportional** to the disturbance, just like in a spring. In this case, the quantity similar to extension or compression of the spring is the change in density. If a region is compressed, the molecules in that region are packed together, and they tend to move out to the adjoining region, thereby increasing the density or creating compression in the adjoining region. Consequently, the air

in the first region undergoes rarefaction. If a region is comparatively rarefied the surrounding air will rush in making the rarefaction move to the adjoining region. Thus, the compression or rarefaction moves from one region to another, making the propagation of a disturbance possible in air.

In solids, similar arguments can be made. In a crystalline solid, atoms or group of atoms are arranged in a periodic lattice. In these, each atom or group of atoms is in equilibrium, due to forces from the surrounding atoms. Displacing one atom, keeping the others fixed, leads to restoring forces, exactly as in a spring. So we can think of atoms in a lattice as end points, with springs between pairs of them.

In the subsequent sections of this chapter we are going to discuss various characteristic properties of waves.

15.2 TRANSVERSE AND LONGITUDINAL WAVES

Mechanical waves can be transverse or longitudinal depending on the relationship between the directions of disturbance or displacement in the medium and that of the propagation of wave. To differentiate between them let us consider the response of a stretched string fixed at one end. If you give a single up-and-down jerk to the free end of this string, as shown in Fig. 15.2, a wave in the form of a single **pulse** travels along the string. We assume that the string is very long as compared to the size of the pulse, so that the pulse dissipates out by the time it reaches the other end and, therefore, its reflection from the other end may be ignored. The formation and propagation of this pulse is possible because the string is under tension. When you pull your end of the string upwards it begins to pull upwards on the adjacent section of the string, because of the tension between the two sections. As the adjacent section begins to move upwards, it begins to pull the next section upwards, and so on. In the meanwhile you have pulled down your end of the string. As each section moves upwards in turn, it begins to be pulled back downwards by neighbouring sections that are already on the way down. The net result is that a distortion in the shape of the string (the pulse) moves along the string with a certain velocity **v**.

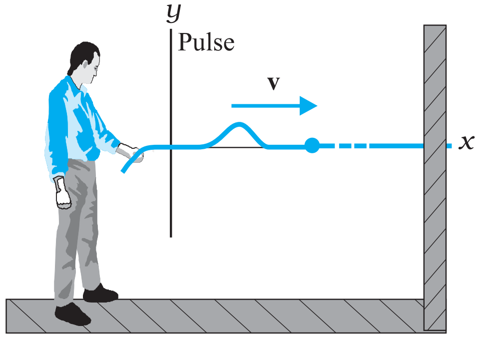


Fig. 15.2 *A single pulse is sent along a stretched string. A typical element of the string (such as that marked with a dot) moves up and then down as the pulse passes through. The element's motion is perpendicular to the direction in which the wave travels.*

If you move your end up and down in a continuous manner, a continuous wave travels along the string with a velocity **v**. However, if the motion of your hand is a sinusoidal function of time, at any given instant of time the wave will have a sinusoidal shape as shown in Fig. 15.3. The wave has the shape of a sine or cosine curve.

The waves shown in Fig. 15.3 can be studied in two ways. Oneway is to monitor the waveforms as they move to the right, i.e. take a 'snapshot' of the string at a given instant of time. Alternatively, we fix our attention to a particular position on the string and monitor the motion of an element at that point as it oscillates up and down while a wave passes through it. We would find that the displacement of every such oscillating string element is transverse (i.e. perpendicular) to the direction of travel of the wave as indicated in Fig. 15.3. Such a wave is said to be a **transverse wave**.

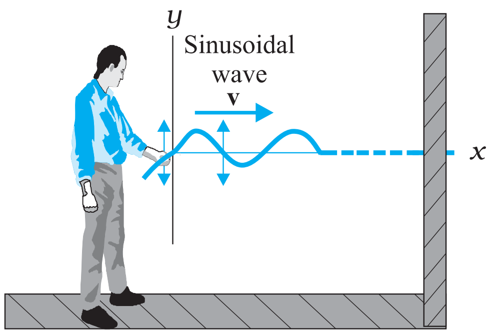


Fig. 15.3 *A sinusoidal wave is sent along the string. A typical element of the string moves up and down continuously as the wave passes. It is a transverse wave.*

Now, let us consider the production of waves in a long air-filled pipe by the movement of a piston as shown in Fig. 15.4. If you suddenly move the piston to the right and then to the left, you are sending a pulse of pressure along the pipe. The motion of the piston to the right pushes

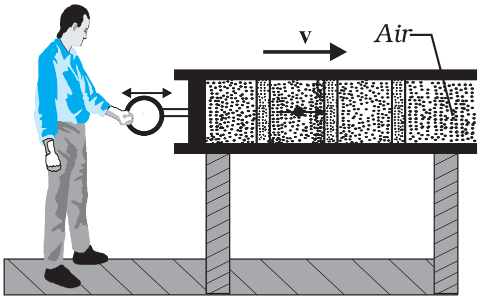


Fig. 15.4 *A sound wave is set up in an air filled pipe by moving a piston back and forth. As the oscillations of an element of air are parallel to the direction in which the wave travels, the wave is a longitudinal wave.*

the elements of air next towards the right, changing the air pressure there. The increased pressure in this region then pushes on the elements of air somewhat farther along the pipe. Moving the piston to the left reduces the air pressure next to it. This causes the elements of air next to it move back to the left and then the farther elements follow. Thus, the motion of the air and the change in air pressure travel towards the right along the pipe as a pulse.

If you push and pull on the piston in a simple harmonic manner, a sinusoidal wave travels along the pipe. It may be noted that the motion of the elements of air is parallel to the direction of propagation of the wave. This motion is said to be **longitudinal** and the wave produced is, therefore, called a **longitudinal wave**. The sound waves produced in air are such pressure waves and are therefore of longitudinal character.

In short, **in transverse waves, the constituents of the medium oscillate perpendicular to the direction of wave propagation and in longitudinal waves they oscillate along the direction of wave propagation**.

A wave, transverse or longitudinal, is said to be **travelling or progressive** if it travels from one point of the medium to another. A progressive wave is to be distinguished from a standing or stationary wave (see Section 15.7). In Fig. 15.3 transverse waves travel from one end of the string to the other end while the longitudinal waves in Fig. 15.4 travel from one end of the pipe to its other end. We note again that in both the cases, it is the wave or the disturbance that moves from end to end, not the material through which the waves propagate.

In transverse waves, the particle motion is normal to the direction of propagation of the wave. Therefore, as the wave propagates, each element of the medium undergoes a shearing strain. Transverse waves can, therefore, be propagated only in those media which can sustain shearing stress, such as solids and strings, and not in fluids. Fluids as well as solids can sustain compressive strain; therefore, longitudinal waves can propagate in all elastic media. For example, in medium like a steel bar, both transverse and longitudinal waves can propagate while air can sustain only longitudinal waves. The waves on the surface of water are of two kinds: **capillary waves** and **gravity waves**. The former are ripples of fairly short wavelength—no more than a few centimetres—and the restoring force that produces them is the surface tension of water. Gravity waves have wavelengths typically ranging from several metres to several hundred metres. The restoring force that produces these waves is the pull of gravity, which tends to keep the water surface at its lowest level. The oscillations of the particles in these waves are not confined to the surface only, but extend with diminishing amplitude to the very bottom. The particle motion in the water waves involves a complicated motion; they not only move up and down but also back and forth. The waves in an ocean are a combination of both longitudinal and transverse waves.

It is found that generally transverse and longitudinal waves travel with different speeds in the same medium.

Example 15.1 Given below are some examples of wave motion. State in each case if the wave motion is transverse, longitudinal or a combination of both:

(a) Motion of a kink in a longitudinal spring produced by displacing one end of the spring sideways.

(b) Waves produced in a cylinder containing a liquid by moving its piston back and forth.

(c) Waves produced by a motorboat sailing in water.

(d) Ultrasonic waves in air produced by a vibrating quartz crystal.

Answer

(a) Transverse and longitudinal
(b) Longitudinal
(c) Transverse and longitudinal
(d) Longitudinal

15.3 DISPLACEMENT RELATION IN A PROGRESSIVE WAVE

To describe the propagation of a wave in a medium (and the motion of any constituent of the medium), we need a function that completely gives the shape of the wave at every instant of time. For example, to completely describe the wave on a string (and the motion of any element along its length) we need a relation which describes the displacement of an element at a particular position as a function of time and also describes the state of vibration of various elements of the string along its length at a given instant of time. For a sinusoidal wave, as shown in Fig. 15.3, this function should be periodic in space as well as in time. Let $y\,(x,\ t)$ denote the transverse displacement of the element at position x at time t. As the wave sweeps through succeeding elements of the string, the elements oscillate parallel to the y-axis. At any time t, the displacement y of the element located at position x is given by

$$y\,(x,\ t) = a \sin\,(kx - \omega t + \phi) \qquad (15.2)$$

One can as well choose a cosine function or a linear combination of sine and cosine functions such as,

$$y\,(x,\ t) = A \sin\,(kx - \omega t) + B \cos\,(kx - \omega t), \qquad (15.3)$$

then in Eq. (15.2),

$$a = \sqrt{A^2 + B^2} \text{ and } \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

The function represented in Eq. (15.2) is periodic in position coordinate x and time t. It represents a transverse wave moving along the x-axis. At any time t, it gives the displacement of the elements of the string as a function of their position. It can tell us the shape of the wave at any given time and show how the wave progresses. Functions, such as that given in Eq. (15.2), represent a progressive wave travelling along the positive direction of the x-axis. On the other hand a function,

$$y(x,\ t) = a \sin\,(kx + \omega t + \phi), \qquad (15.4)$$

represents a wave travelling in the negative direction of x-axis (see Section 15.4). The set of four parameters a, ϕ, k, and ω in Eq. (15.2) completely describe a harmonic wave. The names of these parameters are displayed in Fig. 15.5 and are defined later.

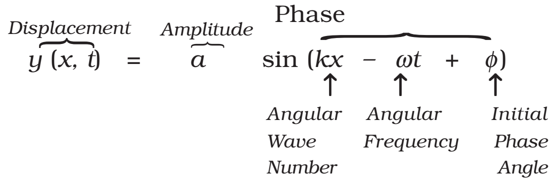


Fig. 15.5 *The names of the quantities in Eq. (15.2) for a progressive wave.*

To understand the definition of the quantities in Eq. (15.2), let us consider the graphs shown in Fig. 15.6. These graphs represent plots of Eq. (15.2) for five different values of time t as the wave travels in positive direction of x-axis. A point of maximum positive displacement in a wave, shown by the arrow, is called *crest*, and a point of maximum negative displacement is called **trough**. The progress of the wave is indicated by the progress of the short arrow pointing to a crest of the wave towards the right. As we move from one plot to another, the short arrow moves to the right with the wave shape,

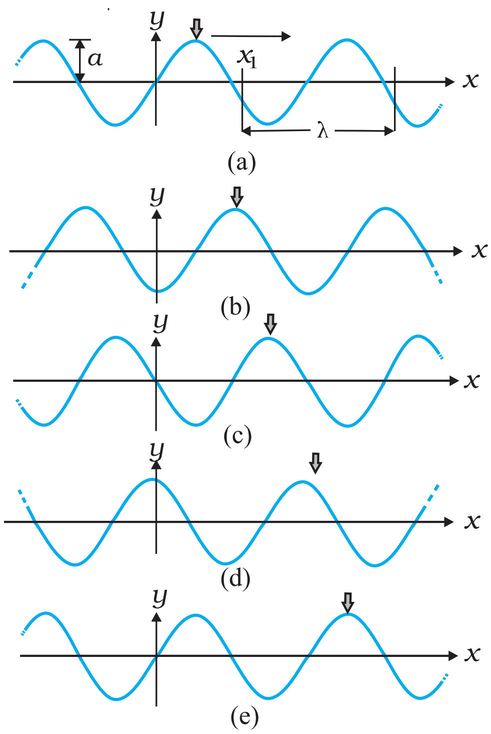


Fig. 15.6 *Plots of Eq. (15.2) for a wave travelling in the positive direction of an x-axis at five different values of time t.*

but the string moves only parallel to y-axis. It can be seen that as we go from plot (a) to (e), a particular element of the string has undergone one complete cycle of changes or completed one full oscillation. During this course of time the short arrow head or the wave has moved by a characteristic distance along the x-axis.

In the context of the above five plots, we will now define various quantities in Eq. (15.2) and shown in Fig.15.5.

15.3.1 Amplitude and Phase

The **amplitude** a of a wave such as that in Figs.15.5 and 15.6 is the magnitude of the maximum displacement of the elements from their equilibrium positions as the wave passes through them. It is depicted in Fig. 15.6 (a). Since a is a magnitude, it is a positive quantity, even if the displacement is negative.

The **phase** of the wave is the argument $(kx - \omega t + \phi)$ of the oscillatory term sin $(kx - \omega t + \phi)$ in Eq. (15.2). It describes the state of motion as the wave sweeps through a string element at a particular position x. It changes linearly with time t. The sine function also changes with time, oscillating between +1 and –1. Its extreme positive value +1 corresponds to a peak of the wave moving through the element; then the value of y at position x is a. Its extreme negative value –1 corresponds to a valley of the wave moving through the element, then the value of y at position x is $-a$. Thus, the sine function and the time dependent phase of a wave correspond to the oscillation of a string element, and the amplitude of the wave determines the extremes of the element's displacement. The constant ϕ is called the **initial phase angle**. The value of ϕ is determined by the initial (t = 0) displacement and velocity of the element (say, at x = 0).

It is always possible to choose origin (x = 0) and the initial instant (t = 0) such that ϕ = 0. There is no loss of generality in working with Eq. (15.2) with ϕ = 0.

15.3.2 Wavelength and Angular Wave Number

The **wavelength** λ of a wave is the distance (parallel to the direction of wave propagation) between the consecutive repetitions of the shape of the wave. It is the minimum distance between two consecutive troughs or crests or two consecutive points in the same phase of wave motion. A typical wavelength is marked in Fig.15.6(a), which is a plot of Eq. (15.2) for $t = 0$ and ϕ = 0. At this time Eq. (15.2) reduces to

$$y(x, 0) = a \sin kx \qquad (15.5)$$

By definition, the displacement y is same at both ends of this wavelength, that is at $x = x_1$ and at $x = x_1 + \lambda$. Thus, by Eq. (15.2),

$$a \sin k x_1 = a \sin k (x_1 + \lambda)$$
$$= a \sin (k x_1 + k \lambda)$$

This condition can be satisfied only when,

$$k \lambda = 2\pi n$$

where n = 1, 2, 3... Since λ is defined as the least distance between points with the same phase, n =1 and

$$k \quad \frac{2}{\;} \qquad (15.6)$$

k is called the **propagation constant** or the **angular wave number** ; its SI unit is radian per metre or rad m^{-1}.*

It may be noted that in Fig. 15.6, as we move from one plot to another, the wave moves to the right by a distance equal to ¼ λ. Thus, by the fifth plot, it has moved to the right by a distance equal to λ.

15.3.3 Period, Angular Frequency and Frequency

Figure15.7 shows a graph of the displacement y, of Eq. (15.2), versus time t at a certain position along the string, taken to be x = 0. If you were to monitor the string, you would see that the

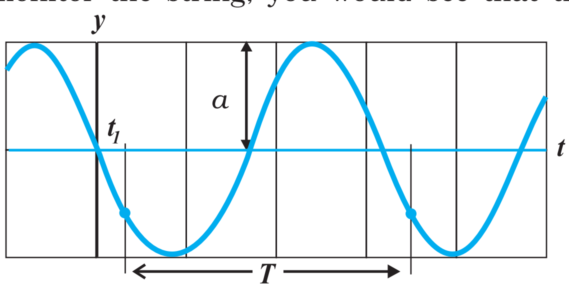


Fig. 15.7 *A graph of the displacement of the string element at x = 0 as a function of time, as the sinusoidal wave of Fig. 14.6 passes through it. The amplitude a is indicated. A typical period T, measured from an arbitrary time t_1 is also indicated.*

* *Here again, 'radian' could be dropped and the units could be written merely as m^{-1}. Thus, k represents 2π times the number of waves (or the total phase difference) that can be accommodated per unit length, with SI units m^{-1}.*

element of the string at that position moves up and down in simple harmonic motion given by Eq. (15.2) with $x = 0$,

$$y\,(0,t) = a \sin(-\omega t)$$

$$= -a \sin \omega t$$

Figure 15.7 is a graph of this equation; it does not show the shape of the wave.

The **period** of oscillation T of a wave is defined as the time any string element takes to move through one complete oscillation. A typical period is marked on the Fig. 15.7. Applying Eq. (15.2) on both ends of this time interval, we get

$$-a \sin \omega t_1 = -a \sin \omega(t_1 + T)$$

$$= -a \sin(\omega t_1 + \omega T)$$

This can be true only if the least value of ωT is 2π, or if

$$\omega = 2\pi/T \qquad (15.7)$$

ω is called the **angular frequency** of the wave, its SI unit is rad s^{-1}.

Look back at the five plots of a travelling wave in Fig. 15.6. The time between two consecutive plots is $T/4$. Thus, by the fifth plot, every string element has made one full oscillation.

The **frequency** ν of a wave is defined as $1/T$ and is related to the angular frequency ω by

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \qquad (15.8)$$

It is the number of oscillations per unit time made by a string element as the wave passes through it. It is usually measured in hertz.

In the discussion above, reference has always been made to a wave travelling along a string or a transverse wave. In a longitudinal wave, the displacement of an element of the medium is parallel to the direction of propagation of the wave. In Eq. (15.2), the displacement function for a longitudinal wave is written as,

$$s(x, t) = a \sin(kx - \omega t + \phi) \qquad (15.9)$$

where $s(x, t)$ is the displacement of an element of the medium in the direction of propagation of the wave at position x and time t. In Eq. (15.9), a is the displacement amplitude; other quantities have the same meaning as in case of a transverse wave except that the displacement function $y\,(x, t)$ is to be replaced by the function $s\,(x, t)$.

Example 15.2 A wave travelling along a string is described by,

$$y(x, t) = 0.005 \sin(80.0\,x - 3.0\,t),$$

in which the numerical constants are in SI units (0.005 m, 80.0 rad m^{-1}, and 3.0 rad s^{-1}). Calculate (a) the amplitude, (b) the wavelength, and (c) the period and frequency of the wave. Also, calculate the displacement y of the wave at a distance $x = 30.0$ cm and time $t = 20$ s ?

Answer On comparing this displacement equation with Eq. (15.2),

$$y\,(x, t) = a \sin(kx - \omega t),$$

we find

(a) the amplitude of the wave is 0.005 m = 5 mm.

(b) the angular wave number k and angular frequency ω are

$$k = 80.0 \text{ m}^{-1} \text{ and } \omega = 3.0 \text{ s}^{-1}$$

We then relate the wavelength λ to k through Eq. (15.6),

$$\lambda = 2\pi/k$$

$$= \frac{2\pi}{80.0 \text{ m}^{-1}}$$

$$= 7.85 \text{ cm}$$

(c) Now we relate T to ω by the relation

$$T = 2\pi/\omega$$

$$= \frac{2\pi}{3.0 \text{ s}^{-1}}$$

$$= 2.09 \text{ s}$$

and frequency, $\nu = 1/T = 0.48$ Hz

The displacement y at $x = 30.0$ cm and time $t = 20$ s is given by

$$y = (0.005 \text{ m}) \sin(80.0 \times 0.3 - 3.0 \times 20)$$

$$= (0.005 \text{ m}) \sin(-36 + 12\pi)$$

$$= (0.005 \text{ m}) \sin(1.699)$$

$$= (0.005 \text{ m}) \sin(97^0) \simeq 5 \text{ mm}$$

15.4 THE SPEED OF A TRAVELLING WAVE

Let us monitor the propagation of a travelling wave represented by Eq. (15.2) along a string. The wave is travelling in the positive direction of x. We find that an element of string at a particular position x moves up and down as a function of time but the waveform advances to

the right. The displacement of various elements of the string at two different instants of time t differing by a small time interval Δt is depicted in Fig. 15.8 (the phase angle ϕ has been taken to be zero). It is observed that during this interval of time the entire wave pattern moves by a distance Δx in the positive direction of x. Thus the wave is travelling to the right, in the positive direction of x. The ratio $\Delta x/\Delta t$ is the wave speed v.

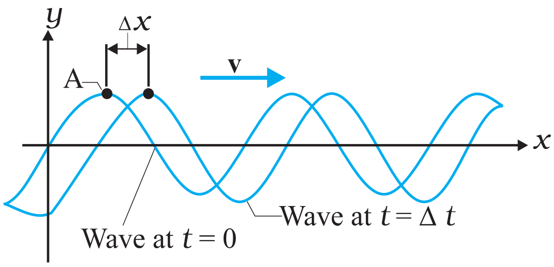


Fig. 15.8 *The plots of Eq.(15.2) at two instants of time differing by an interval Δt, at $t = 0$ and then at $t = \Delta t$. As the wave moves to the right at velocity* **v***, the entire curve shifts a distance Δx during Δt. The point A rides the waveform but the string element moves only up and down.*

As the wave moves (see Fig. 15.8), each point of the moving waveform represents a particular phase of the wave and retains its displacement y. It may be noted that the points on the string do not retain their displacement, but the points on the waveform do. Let us consider a point like A marked on a peak of the waveform. If a point like A on the waveform retains its displacement as it moves, it follows from Eq. (15.2) that this is possible only when the argument is constant. It, therefore, follows that

$$kx - \omega t = \text{constant} \qquad (15.10)$$

Note that in the argument both x and t are changing; therefore, to keep the argument constant, if t increases, x must also increase. This is possible only when the wave is moving in the positive direction of x.

To find the wave speed v, let us differentiate Eq. (15.10) with respect to time ;

$$\frac{\mathrm{d}}{\mathrm{d}t} \quad kx \quad t \quad 0$$

or $$k \frac{\mathrm{d}x}{\mathrm{d}t} \qquad 0$$

or $$\frac{\mathrm{d}x}{\mathrm{d}t} \quad \frac{}{k} \quad v \qquad (15.11)$$

Making use of Eqs. (15.6)-(15.8), we can write,

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda \nu \qquad (15.12)$$

Equation (15.11) is a general relation valid for all progressive waves. It merely states that the wave moves a distance of one wavelength in one period of oscillation. The speed of a wave is related to its wavelength and frequency by the Eq. (15.12), **but it is determined by the properties of the medium.** If a wave is to travel in a medium like air, water, steel, or a stretched string, it must cause the particles of that medium to oscillate as it passes through it. For this to happen, the medium must possess mass and elasticity. Therefore the linear mass density (or mass per unit length, in case of linear systems like a stretched string) and the elastic properties determine how fast the wave can travel in the medium. Conversely, it should be possible to calculate the speed of the wave through the medium in terms of these properties. In subsequent sub-sections of this chapter, we will obtain specific expressions for the speed of mechanical waves in some media.

15.4.1 Speed of a Transverse Wave on Stretched String

The speed of transverse waves on a string is determined by two factors, (i) the linear mass density or mass per unit length, μ, and (ii) the tension T. The mass is required so that there is kinetic energy and without tension no disturbance can be propagated in the string. The exact derivation of the relationship between the speed of wave in a stretched string and the two parameters mentioned above is outside the scope of this book. However, we take recourse to a simpler procedure. In dimensional analysis, we have already learnt (Chapter 2) how to get a relationship between different quantities which are interrelated. Such a relation is, however, uncertain to the extent of a constant factor.

The linear mass density, μ, of a string is the mass m of the string divided by its length l. Therefore its dimension is $[ML^{-1}]$. The tension T

has the dimension of force — namely, [M L T^{-2}]. Our goal is to combine μ and T in such a way as to generate v [dimension (L T^{-1})]. If we examine the dimensions of these quantities, it can be seen that the ratio T/μ has the dimension

$$\frac{[\mathrm{MLT}^{-2}]}{[\mathrm{ML}^{-1}]} = [\mathrm{L}^2\,\mathrm{T}^{-2}]$$

Therefore, if v depends only on T and μ , the relation between them must be

$$v = C\sqrt{\frac{T}{\mu}} \qquad (15.13)$$

Here C is a dimensionless constant that cannot be determined by dimensional analysis. By adopting a more rigorous procedure it can be shown that the constant C is indeed equal to unity. The speed of transverse waves on a stretched string is, therefore, given by

$$v = \sqrt{\frac{T}{\mu}} \qquad (15.14)$$

Equation (15.14) tells us :

The speed of a wave along a stretched ideal string depends only on the tension and the linear mass density of the string and does not depend on the frequency of the wave.

The **frequency** of the wave is determined by the source that generates the wave. The **wavelength** is then fixed by Eq. (15.12) in the form,

$$\lambda = \frac{v}{\nu} \qquad (15.15)$$

Example 15.3 A steel wire 0.72 m long has a mass of 5.0 ×10^{-3} kg. If the wire is under a tension of 60 N, what is the speed of transverse waves on the wire ?

Answer Mass per unit length of the wire,

$$\mu = \frac{5.0\times10^{-3}\ \mathrm{kg}}{0.72\ \mathrm{m}}$$

$$= 6.9\times10^{-3}\ \mathrm{kg\ m^{-1}}$$

Tension, T = 60 N

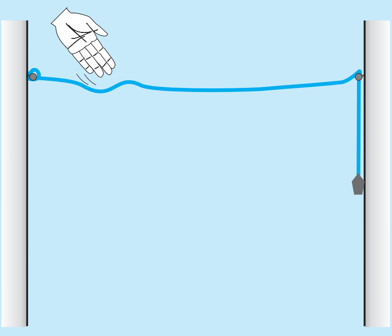
The speed of wave on the wire is given by

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{60\ \mathrm{N}}{6.9\times10^{-3}\mathrm{kg\ m^{-1}}}} = 93\ \mathrm{m\ s^{-1}}$$

15.4.2 Speed of a Longitudinal Wave Speed of Sound

In a longitudinal wave the constituents of the medium oscillate forward and backward in the direction of propagation of the wave. We have already seen that the sound waves travel in the form of compressions and rarefactions of small

Propagation of a pulse on a rope



You can easily see the motion of a pulse on a rope. You can also see its reflection from a rigid boundary and measure its velocity of travel. You will need a rope of diameter 1 to 3 cm, two hooks and some weights. You can perform this experiment in your classroom or laboratory.

Take a long rope or thick string of diameter 1 to 3 cm, and tie it to hooks on opposite walls in a hall or laboratory. Let one end pass on a hook and hang some weight (about 1 to 5 kg) to it. The walls may be about 3 to 5 m apart.

Take a stick or a rod and strike the rope hard at a point near one end. This creates a pulse on the rope which now travels on it. You can see it reaching the end and reflecting back from it. You can check the phase relation between the incident pulse and reflected pulse. You can easily watch two or three reflections before the pulse dies out. You can take a stopwatch and find the time for the pulse to travel the distance between the walls, and thus measure its velocity. Compare it with that obtained from Eq. (15.14).

This is also what happens with a thin metallic string of a musical instrument. The major difference is that the velocity on a string is fairly high because of low mass per unit length, as compared to that on a thick rope. The low velocity on a rope allows us to watch the motion and make measurements beautifully.

volume elements of air. The property that determines the extent to which the volume of an element of a medium changes when the pressure on it changes, is the **bulk modulus** B, defined (see Chapter 9) as,

$$B = -\frac{\Delta P}{\Delta V / V} \tag{15.16}$$

Here $\Delta V/V$ is the fractional change in volume produced by a change in pressure ΔP. The SI unit for pressure is N m^{-2} or pascal (Pa). Now since the longitudinal waves in a medium travel in the form of compressions and rarefactions or changes in density, the inertial property of the medium, which could be involved in the process, is the density ρ. The dimension of density is [ML^{-3}]. Thus, the dimension of the ratio B/ρ is,

$$\frac{\left[\mathrm{M\,L^{-1}\,T^{-2}}\right]}{\left[\mathrm{M\,L^{-3}}\right]} = \left[\mathrm{L^{2}\;T^{-2}}\right] \tag{15.17}$$

Therefore, on the basis of dimensional analysis the most appropriate expression for the speed of longitudinal waves in a medium is

$$v = C\sqrt{\frac{B}{\rho}} \tag{15.18}$$

where C is a dimensionless constant and can be shown to be unity. Thus the speed of longitudinal waves in a medium is given by,

$$v = \sqrt{\frac{B}{\rho}} \tag{15.19}$$

The speed of propagation of a longitudinal wave in a fluid therefore depends only on the bulk modulus and the density of the medium.

When a solid bar is struck a blow at one end, the situation is somewhat different from that of a fluid confined in a tube or cylinder of constant cross section. For this case, the relevant modulus of elasticity is the Young's modulus, since the sideway expansion of the bar is negligible and only longitudinal strain needs to be considered. It can be shown that the speed of a longitudinal wave in the bar is given by,

$$v = \sqrt{\frac{Y}{\rho}} \tag{15.20}$$

where Y is the Young's modulus of the material of the bar.

Table 15.1 gives the speed of sound in various media.

Table 15.1 Speed of Sound in some Media

Medium	Speed (m s^{-1})
Gases	
Air (0 °C)	331
Air (20 °C)	343
Helium	965
Hydrogen	1284
Liquids	
Water (0 °C)	1402
Water (20 °C)	1482
Seawater	1522
Solids	
Aluminium	6420
Copper	3560
Steel	5941
Granite	6000
Vulcanised Rubber	54

It may be noted that although the densities of liquids and solids are much higher than those of the gases, the speed of sound in them is higher. It is because liquids and solids are less compressible than gases, i.e. have much greater bulk modulus.

In the case of an ideal gas, the relation between pressure P and volume V is given by (see Chapter 11)

$$PV = Nk_BT \tag{15.21}$$

where N is the number of molecules in volume V, k_B is the Boltzmann constant and T the temperature of the gas (in Kelvin). Therefore, for an isothermal change it follows from Eq.(15.21) that

$$V\Delta P + P\Delta V = 0$$

or $$-\frac{\Delta P}{\Delta V/V} = P$$

Hence, substituting in Eq. (15.16), we have

$$B = P$$

Therefore, from Eq. (15.19) the speed of a longitudinal wave in an ideal gas is given by,

$$v = \sqrt{\frac{P}{\rho}} \tag{15.22}$$

This relation was first given by Newton and is known as Newton's formula.

> **Example 15.4** Estimate the speed of sound in air at standard temperature and pressure. The mass of 1 mole of air is 29.0×10^{-3} kg.

Answer We know that 1 mole of any gas occupies 22.4 litres at STP. Therefore, density of air at STP is :

ρ_o = (mass of one mole of air)/ (volume of one mole of air at STP)

$$\frac{29.0 \quad 10^{3} \text{ kg}}{22.4 \quad 10^{3} \text{ m}^3}$$

$$= 1.29 \text{ kg m}^{-3}$$

According to Newton's formula for the speed of sound in a medium, we get for the speed of sound in air at STP,

$$\upsilon = \left[\frac{1.01 \times 10^5 \text{ N m}^{-2}}{1.29 \text{ kg m}^{-3}}\right]^{1/2} = 280 \text{ m s}^{-1} \quad (15.23)$$

The result shown in Eq.(15.23) is about 15% smaller as compared to the experimental value of 331 m s^{-1} as given in Table 15.1. Where did we go wrong ? If we examine the basic assumption made by Newton that the pressure variations in a medium during propagation of sound are isothermal, we find that this is not correct. It was pointed out by Laplace that the pressure variations in the propagation of sound waves are so fast that there is little time for the heat flow to maintain constant temperature. These variations, therefore, are adiabatic and not isothermal. For adiabatic processes the ideal gas satisfies the relation,

$$PV^{\gamma} = \text{constant}$$

i.e. $\Delta(PV^{\gamma}) = 0$

or $P\gamma\, V^{\gamma-1}\Delta V + V^{\gamma}\Delta P = 0$

Thus for an ideal gas the adiabatic bulk modulus is given by,

$$B_{ad} = -\frac{\Delta P}{\Delta V/V} = \gamma P$$

where γ is the ratio of two specific heats, C_p/C_v. The speed of sound is, therefore, given by,

$$\upsilon = \sqrt{\frac{\gamma P}{\rho}} \qquad (15.24)$$

This modification of Newton's formula is referred to as the **Laplace correction**. For air $\gamma = 7/5$. Now using Eq. (15.24) to estimate the speed of sound in air at STP, we get a value 331.3 m s^{-1}, which agrees with the measured speed.

15.5 THE PRINCIPLE OF SUPERPOSITION OF WAVES

Let us consider that two waves are travelling simultaneously along the same stretched string in opposite directions. The sequence of pictures shown in Fig. 15.9 depicts the state of displacement of various elements of the string at different time instant. Each picture depicts the resultant waveform in the string at a given instant of time. It is observed that the **net displacement of any element of the string at a given time is the algebraic sum of the displacements due to each wave**. This way of

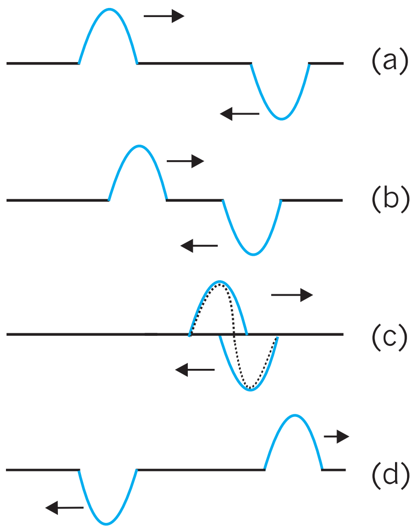


Fig. 15.9 *A sequence of pictures depicting two pulses travelling in opposite directions along a stretched string. They meet and pass through each other and move on independently as shown by the sequence of time snapshots (a) through (d). The total disturbance is the algebraic sum of the displacements due to each pulse. When the two disturbances overlap they give a complicated pattern as shown in (c). In region (d) they have passed each other and proceed unchanged.*

addition of individual waveforms to determine the net waveform is called the **principle of superposition**. To put this rule in a mathematical form, let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that any element of the string would experience if each wave travelled alone. The displacement $y(x,t)$ of an element of the string when the waves overlap is then given by,

$$y(x, t) = y_1(x, t) + y_2(x, t) \quad (15.25)$$

The principle of superposition can also be expressed by stating that **overlapping waves algebraically add to produce a resultant wave (or a net wave)**. The principle implies that the overlapping waves do not, in any way, alter the travel of each other.

If we have two or more waves moving in the medium the resultant waveform is the sum of wave functions of individual waves. That is, if the wave functions of the moving waves are

$$y_1 = f_1(x-vt),$$

$$y_2 = f_2(x-vt),$$

....

....

$$y_n = f_n(x-vt)$$

then the wave function describing the disturbance in the medium is

$$y = f_1(x-vt) + f_2(x-vt) + \ldots + f_n(x-vt)$$

$$= \sum_{i=1}^{n} f_i(x-vt) \quad (15.26)$$

As illustrative examples of this principle we shall study the phenomena of interference and reflection of waves.

Let a wave travelling along a stretched string be given by,

$$y_1(x, t) = a \sin(kx - \omega t) \quad (15.27)$$

and another wave, shifted from the first by a phase ϕ,

$$y_2(x, t) = a \sin(kx - \omega t + \phi) \quad (15.28)$$

Both the waves have the same angular frequency, same angular wave number k (same wavelength) and the same amplitude a. They travel in the positive direction of x-axis, with the same speed. Their phases at a given distance and time differ by a constant angle ϕ. These waves are said to be out of phase by ϕ or have a phase difference ϕ.

Now, applying the superposition principle, the resultant wave is the algebraic sum of the two constituent waves and has displacement

$$y(x, t) = a \sin(kx - \omega t) + a \sin(kx - \omega t + \phi) \quad (15.29)$$

We now use the trigonometric relation

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta) \quad (15.30)$$

Applying this relation to Eq. (15.29) we have

$$y(x, t) = [2a\cos\frac{1}{2}\phi]\sin(kx - \omega t + \frac{1}{2}\phi) \quad (15.31)$$

Equation (15.31) shows that the resultant wave is also a sinusoidal wave, travelling in the positive direction of x-axis.

The resultant wave differs from the constituent waves in two respects: (1) its phase angle is $(½)\phi$ and (2) its amplitude is the quantity in brackets in Eq. (15.31) viz.,

$$A(\phi) = 2a\cos(½)\phi \quad (15.32)$$

If $\phi = 0$, i.e. the two waves are in phase, Eq. (15.31) reduces to

$$A(0) = 2a\sin(kx - \omega t) \quad (15.33)$$

The amplitude of the resultant wave is $2a$, which is the largest possible value of $A(\phi)$.

If $\phi = \pi$, the two waves are completely out of phase, the amplitude of the resultant wave given by Eq. (15.32) reduces to zero. We then have for all x and t,

$$y(x, t) = 0 \quad (15.34)$$

These cases are shown in Fig. 15.10.

15.6 REFLECTION OF WAVES

In previous sections we have discussed wave propagation in unbounded media. What happens when a pulse or a travelling wave encounters a rigid boundary ? It is a common experience that under such a situation the pulse or the wave gets reflected. An everyday example of the reflection of sound waves from a rigid boundary is the phenomenon of echo. If the boundary is not completely rigid or is an interface between two different elastic media, the effect of boundary conditions on

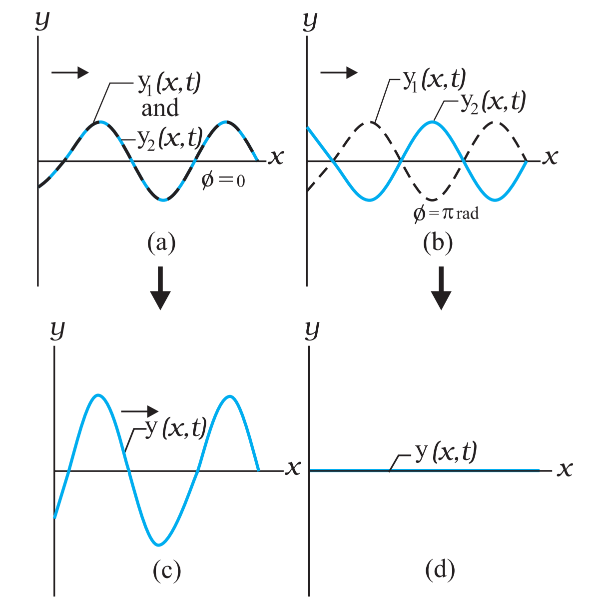


Fig. 15.10 *Two identical sinusoidal waves, $y_1(x, t)$ and $y_2(x, t)$, travel along a stretched string in the positive direction of x-axis. They give rise to a resultant wave y (x, t). The phase difference between the two waves is (a) 0 and (b) π or 180°. The corresponding resultant waves are shown in (c) and (d).*

an incident pulse or a wave is somewhat complicated. A part of the wave is reflected and a part is transmitted into the second medium. If a wave is incident obliquely on the boundary between two different media the transmitted wave is called the **refracted wave**. The incident and refracted waves obey Snell's law of refraction, and the incident and reflected waves obey the usual laws of reflection.

To illustrate the reflection of waves at a boundary, we consider two situations. First, a string is fixed to a rigid wall at its left end, as shown in Fig. 15.11(a). Second, the left end of the string is tied to a ring, which slides up and down without any friction on a rod, as shown in Fig. 15.11(b). A pulse is allowed to propagate in both these strings, the pulse, on reaching the left end, gets reflected; the state of disturbance in the string at various times is shown in Fig. 15.11.

In Fig. 15.11(a), the string is fixed to the wall at its left end. When the pulse arrives at that end, it exerts an upward force on the wall. By Newton's third law, the wall exerts an opposite force of equal magnitude on the string. This second force generates a pulse at the support (the wall), which travels back along the string in the direction opposite to that of the incident pulse. In a reflection of this kind, there must be no displacement at the support as the string is fixed there. The reflected and incident pulses must have opposite signs, so as to cancel each other at that point. Thus, in case of a travelling wave, the reflection at a rigid boundary will take place with a phase reversal or with a phase difference of π or 180°.

In Fig. 15.11(b), the string is fastened to a ring, which slides without friction on a rod. In this case, when the pulse arrives at the left end, the ring moves up the rod. As the ring moves, it pulls on the string, stretching the string and producing a reflected pulse with the same sign and amplitude as the incident pulse. Thus, in such a reflection, the incident and reflected pulses reinforce each other, creating the

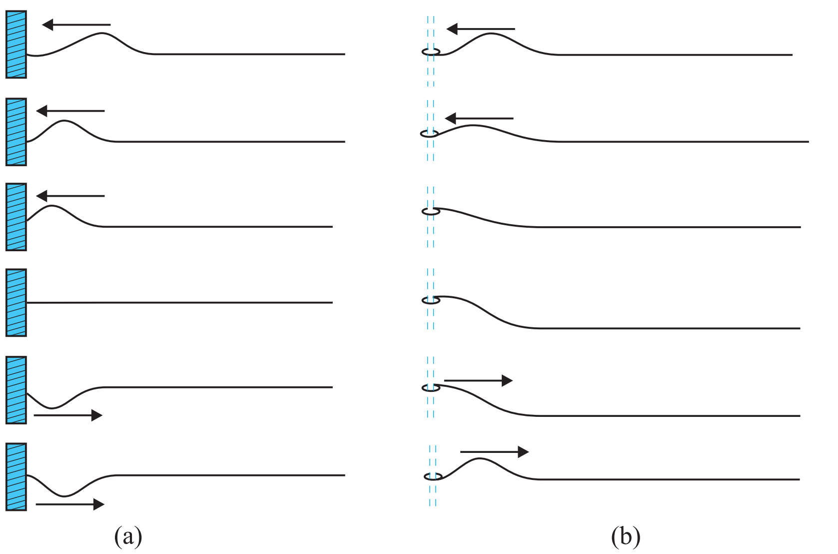


Fig. 15.11 *(a) A pulse incident from the right is reflected at the left end of the string, which is tied to a wall. Note that the reflected pulse is inverted from the incident pulse. (b) Here the left end is tied to a ring that can slide up and down without friction on the rod. Now the reflected pulse is not inverted by reflection.*

maximum displacement at the end of the string: the maximum displacement of the ring is twice the amplitude of either of the pulses. Thus, the reflection is without any additional phase shift. In case of a travelling wave the reflection at an open boundary, such as the open end of an organ pipe, the reflection takes place without any phase change.

We can thus, summarise the reflection of waves at a boundary or interface between two media as follows:

A travelling wave, at a rigid boundary or a closed end, is reflected with a phase reversal but the reflection at an open boundary takes place without any phase change.

To express the above statement mathematically, let the incident wave be represented by

$$y_i(x, t) = a \sin (kx - \omega t),$$

then, for reflection at a rigid boundary the reflected wave is represented by,

$$y_r(x, t) = a \sin (kx + \omega t + \pi).$$

$$= - a \sin (kx + \omega t) \quad (15.35)$$

For reflection at an open boundary, the reflected wave is represented by

$$y_r (x, t) = a \sin (kx + \omega t). \quad (15.36)$$

15.6.1 Standing Waves and Normal Modes

In the previous section we have considered a system which is bounded at one end. Let us now consider a system which is bounded at both the ends such as a stretched string fixed at the ends or an air column of finite length. In such a system suppose that we send a continuous sinusoidal wave of a certain frequency, say, toward the right. When the wave reaches the right end, it gets reflected and begins to travel back. The left-going wave then overlaps the wave, travelling to the right. When the left-going wave reaches the left end, it gets reflected again and the newly reflected wave begins to travel to the right, overlapping the left-going wave. This process will continue and, therefore, very soon we have many overlapping waves, which interfere with one another. In such a system, at any point x and at any time t, there are always two waves, one moving to the left and another to the right. We, therefore, have

$y_1(x, t) = a \sin (kx - \omega t)$ (wave travelling in the positive direction of x-axis)

and $y_2(x, t) = a \sin (kx + \omega t)$ (wave travelling in the negative direction of x-axis).

The principle of superposition gives, for the combined wave

$$y (x, t) = y_1(x, t) + y_2(x, t)$$

$$= a \sin (kx - \omega t) + a \sin (kx + \omega t)$$

$$= (2a \sin kx) \cos \omega t \quad (15.37)$$

The wave represented by Eq. (15.37) does not describe a travelling wave, as the waveform or the disturbance does not move to either side. Here, the quantity $2a \sin kx$ within the brackets is the amplitude of oscillation of the element of the string located at the position x. In a travelling wave, in contrast, the amplitude of the wave is the same for all elements. Equation (15.37), therefore, represents a **standing wave**, a wave in which the waveform does not move. The formation of such waves is illustrated in Fig. 15.12.

It is seen that the points of maximum or minimum amplitude stay at one position.

The amplitude is zero for values of kx that give $\sin kx = 0$. Those values are given by

$$kx = n\pi, \text{ for } n = 0, 1, 2, 3, ...$$

Substituting $k = 2\pi/\lambda$ in this equation, we get

$$x = n\frac{\lambda}{2}, \text{ for } n = 0, 1, 2, 3, ... \quad (15.38)$$

The positions of zero amplitude are called **nodes**. Note that a distance of $\frac{\lambda}{2}$ or half a wavelength separates two consecutive nodes.

The amplitude has a maximum value of $2a$, which occurs for the values of kx that give $|\sin kx| = 1$. Those values are

$$kx = (n + ½)\pi \text{ for } n = 0, 1, 2, 3, ...$$

Substituting $k = 2\pi/\lambda$ in this equation, we get

$$x = (n + ½)\frac{\lambda}{2} \text{ for } n = 0, 1, 2, 3, ... \quad (15.39)$$

as the positions of maximum amplitude. These are called the **antinodes**. The antinodes are separated by $\lambda/2$ and are located half way between pairs of nodes.

For a stretched string of length L, fixed at both ends, the two ends of the string have to be nodes.

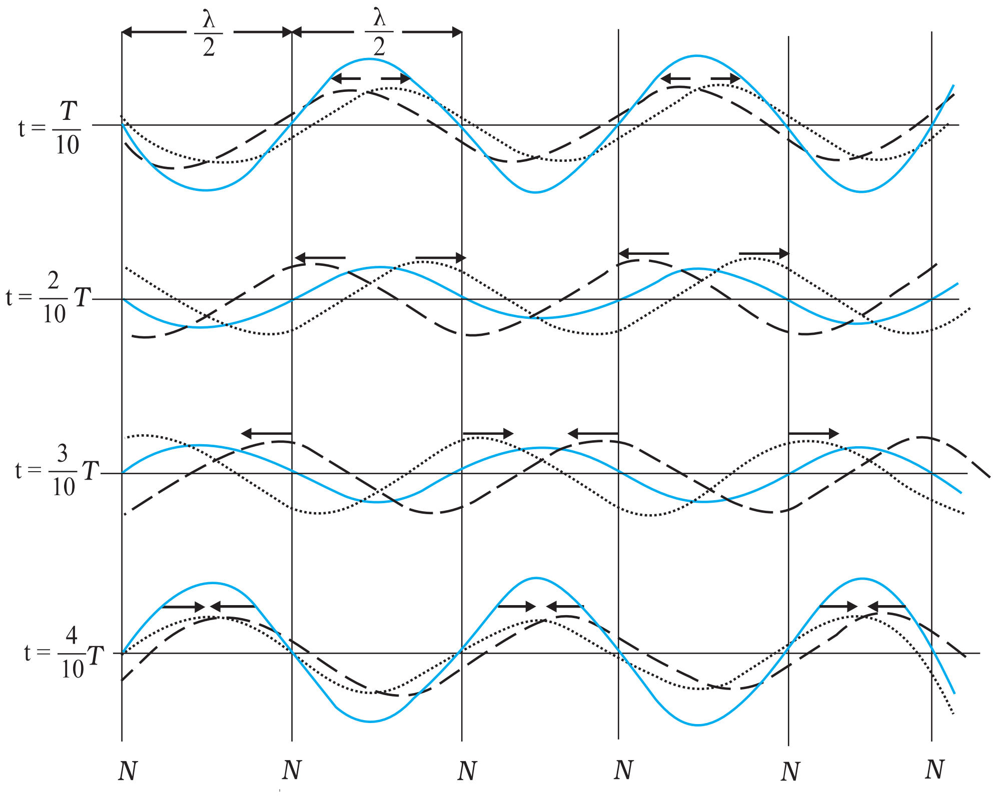


Fig. 15.12 *The formation of a standing wave in a stretched string. Two sinusoidal waves of same amplitude travel along the string in opposite directions. The set of pictures represent the state of displacements at four different times. The displacement at positions marked as N is zero at all times. These positions are called nodes.*

If one of the ends is chosen as position $x = 0$, then the other end is $x = L$. In order that this end is a node; the length L must satisfy the condition

$$L = n\,\frac{\lambda}{2},\ \text{for } n = 1, 2, 3, \ldots \qquad (15.40)$$

This condition shows that standing waves on a string of length L have restricted wavelength given by

$$\lambda = \frac{2L}{n},\ \text{for } n = 1, 2, 3, \ldots \text{ etc.} \qquad (15.41)$$

The frequencies corresponding to these wavelengths follow from Eq. (15.12) as

$$\nu = n\,\frac{v}{2L},\ \text{for } n = 1, 2, 3, \ldots \text{ etc.} \qquad (15.42)$$

where v is the speed of travelling waves on the string. The set of frequencies given by Eq. (15.42) are called the natural frequencies or **modes** of oscillation of the system. This equation tells us that the natural frequencies of a string are integral multiples of the lowest frequency $\nu = \dfrac{v}{2L}$, which corresponds to $n = 1$. The oscillation mode with that lowest frequency is called the **fundamental mode** or the **first harmonic**. The **second harmonic** is the oscillation mode with $n = 2$. The third harmonic corresponds to $n = 3$ and so on. The frequencies associated with these modes are often labelled as ν_1, ν_2, ν_3 and so on. The collection of all possible modes is called the **harmonic series** and n is called the harmonic number.

Some of the harmonics of a stretched string fixed at both the ends are shown in Fig. 15.13. According to the principle of superposition, a stretched string tied at both ends can vibrate simultaneously in more than one modes. Which mode is strongly excited depends on where the

string is plucked or bowed. Musical instruments like sitar and violin are designed on this principle.

We now study the modes of vibration of a system closed at one end, with the other end being free. Air columns such as glass tubes partially filled with water provide examples of such systems. In these, the length of the air column can be adjusted by changing the water level in the tube. In such systems, the end of the air column in touch with the water suffers no displacement as the reflected and incident waves are exactly out of phase. For this reason the pressure changes here are the largest, since when the compressional part is reflected the pressure increase is doubled, and when the rarefaction is reflected the decrease in pressure is doubled. On the other hand, at the open end, there is maximum displacement and minimum pressure change. The two waves travelling in opposite directions are in phase here, so there are no pressure changes. Now if the length of the air column is L, then the open end, $x = L$, is an antinode and therefore, it follows from Eq. (15.39) that

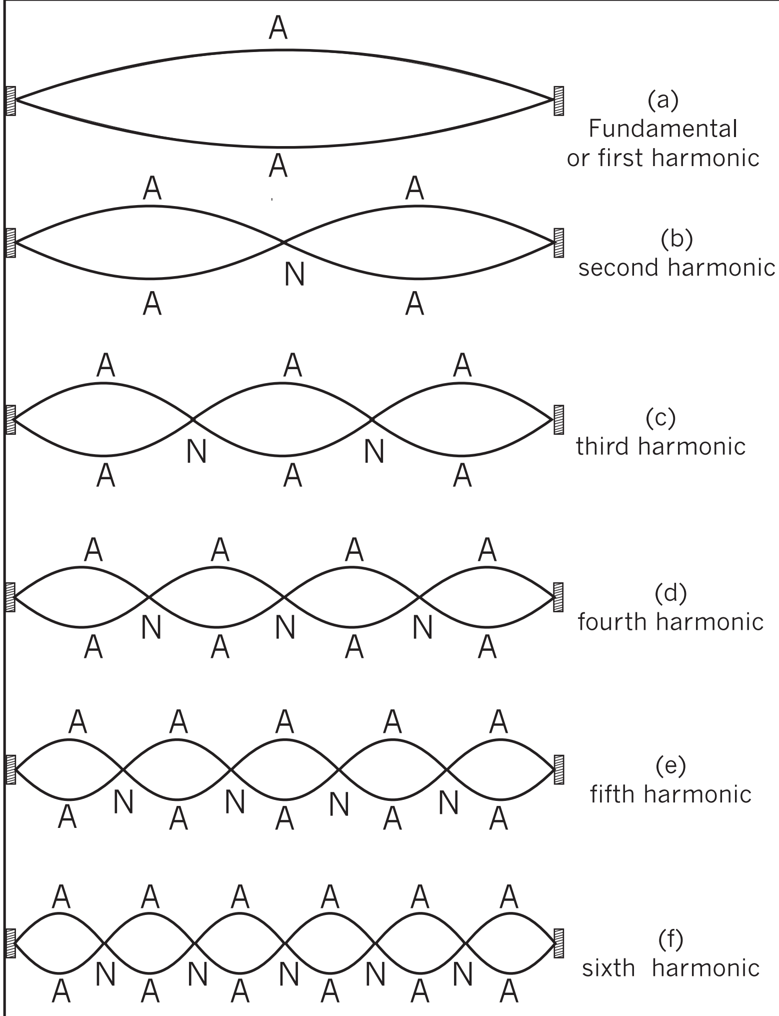


Fig. 15.13 *Stationary waves in a stretched string fixed at both ends. Various modes of vibration are shown.*

$$L = (n + \frac{1}{2})\frac{\ }{2}, \text{ for } n = 0, 1, 2, 3, \ldots$$

The modes, which satisfy the condition

$$\lambda = \frac{2L}{(n + 1/2)}, \text{ for } n = 0, 1, 2, 3, \ldots \qquad (15.43)$$

are sustained in such an air column. The corresponding frequencies of various modes of such an air column are given by,

$$\nu = (n + \frac{1}{2})\frac{v}{2L}, \text{ for } n = 0, 1, 2, 3, \ldots \qquad (15.44)$$

Some of the normal modes in an air column with the open end are shown in Fig. 15.14. The fundamental frequency is $\frac{v}{4L}$ and the higher frequencies are **odd harmonics** of the fundamental frequency, i.e. $3\frac{v}{4L}$, $5\frac{v}{4L}$, etc.

In the case of a pipe open at both ends, there will be antinodes at both ends, and **all harmonics** will be generated.

Normal modes of a circular membrane rigidly clamped to the circumference as in a tabla are determined by the boundary condition that no

point on the circumference of the membrane vibrates. Estimation of the frequencies of normal modes of this system is more complex. This problem involves wave propagation in two dimensions. However, the underlying physics is the same.

We have seen above that in a string, fixed at both ends, standing waves are produced only at certain frequencies as given by Eq. (15.42) or the system **resonates** at these frequencies. Similarly an air column open at one end resonates at frequencies given by Eq. (15.44).

Example 15.5 A pipe, 30.0 cm long, is open at both ends. Which harmonic mode of the pipe rasonates a 1.1 kHz source? Will resonance with the same source be observed if one end of the pipe is closed ? Take the speed of sound in air as 330 m s^{-1}.

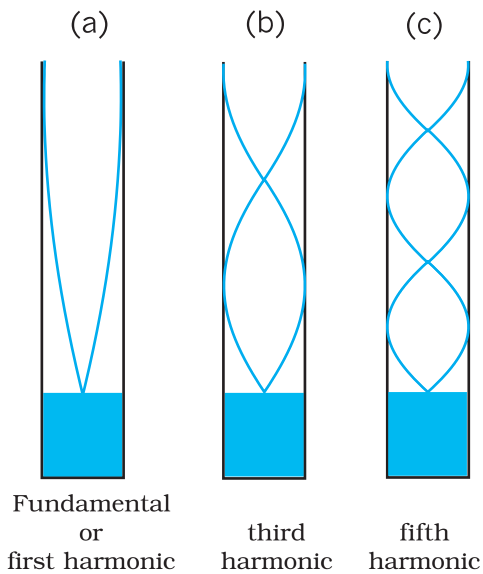
Answer The first harmonic frequency is given by

$$\nu_1 = \frac{\upsilon}{\lambda_1} = \frac{\upsilon}{2L} \quad \text{(open pipe)}$$

where L is the length of the pipe. The frequency of its nth harmonic is:

$$\nu_n = \frac{n\upsilon}{2L}, \text{ for } n = 1, 2, 3, \ldots \text{ (open pipe)}$$

First few modes of an open pipe are shown in Fig. 15.14.



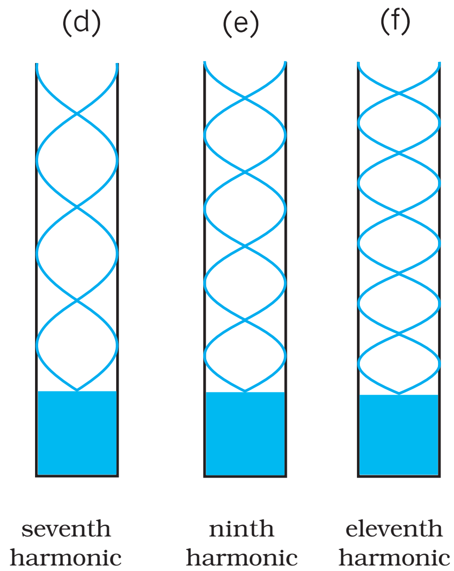


Fig. 15.14 *Some of the normal modes of vibration of an air column open at one end.*

For $L = 30.0$ cm, $\upsilon = 330$ m s^{-1},

$$\nu_n = \frac{n \times 330 \text{ (m s}^{-1})}{0.6 \text{ (m)}} = 550\, n \text{ s}^{-1}$$

Clearly, a source of frequency 1.1 kHz will resonate at υ_2, i.e. the **second harmonic**.

Now if one end of the pipe is closed (Fig. 15.15), it follows from Eq. (14.50) that the fundamental frequency is

$$\nu_1 = \frac{\upsilon}{\lambda_1} = \frac{\upsilon}{4L} \quad \text{(pipe closed at one end)}$$

and only the odd numbered harmonics are present :

$$\nu_3 = \frac{3\upsilon}{4L},\ \nu_5 = \frac{5\upsilon}{4L}, \text{ and so on.}$$

For L = 30 cm and υ = 330 m s^{-1}, the fundamental frequency of the pipe closed at one end is 275 Hz and the source frequency corresponds to its fourth harmonic. Since this harmonic is not a possible mode, no resonance will be observed with the source, the moment one end is closed.

15.7 BEATS

If we listen, a few minutes apart, two sounds of very close frequencies, say 256 Hz and 260 Hz, we will not be able to discriminate between

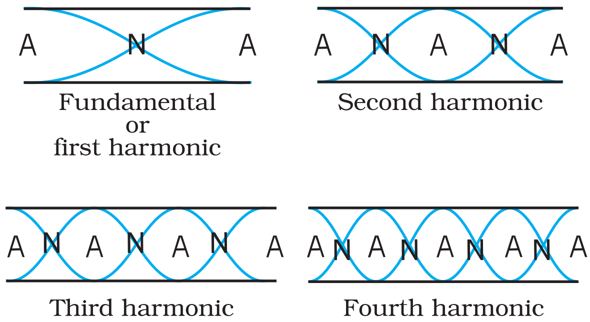


Fig. 15.15 *Standing waves in an open pipe, first four harmonics are depicted.*

them. However, if both these sounds reach our ears simultaneously, what we hear is a sound of frequency 258 Hz, the **average** of the two combining frequencies. In addition we hear a striking variation in the intensity of sound — it increases and decreases in slow, wavering **beats** that repeat at a frequency of 4 Hz, the **difference** between the frequencies of two incoming sounds. The phenomenon of wavering of sound intensity when two waves of nearly same frequencies and amplitudes travelling in the same direction, are superimposed on each other is called **beats**.

Let us find out what happens when two waves having slightly different frequencies are superposed on each other. Let the time dependent variations of the displacements due to two sound waves at a particular location be

$$s_1 = a \cos \omega_1 t \quad \text{and} \quad s_2 = a \cos \omega_2 t \qquad (15.45)$$

where $\omega_1 > \omega_2$. We have assumed, for simplicity, that the waves have same amplitude and phase. According to the superposition principle, the resultant displacement is

$$s = s_1 + s_2 = a (\cos \omega_1 t + \cos \omega_2 t)$$

$$= 2\, a \cos \frac{(\omega_1 - \omega_2)t}{2} \cos \frac{(\omega_1 + \omega_2)t}{2} \qquad (15.46)$$

If we write $\omega_b = \dfrac{(\omega_1 - \omega_2)}{2}$ and $\omega_a = \dfrac{(\omega_1 + \omega_2)}{2}$

then Eq. (15.46) can be written as

$$s = [2\, a \cos \omega_b t] \cos \omega_a t \qquad (15.47)$$

If $|\omega_1 - \omega_2| << \omega_1, \omega_2$, $\omega_a >> \omega_b$, then in Eq. (15.47) the main time dependence arises from cosine function whose angular frequency is ω_a. The quantity in the brackets can be regarded as the amplitude of this function (which is not a constant but, has a small variation of angular frequency ω_b). It becomes maximum whenever $\cos \omega_b t$ has the value +1 or –1, which happens twice in each repetition of cosine function. Since ω_1 and ω_2 are very close, ω_a cannot be differentiated easily from either of them. Thus, the result of superposition of two waves having nearly the same

Musical Pillars

Temples often have some pillars portraying human figures playing musical instruments, but seldom do these pillars themselves produce music. At the Nellaiappar temple in Tamil Nadu, gentle taps on a cluster of pillars carved out of a single piece of rock produce the basic notes of Indian classical music, viz. Sa, Re, Ga, Ma, Pa, Dha, Ni, Sa. Vibrations of these pillars depend on elasticity of the stone used, its density and shape.

Musical pillars are categorised into three types: The first is called the **Shruti Pillar**, as it can produce the basic notes — the "swaras". The second type is the **Gana Thoongal**, which generates the basic tunes that make up the "ragas". The third variety is the **Laya Thoongal** pillars that produce "taal" (beats) when tapped. The pillars at the Nellaiappar temple are a combination of the Shruti and Laya types.

Archaeologists date the Nelliappar temple to the 7th century and claim it was built by successive rulers of the Pandyan dynasty.

The musical pillars of Nelliappar and several other temples in southern India like those at Hampi (picture), Kanyakumari, and Thiruvananthapuram are unique to the country and have no parallel in any other part of the world.

Reflection of sound in an open pipe

When a high pressure pulse of air traveling down an open pipe reaches the other end, its momentum drags the air out into the open, where pressure falls rapidly to the atmospheric pressure. As a result the air following after it in the tube is pushed out. The low pressure at the end of the tube draws air from further up the tube. The air gets drawn towards the open end forcing the low pressure region to move upwards. As a result a pulse of *high* pressure air travelling *down* the tube turns into a pulse of *low* pressure air travelling *up* the tube. We say a pressure wave has been reflected at the open end with a change in phase of 180^0. Standing waves in an open pipe organ like the flute is a result of this phenomenon.

Compare this with what happens when a pulse of high pressure air arrives at a closed end: it collides and as a result pushes the air back in the opposite direction. Here, we say that the pressure wave is reflected, with no change in phase.

frequencies is a wave with nearly same angular frequency but its amplitude is not constant. Thus the intensity of resultant sound varies with an angular frequency $\omega_{beat} = 2\ \omega_b = \omega_1 - \omega_2$. Now using the relation,

$$\omega = 2\pi\nu$$

the beat frequency, ν_{beat}, is given by

$$\nu_{beat} = \nu_1 - \nu_2 \qquad (15.48)$$

Thus we hear a waxing and waning of sound with a frequency equal to the difference between the frequencies of the superposing waves. The time-displacement graphs of two waves of frequency 11 Hz and 9 Hz is shown in Figs. 15.16(a) and 15.16(b). The result of their 'superposition' is shown in Fig. 15.16(c).

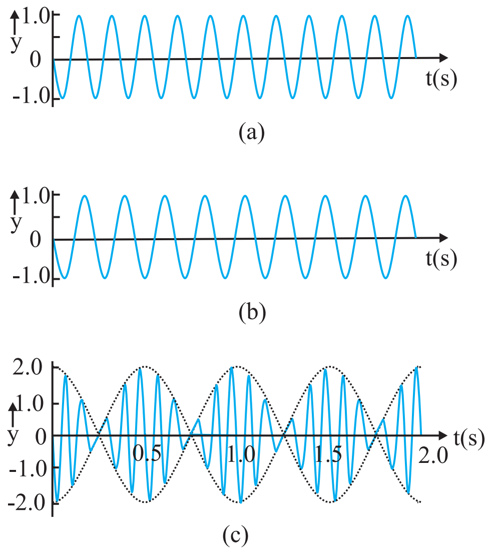


Fig. 15.16 *(a) Plot of a harmonic wave of frequency 11 Hz. (b) Plot of a harmonic wave of frequency 9 Hz. (c) Superposition of (a) and (b), showing clearly the beats in the slow (2 Hz) of the total disturbance.*

Musicians use the beat phenomenon in tuning their instruments. If an instrument is sounded against a standard frequency and tuned until the beat disappears, then the instrument is in tune with that standard.

▶ ***Example 15.6*** Two sitar strings A and B playing the note '*Dha*' are slightly out of tune and produce beats of frequency 5 Hz. The tension of the string B is slightly increased and the beat frequency is found to decrease to 3 Hz. What is the original frequency of B if the frequency of A is 427 Hz ?

Answer Increase in the tension of a string increases its frequency. If the original frequency of B (ν_B) were greater than that of A (ν_A), further increase in ν_B should have resulted in an increase in the beat frequency. But the beat frequency is found to decrease. This shows that $\nu_B < \nu_A$. Since $\nu_A - \nu_B = 5$ Hz, and $\nu_A = 427$ Hz, we get $\nu_B = 422$ Hz. ◀

15.8 DOPPLER EFFECT

It is an everyday experience that the pitch (or frequency) of the whistle of a fast moving train

decreases as it recedes away. When we approach a stationary source of sound with high speed, the pitch of the sound heard appears to be higher than that of the source. As the observer recedes away from the source, the observed **pitch** (or frequency) becomes lower than that of the source. This motion-related frequency change is called **Doppler effect**. The Austrian physicist Johann Christian Doppler first proposed the effect in 1842. Buys Ballot in Holland tested it experimentally in 1845. Doppler effect is a wave phenomenon, it holds not only for sound waves but also for electromagnetic waves. However, here we shall consider only sound waves.

We shall analyse changes in frequency under three different situations: (1) observer is stationary but the source is moving, (2) observer is moving but the source is stationary, and (3) both the observer and the source are moving. The situations (1) and (2) differ from each other because of the absence or presence of relative motion between the observer and the medium. Most waves require a medium for their propagation; however, electromagnetic waves do not require any medium for propagation. If there is no medium present, the Doppler shifts are same irrespective of whether the source moves or the observer moves, since there is no way of distinction between the two situations.

15.8.1 Source Moving ; Observer Stationary

Let us choose the convention to take the direction from the observer to the source as the positive direction of velocity. Consider a source S moving with velocity v_s and an observer who is stationary in a frame in which the medium is also at rest. Let the speed of a wave of angular frequency ω and period T_o, both measured by an observer at rest with respect to the medium, be v. We assume that the observer has a detector that counts every time a wave crest reaches it. As shown in Fig. 15.17, at time $t = 0$ the source is at point S_1, located at a distance L from the observer, and emits a crest. This reaches the observer at time $t_1 = L/v$. At time $t = T_0$ the source has moved a distance $v_s T_0$ and is at point S_2, located at a distance $(L + v_s T_0)$ from the observer. At S_2, the source emits a second crest. This reaches the observer at

$$t_2 = T_0 + \frac{(L + v_s T_0)}{v}$$

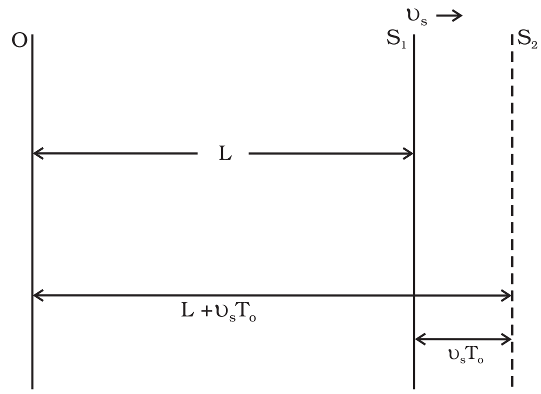


Fig. 15.17 *A source moving with velocity v_s emits a wave crest at the point S_1. It emits the next wave crest at S_2 after moving a distance $v_s T_0$.*

At time $n\,T_0$, the source emits its $(n+1)^{th}$ crest and this reaches the observer at time

$$t_{n+1} \quad n\,T_0 \quad \frac{L \quad n\ {}_s T_0}{v}$$

Hence, in a time interval

$$nT_0 \quad \frac{L \quad n v_s T_0}{v} \quad \frac{L}{v}$$

the observer's detector counts n crests and the observer records the period of the wave as T given by

$$T \quad nT_0 \quad \frac{L \quad n v_s T_0}{v} \quad \frac{L}{v} \;/n$$

$$= T_0 + \frac{v_s T_0}{v}$$

$$= T_0\left(1 + \frac{v_s}{v}\right) \qquad (15.49)$$

Equation (15.49) may be rewritten in terms of the frequency ν_o that would be measured if the source and observer were stationary, and the frequency ν observed when the source is moving, as

$$\nu = \nu_0\left(1 + \frac{v_s}{v}\right)^{-1} \qquad (15.50)$$

If v_s is small compared with the wave speed v, taking binomial expansion to terms in first order in v_s/v and neglecting higher power, Eq. (15.50) may be approximated, giving

$$\nu = \nu_0\left(1 - \frac{v_s}{v}\right) \qquad (15.51)$$

For a source approaching the observer, we replace v_s by $-v_s$ to get

$$\nu = \nu_0\left(1+\frac{\upsilon_s}{\upsilon}\right) \qquad (15.52)$$

The observer thus measures a lower frequency when the source recedes from him than he does when it is at rest. He measures a higher frequency when the source approaches him.

15.8.2 Observer Moving; Source Stationary

Now to derive the Doppler shift when the observer is moving with velocity υ_o towards the source and the source is at rest, we have to proceed in a different manner. We work in the reference frame of the moving observer. In this reference frame the source and medium are approaching at speed υ_o and the speed with which the wave approaches is $\upsilon_o + \upsilon$. Following a similar procedure as in the previous case, we find that the time interval between the arrival of the first and the $(n+1)$ th crests is

$$t_{n+1} - t_1 = n\,T_0 - \frac{n\upsilon_0 T_0}{\upsilon_0 + \upsilon}$$

The observer thus, measures the period of the wave to be

$$= T_0\left(1-\frac{\upsilon_0}{\upsilon_0+\upsilon}\right)$$

$$T_0 \quad 1 \quad \frac{\upsilon_0}{\upsilon}^{\;-1}$$

giving

$$\nu = {}_0\left(1+\frac{\upsilon_0}{\upsilon}\right) \qquad (15.53)$$

If $\frac{\upsilon_0}{\upsilon}$ is small, the Doppler shift is almost same whether it is the observer or the source moving since Eq. (15.53) and the approximate relation Eq. (15.51) are the same.

15.8.3 Both Source and Observer Moving

We will now derive a general expression for Doppler shift when both the source and the observer are moving. As before, let us take the direction from the observer to the source as the positive direction. Let the source and the observer be moving with velocities υ_s and υ_o respectively as shown in Fig.15.18. Suppose at time $t = 0$, the observer is at O_1 and the source is at S_1, O_1 being to the left of S_1. The source emits a wave of velocity υ, of frequency υ and

Application of Doppler effect

The change in frequency caused by a moving object due to Doppler effect is used to measure their velocities in diverse areas such as military, medical science, astrophysics, etc. It is also used by police to check over-speeding of vehicles.

A sound wave or electromagnetic wave of known frequency is sent towards a moving object. Some part of the wave is reflected from the object and its frequency is detected by the monitoring station. This change in frequency is called **Doppler shift**.

It is used at airports to guide aircraft, and in the military to detect enemy aircraft. Astrophysicists use it to measure the velocities of stars.

Doctors use it to study heart beats and blood flow in different part of the body. Here they use ulltrasonic waves, and in common practice, it is called **sonography**. Ultrasonic waves enter the body of the person, some of them are reflected back, and give information about motion of blood and pulsation of heart valves, as well as pulsation of the heart of the foetus. In the case of heart, the picture generated is called **echocardiogram**.

period T_0 all measured by an observer at rest with respect to the medium. Let L be the distance between O_1 and S_1 at $t = 0$, when the source emits the first crest. Now, since the observer is moving, the velocity of the wave relative to the observer is $\upsilon + \upsilon_0$. Therfore the first crest reaches the observer at time $t_1 = L/(\upsilon+\upsilon_0)$. At time $t = T_0$, both the observer and the source have moved to their new positions O_2 and S_2 respectively. The new distance between the observer and the source, $O_2\,S_2$, would be $L+(\upsilon_s - \upsilon_0)\,T_0]$. At S_2, the source emits a second crest. This reaches the observer at time.

$$t_2 = T_o + [L + (\upsilon_s - \upsilon_o)T_o)]\;/(\upsilon + \upsilon_o)$$

At time nT_o the source emits its $(n+1)$ th crest and this reaches the observer at time

$$t_{n+1} = nT_o + [L + n\,(\upsilon_s - \upsilon_o)T_o)]\;/(\upsilon + \upsilon_o)$$

Hence, in a time interval $t_{n+1} - t_1$, i.e.,

$$nT_o + [L + n\,(\upsilon_s - \upsilon_o)T_o)]\;/(\upsilon + \upsilon_o) - L\,/(\upsilon + \upsilon_o),$$

the observer counts n crests and the observer records the period of the wave as equal to T given by

$$T = T_0\left(1+\frac{\upsilon_s - \upsilon_o}{\upsilon + \upsilon_0}\right) = T_0\left(\frac{\upsilon+\upsilon_s}{\upsilon+\upsilon_0}\right) \qquad (15.54)$$

The frequency ν observed by the observer is given by

$$\nu = \nu_0\left(\frac{\upsilon+\upsilon_0}{\upsilon+\upsilon_s}\right) \qquad (15.55)$$

Consider a passenger sitting in a train moving on a straight track. Suppose she hears a whistle sounded by the driver of the train. What frequency will she measure or hear? Here both the observer and the source are moving with the same velocity, so there will be no shift in frequency and the passanger will note the natural frequency. But an observer outside who is stationary with respect to the track will note a higher frequency if the train is approaching him and a lower frequency when it recedes from him.

Note that we have defined the direction from the observer to the source as the positive

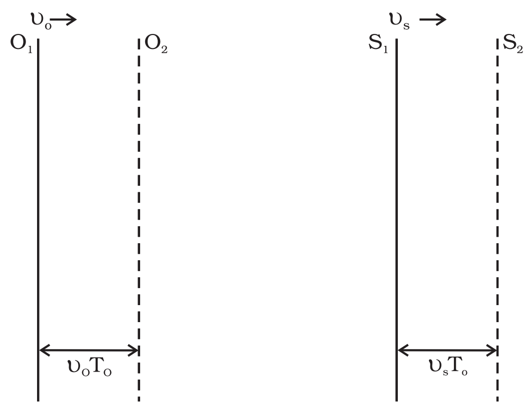


Fig. 15.18 *The observer O and the source S, both moving respectively with velocities υ_0 and υ_s . They are at position O_1 and S_1 at time t = 0, when the source emits the first crest of a sound, whose velocity is υ with respect to the medium. After one period, $t = T_0$, they have moved to O_2 and S_2, respectively through distances $\upsilon_0 T_0$ and $\upsilon_s T_0$, when the source emits the next crest.*

direction. Therefore, if the observer is moving towards the source, υ_0 has a positive (numerical) value whereas if O is moving away from S, υ_0 has a negative value. On the other hand, if S is moving away from O, υ_s has a positive value whereas if it is moving towards O, υ_s has a negative value. The sound emitted by the source travels in all directions. It is that part of sound coming towards the observer which the observer receives and detects. Therefore the relative velocity of sound with respect to the observer is $\upsilon + \upsilon_0$ in all cases.

Example 15.7 A rocket is moving at a speed of 200 m s^{-1} towards a stationary target. While moving, it emits a wave of frequency 1000 Hz. Some of the sound reaching the target gets reflected back to the rocket as an echo. Calculate (1) the frequency of the sound as detected by the target and (2) the frequency of the echo as detected by the rocket.

Answer (1) The observer is at rest and the source is moving with a speed of 200 m s^{-1}. Since this is comparable with the velocity of sound, 330 m s^{-1}, we must use Eq. (15.50) and not the approximate Eq. (15.51). Since the source is approaching a stationary target, $\upsilon_o = 0$, and υ_s must be replaced by $-\upsilon_s$. Thus, we have

$$\nu = \nu_0\left(1-\frac{\upsilon_s}{\upsilon}\right)^{-1}$$

$$\nu = 1000 \text{ Hz} \times [1 - 200 \text{ m s}^{-1}/330 \text{ m s}^{-1}]^{-1}$$

$$\simeq 2540 \text{ Hz}$$

(2) The target is now the source (because it is the source of echo) and the rocket's detector is now the detector or observer (because it detects echo). Thus, $\upsilon_s = 0$ and υ_o has a positive value. The frequency of the sound emitted by the source (the target) is ν, the frequency intercepted by the target and not ν_o. Therefore, the frequency as registered by the rocket is

$$\nu' = \nu\left(\frac{\upsilon+\upsilon_0}{\upsilon}\right)$$

$$= 2540 \text{ Hz} \times \frac{200 \text{ m s}^{-1} + 330 \text{ m s}^{-1}}{330 \text{ m s}^{-1}}$$

$$\simeq 4080 \text{ Hz}$$

SUMMARY

1. *Mechanical waves* can exist in material media and are governed by Newton's Laws.
2. *Transverse waves* are waves in which the particles of the medium oscillate perpendicular to the direction of wave propagation.
3. *Longitudinal waves* are waves in which the particles of the medium oscillate along the direction of wave propagation.
4. *Progressive wave* is a wave that moves from one point of medium to another.
5. *The displacement* in a sinusoidal wave propagating in the positive x direction is given by

   $$y\,(x,\ t) = a\ \sin\,(kx - \omega t + \phi)$$

   where *a is the* amplitude of the wave, *k* is the angular wave number, *ω is the* angular frequency, $(kx - \omega t + \phi)$ *is the* phase, *and* ϕ *is the* phase constant or phase angle.
6. *Wavelength* λ of a progressive wave is the distance between two consecutive points of the same phase at a given time. In a stationary wave, it is twice the distance between two consecutive nodes or anti nodes.
7. *Period* T of oscillation of a wave is defined as the time any element of the medium takes to move through one complete oscillation. It is related to the *angular frequency* ω through the relation

   $$T = \frac{2\pi}{\omega}$$

8. *Frequency* ν of a wave is defined as $1/T$ and is related to angular frequency by

   $$f = \frac{\omega}{2\pi}$$

9. *Speed* of a progressive wave is given by $\upsilon \quad \frac{}{k} \quad \frac{}{T}$
10. *The speed of a transverse wave* on a stretched string is set by the properties of the string. The speed on a string with tension T and linear mass density μ is

    $$\upsilon = \sqrt{\frac{T}{\mu}}$$

11. *Sound waves* are longitudinal mechanical waves that can travel through solids, liquids, or gases. The speed υ of sound wave in a fluid having *bulk modulus* B and density ρ is

    $$\upsilon = \sqrt{\frac{B}{\rho}}$$

    The speed of longitudinal waves in a metallic bar is

    $$\upsilon = \sqrt{\frac{Y}{\rho}}$$

    For gases, since $B = \gamma P$, the speed of sound is

    $$\upsilon = \sqrt{\frac{\gamma P}{\rho}}$$

12. When two or more waves traverse the same medium, the displacement of any element of the medium is the algebraic sum of the displacements due to each wave. This is known as the *principle of superposition* of waves

$$y = \sum_{i=1}^{n} f_i(x - vt)$$

13. Two sinusoidal waves on the same string exhibit *interference*, adding or cancelling according to the principle of superposition. If the two are travelling in the same direction and have the same amplitude a and frequency but differ in phase by a *phase constant* ϕ, the result is a single wave with the same frequency ω:

$$y(x, t) = 2a \cos\left(\frac{1}{2}\phi\right) \sin\left(kx - \omega t + \frac{1}{2}\phi\right)$$

If $\phi = 0$ or an integral multiple of 2π, the waves are exactly in phase and the interference is constructive; if $\phi = \pi$, they are exactly out of phase and the interference is destructive.

14. A travelling wave, at a rigid boundary or a closed end, is reflected with a phase reversal but the reflection at an open boundary takes place without any phase change.

For an incident wave

$$y_i(x, t) = a \sin(kx - \omega t)$$

the reflected wave at a rigid boundary is

$$y_r(x, t) = -a \sin(kx + \omega t)$$

For reflection at an open boundary

$$y_r(x,t) = a \sin(kx + \omega t)$$

15. The interference of two identical waves moving in opposite directions produces *standing waves.* For a string with fixed ends, the standing wave is given by

$$y(x, t) = [2a \sin kx] \cos \omega t$$

Standing waves are characterised by fixed locations of zero displacement called *nodes* and fixed locations of maximum displacements called *antinodes.* The separation between two consecutive nodes or antinodes is $\lambda/2$.

A stretched string of length L fixed at both the ends vibrates with frequencies given by

$$\nu = n\frac{v}{2L}, \quad n = 1, 2, 3, \ldots$$

The set of frequencies given by the above relation are called the *normal modes* of oscillation of the system. The oscillation mode with lowest frequency is called the *fundamental mode* or the *first harmonic.* The *second harmonic* is the oscillation mode with $n = 2$ and so on.

A pipe of length L with one end closed and other end open (such as air columns) vibrates with frequencies given by

$$\nu = \left(n + \frac{1}{2}\right)\frac{v}{2L}, \quad n = 0, 1, 2, 3, \ldots$$

The set of frequencies represented by the above relation are the *normal modes* of oscillation of such a system. The lowest frequency given by $v/4L$ is the fundamental mode or the first harmonic.

16. A string of length L fixed at both ends or an air column closed at one end and open at the other end, vibrates with frequencies called its normal modes. Each of these frequencies is a *resonant frequency* of the system.

17. *Beats* arise when two waves having slightly different frequencies, ν_1 and ν_2 and comparable amplitudes, are superposed. The beat frequency is

$$\nu_{beat} = \nu_1 \sim \nu_2$$

18. The *Doppler effect* is a change in the observed frequency of a wave when the source and the observer O moves relative to the medium. For sound the observed frequency ν is given in terms of the source frequency ν_o by

$$\nu = \nu_o \left(\frac{\upsilon + \upsilon_0}{\upsilon + \upsilon_S} \right)$$

here υ is the speed of sound through the medium, υ_o is the velocity of observer relative to the medium, and υ_s is the source velocity relative to the medium. In using this formula, velocities in the direction OS should be treated as positive and those opposite to it should be taken to be negative.

Physical quantity	Symbol	Dimensions	Unit	Remarks
Wavelength	λ	[L]	m	Distance between two consecutive points with the same phase.
Propagation constant	k	$[L^{-1}]$	m^{-1}	$k = \frac{2\pi}{\lambda}$
Wave speed	υ	$[LT^{-1}]$	$m\ s^{-1}$	$\nu = \upsilon\lambda$
Beat frequency	ν_{beat}	$[T^{-1}]$	s^{-1}	Difference of two close frequencies of superposing waves.

POINTS TO PONDER

1. A wave is not motion of matter as a whole in a medium. A wind is different from the sound wave in air. The former involves motion of air from one place to the other. The latter involves compressions and rarefactions of layers of air.
2. In a wave, energy and *not the matter* is transferred from one point to the other.
3. Energy transfer takes place because of the coupling through elastic forces between neighbouring oscillating parts of the medium.
4. Transverse waves can propagate only in medium with shear modulus of elasticity, Longitudinal waves need bulk modulus of elasticity and are therefore, possible in all media, solids, liquids and gases.
5. In a harmonic progressive wave of a given frequency all particles have the same amplitude but different phases at a given instant of time. In a stationary wave, all particles between two nodes have the same phase at a given instant but have different amplitudes.
6. Relative to an observer at rest in a medium the speed of a mechanical wave in that medium (υ) depends only on elastic and other properties (such as mass density) of the medium. It does not depend on the velocity of the source.
7. For an observer moving with velocity υ_o relative to the medium, the speed of a wave is obviously different from υ and is given by $\upsilon \pm \upsilon_o$.

EXERCISES

15.1 A string of mass 2.50 kg is under a tension of 200 N. The length of the stretched string is 20.0 m. If the transverse jerk is struck at one end of the string, how long does the disturbance take to reach the other end?

15.2 A stone dropped from the top of a tower of height 300 m high splashes into the water of a pond near the base of the tower. When is the splash heard at the top given that the speed of sound in air is 340 m s^{-1} ? (g = 9.8 m s^{-2})

15.3 A steel wire has a length of 12.0 m and a mass of 2.10 kg. What should be the tension in the wire so that speed of a transverse wave on the wire equals the speed of sound in dry air at 20 °C = 343 m s^{-1}.

15.4 Use the formula $v = \sqrt{\dfrac{\gamma P}{\rho}}$ to explain why the speed of sound in air

(a) is independent of pressure,
(b) increases with temperature,
(c) increases with humidity.

15.5 You have learnt that a travelling wave in one dimension is represented by a function $y = f(x, t)$ where x and t must appear in the combination $x - v\,t$ or $x + v\,t$, i.e. $y = f(x \pm v\,t)$. Is the converse true? Examine if the following functions for y can possibly represent a travelling wave :

(a) $(x - vt)^2$
(b) $\log[(x + vt)/x_0]$
(c) $1/(x + vt)$

15.6 A bat emits ultrasonic sound of frequency 1000 kHz in air. If the sound meets a water surface, what is the wavelength of (a) the reflected sound, (b) the transmitted sound? Speed of sound in air is 340 m s^{-1} and in water 1486 m s^{-1}.

15.7 A hospital uses an ultrasonic scanner to locate tumours in a tissue. What is the wavelength of sound in the tissue in which the speed of sound is 1.7 km s^{-1}? The operating frequency of the scanner is 4.2 MHz.

15.8 A transverse harmonic wave on a string is described by

$$y(x, t) = 3.0 \sin (36\ t + 0.018\ x + \pi/4)$$

where x and y are in cm and t in s. The positive direction of x is from left to right.

(a) Is this a travelling wave or a stationary wave ?
If it is travelling, what are the speed and direction of its propagation ?
(b) What are its amplitude and frequency ?
(c) What is the initial phase at the origin ?
(d) What is the least distance between two successive crests in the wave ?

15.9 For the wave described in Exercise 15.8, plot the displacement (y) versus (t) graphs for x = 0, 2 and 4 cm. What are the shapes of these graphs? In which aspects does the oscillatory motion in travelling wave differ from one point to another: amplitude, frequency or phase ?

15.10 For the travelling harmonic wave

$$y(x, t) = 2.0 \cos 2\pi (10t - 0.0080\ x + 0.35)$$

where x and y are in cm and t in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of

(a) 4 m,
(b) 0.5 m,
(c) $\lambda/2$,
(d) $3\lambda/4$

15.11 The transverse displacement of a string (clamped at its both ends) is given by

$$y(x,\ t) = 0.06 \sin \frac{2}{3}x \quad \cos (120\ \pi t)$$

where x and y are in m and t in s. The length of the string is 1.5 m and its mass is 3.0×10^{-2} kg.

Answer the following :

(a) Does the function represent a travelling wave or a stationary wave?

(b) Interpret the wave as a superposition of two waves travelling in opposite directions. What is the wavelength, frequency, and speed of each wave ?

(c) Determine the tension in the string.

15.12 (i) For the wave on a string described in Exercise 15.11, do all the points on the string oscillate with the same (a) frequency, (b) phase, (c) amplitude? Explain your answers. (ii) What is the amplitude of a point 0.375 m away from one end?

15.13 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represent (i) a travelling wave, (ii) a stationary wave or (iii) none at all:

(a) $y = 2 \cos (3x) \sin (10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3 \sin (5x - 0.5t) + 4 \cos (5x - 0.5t)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

15.14 A wire stretched between two rigid supports vibrates in its fundamental mode with a frequency of 45 Hz. The mass of the wire is 3.5×10^{-2} kg and its linear mass density is 4.0×10^{-2} kg m^{-1}. What is (a) the speed of a transverse wave on the string, and (b) the tension in the string?

15.15 A metre-long tube open at one end, with a movable piston at the other end, shows resonance with a fixed frequency source (a tuning fork of frequency 340 Hz) when the tube length is 25.5 cm or 79.3 cm. Estimate the speed of sound in air at the temperature of the experiment. The edge effects may be neglected.

15.16 A steel rod 100 cm long is clamped at its middle. The fundamental frequency of longitudinal vibrations of the rod are given to be 2.53 kHz. What is the speed of sound in steel?

15.17 A pipe 20 cm long is closed at one end. Which harmonic mode of the pipe is resonantly excited by a 430 Hz source ? Will the same source be in resonance with the pipe if both ends are open? (speed of sound in air is 340 m s^{-1}).

15.18 Two sitar strings A and B playing the note '*Ga*' are slightly out of tune and produce beats of frequency 6 Hz. The tension in the string A is slightly reduced and the beat frequency is found to reduce to 3 Hz. If the original frequency of A is 324 Hz, what is the frequency of B?

15.19 Explain why (or how):

(a) in a sound wave, a displacement node is a pressure antinode and vice versa,

(b) bats can ascertain distances, directions, nature, and sizes of the obstacles without any "eyes",

(c) a violin note and sitar note may have the same frequency, yet we can distinguish between the two notes,

(d) solids can support both longitudinal and transverse waves, but only longitudinal waves can propagate in gases, and

(e) the shape of a pulse gets distorted during propagation in a dispersive medium.

15.20 A train, standing at the outer signal of a railway station blows a whistle of frequency 400 Hz in still air. (i) What is the frequency of the whistle for a platform observer when the train (a) approaches the platform with a speed of 10 m s^{-1}, (b) recedes from the platform with a speed of 10 m s^{-1}? (ii) What is the speed of sound in each case ? The speed of sound in still air can be taken as 340 m s^{-1}.

15.21 A train, standing in a station-yard, blows a whistle of frequency 400 Hz in still air. The wind starts blowing in the direction from the yard to the station with at a speed of 10 m s^{-1}. What are the frequency, wavelength, and speed of sound for an observer standing on the station's platform? Is the situation exactly identical to the case when the air is still and the observer runs towards the yard at a speed of 10 m s^{-1}? The speed of sound in still air can be taken as 340 m s^{-1}

Additional Exercises

15.22 A travelling harmonic wave on a string is described by

$$y(x, t) = 7.5 \sin (0.0050x + 12t + \pi/4)$$

(a) what are the displacement and velocity of oscillation of a point at x = 1 cm, and t = 1 s? Is this velocity equal to the velocity of wave propagation?

(b) Locate the points of the string which have the same transverse displacements and velocity as the x = 1 cm point at t = 2 s, 5 s and 11 s.

15.23 A narrow sound pulse (for example, a short pip by a whistle) is sent across a medium. (a) Does the pulse have a definite (i) frequency, (ii) wavelength, (iii) speed of propagation? (b) If the pulse rate is 1 after every 20 s, (that is the whistle is blown for a split of second after every 20 s), is the frequency of the note produced by the whistle equal to 1/20 or 0.05 Hz ?

15.24 One end of a long string of linear mass density 8.0×10^{-3} kg m^{-1} is connected to an electrically driven tuning fork of frequency 256 Hz. The other end passes over a pulley and is tied to a pan containing a mass of 90 kg. The pulley end absorbs all the incoming energy so that reflected waves at this end have negligible amplitude. At t = 0, the left end (fork end) of the string x = 0 has zero transverse displacement (y = 0) and is moving along positive y-direction. The amplitude of the wave is 5.0 cm. Write down the transverse displacement y as function of x and t that describes the wave on the string.

15.25 A SONAR system fixed in a submarine operates at a frequency 40.0 kHz. An enemy submarine moves towards the SONAR with a speed of 360 km h^{-1}. What is the frequency of sound reflected by the submarine ? Take the speed of sound in water to be 1450 m s^{-1}.

15.26 Earthquakes generate sound waves inside the earth. Unlike a gas, the earth can experience both transverse (S) and longitudinal (P) sound waves. Typically the speed of S wave is about 4.0 km s^{-1}, and that of P wave is 8.0 km s^{-1}. A seismograph records P and S waves from an earthquake. The first P wave arrives 4 min before the first S wave. Assuming the waves travel in straight line, at what distance does the earthquake occur ?

15.27 A bat is flitting about in a cave, navigating via ultrasonic beeps. Assume that the sound emission frequency of the bat is 40 kHz. During one fast swoop directly toward a flat wall surface, the bat is moving at 0.03 times the speed of sound in air. What frequency does the bat hear reflected off the wall ?

ANSWERS

Chapter 9

9.1 1.8

9.2 (a) From the given graph for a stress of 150×10^6 N m^{-2} the strain is 0.002

(b) Approximate yield strength of the material is 3×10^8 N m^{-2}

9.3 (a) Material A

(b) Strength of a material is determined by the amount of stress required to cause fracture: material A is stronger than material B.

9.4 (a) False (b) True

9.5 1.5×10^{-4} m (steel); 1.3×10^{-4} m (brass)

9.6 Deflection = 4×10^{-6} m

9.7 2.8×10^{-6}

9.8 0.127

9.9 7.07×10^4 N

9.10 $D_{copper}/D_{iron} = 1.25$

9.11 1.539×10^{-4} m

9.12 2.026×10^9 Pa

9.13 1.034×10^3 kg/m^3

9.14 0.0027

9.15 0.058 cm^3

9.16 2.2×10^6 N/m^2

9.17 Pressure at the tip of anvil is 2.5×10^{11} Pa

9.18 (a) 0.7 m (b) 0.43 m from steel wire

9.19 Approximately 0.01 m

9.20 260 kN

9.21 2.51×10^{-4} m^3

Chapter 10

10.3 (a) decreases (b) η of gases increases, η of liquid decreases with temperature (c) shear strain, rate of shear strain (d) conservation of mass, Bernoulli's equation (e) greater.

10.5 6.2×10^6 Pa

10.6 10.5 m

10.7 Pressure at that depth in the sea is about 3×10^7 Pa. The structure is suitable since it can withstand far greater pressure or stress.

10.8 6.92×10^5 Pa

10.9 0.800

10.10 Mercury will rise in the arm containing spirit; the difference in levels of mercury will be 0.221 cm.

10.11 No, Bernoulli's principle applies to streamline flow only.

10.12 No, unless the atmospheric pressures at the two points where Bernoulli's equation is applied are significantly different.

10.13 9.8×10^2 Pa (The Reynolds number is about 0.3 so the flow is laminar).

10.14 1.5×10^3 N

10.15 Fig (a) is incorrect [Reason: at a constriction (i.e. where the area of cross-section of the tube is smaller), flow speed is larger due to mass conservation. Consequently pressure there is smaller according to Bernoulli's equation. We assume the fluid to be incompressible].

10.16 0.64 m s^{-1}

10.17 2.5×10^{-2} N m^{-1}

10.18 4.5×10^{-2} N for (b) and (c), the same as in (a).

10.19 Excess pressure = 310 Pa, total pressure = 1.0131×10^5 Pa. However, since data are correct to three significant figures, we should write total pressure inside the drop as 1.01×10^5 Pa.

10.20 Excess pressure inside the soap bubble = 20.0 Pa; excess pressure inside the air bubble in soap solution = 10.0 Pa. Outside pressure for air bubble = $1.01 \times 10^5 + 0.4 \times 10^3 \times 9.8$

$\times$ 1.2 = 1.06 $\times$ 10^5 Pa. The excess pressure is so small that up to three significant figures, total pressure inside the air bubble is 1.06 $\times$ 10^5 Pa.

10.21 55 N (Note, the base area does not affect the answer)

10.22 (a) absolute pressure = 96 cm of Hg; gauge pressure = 20 cm of Hg for (a), absolute pressure = 58 cm of Hg, gauge pressure = -18 cm of Hg for (b); (b) mercury would rise in the left limb such that the difference in its levels in the two limbs becomes19 cm.

10.23 Pressure (and therefore force) on the two equal base areas are identical. But force is exerted by water on the sides of the vessels also, which has a nonzero vertical component when the sides of the vessel are not perfectly normal to the base. This net vertical component of force by water on sides of the vessel is greater for the first vessel than the second. Hence the vessels weigh different even when the force on the base is the same in the two cases.

10.24 0.2 m

10.25 (a) The pressure drop is greater (b) More important with increasing flow velocity.

10.26 (a) 0.98 m s^{-1}; (b) 1.24 $\times$ 10^{-5} m^3 s^{-1}

10.27 4393 kg

10.28 5.8 cm s^{-1}, 3.9 $\times$ 10^{-10} N

10.29 5.34 mm

10.30 For the first bore, pressure difference (between the concave and convex side) = 2 $\times$ 7.3 $\times$ 10^{-2} / 3 $\times$ 10^{-3} = 48.7 Pa. Similarly for the second bore, pressure difference = 97.3 Pa. Consequently, the level difference in the two bores is [48.7 / (10^3 x 9.8)] m = 5.0 mm.

The level in the narrower bore is higher. (Note, for zero angle of contact, the radius of the meniscus equals radius of the bore. The concave side of the surface in each bore is at 1 atm).

10.31 (b) 8 km. If we consider the variation of g with altitude the height is somewhat more, about 8.2 km.

Chapter 11

11.1 Neon: –248.58 °C = – 415.44 °F;

CO_2: – 56.60 °C = – 69.88 °F

(use $t_F = \frac{9}{5}t_c + 32$)

11.2 T_A = (4/7) T_B

11.3 384.8 K

11.4 (a) Triple-point has a *unique* temperature; fusion point and boiling point temperatures depend on pressure; (b) The other fixed point is the absolute zero itself; (c) Triple-point is 0.01°C, not 0 °C; (d) 491.69.

11.5 (a) T_A = 392.69 K, T_B = 391.98 K; (b) The discrepancy arises because the gases are not perfectly ideal. To reduce the discrepancy, readings should be taken for lower and

lower pressures and the plot between temperature measured versus absolute pressure of the gas at triple point should be extrapolated to obtain temperature in the limit pressure tends to zero, when the gases approach ideal gas behaviour.

11.6 Actual length of the rod at 45.0 °C = (63.0 + 0.0136) cm = 63.0136 cm. (However, we should say that change in length up to three significant figures is 0.0136 cm, but the total length is 63.0 cm, up to three significant places. Length of the same rod at 27.0 °C = 63.0 cm.

11.7 When the shaft is cooled to temperature – 69^0C the wheel can slip on the shaft.

11.8 The diameter increases by an amount = 1.44×10^{-2} cm.

11.9 3.8×10^2 N

11.10 Since the ends of the combined rod are not clamped, each rod expands freely.

Δl_{brass} = 0.21 cm, Δl_{steel} = 0.126 cm = 0.13 cm

Total change in length = 0.34 cm. No 'thermal stress' is developed at the junction since the rods freely expand.

11.11 0.0147 = 1.5×10^{-2}

11.12 103 °C

11.13 1.5 kg

11.14 0.43 J g^{-1} K^{-1} ; smaller

11.15 The gases are diatomic, and have other degrees of freedom (i.e. have other modes of motion) possible besides the translational degrees of freedom. To raise the temperature of the gas by a certain amount, heat is to be supplied to increase the average energy of all the modes. Consequently, molar specific heat of diatomic gases is more than that of monatomic gases. It can be shown that if only rotational modes of motion are considered, the molar specific heat of diatomic gases is nearly (5/2) R which agrees with the observations for all the gases listed in the table, except chlorine. The higher value of molar specific heat of chlorine indicates that besides rotational modes, vibrational modes are also present in chlorine at room temperature.

11.16 (a) At the triple point temperature = – 56.6 °C and pressure = 5.11 atm.

(b) Both the boiling point and freezing point of CO_2 decrease if pressure decreases.

(c) The critical temperature and pressure of CO_2 are 31.1 °C and 73.0 atm respectively. Above this temperature, CO_2 will not liquefy even if compressed to high pressures.

(d) (a) vapour (b) solid (c) liquid

11.17 (a) No, vapour condenses to solid directly.

(b) It condenses to solid directly without passing through the liquid phase.

(c) It turns to liquid phase and then to vapour phase. The fusion and boiling points are where the horizontal line on *P-T* diagram at the constant pressure of 10 atm intersects the fusion and vaporization curves.

(d) It will not exhibit any clear transition to the liquid phase, but will depart more and more from ideal gas behaviour as its pressure increases.

11.18 4.3 g/min

11.19 3.7 kg

11.20 238 °C

11.22 9 min

Chapter 12

12.1 16 g per min

12.2 934 J

12.4 2.64

12.5 16.9 J

12.6 (a) 0.5 atm (b) zero (c) zero (assuming the gas to be ideal) (d) No, since the process (called free expansion) is rapid and cannot be controlled. The intermediate states are non-equilibrium states and do not satisfy the gas equation. In due course, the gas does return to an equilibrium state.

12.7 15%, 3.1×10^{9} J

12.8 25 W

12.9 450 J

12.10 10.4

Chapter 13

13.1 4×10^{-4}

13.3 (a) The dotted plot corresponds to 'ideal' gas behaviour; (b) $T_1 > T_2$; (c) 0.26 J K^{-1}; (d) No, 6.3×10^{-5} kg of H_2 would yield the same value

13.4 0.14 kg

13.5 5.3×10^{-6} m^3

13.6 6.10×10^{26}

13.7 (a) 6.2×10^{-21} J (b) 1.24×10^{-19} J (c) 2.1×10^{-16} J

13.8 Yes, according to Avogadro's law. No, v_{rms} is largest for the lightest of the three gases; neon.

13.9 2.52×10^{3} K

13.10 Use the formula for mean free path :

$$\bar{l} \quad \frac{1}{\sqrt{2}\ nd^2}$$

where d is the diameter of a molecule. For the given pressure and temperature $N/V = 5.10 \times 10^{25}\ \text{m}^{-3}$ and $= 1.0 \times 10^{-7}$ m. $v_{rms} = 5.1 \times 10^2\ \text{m s}^{-1}$.

collisional frequency = $\frac{v_{rms}}{\bar{l}} \quad 5.1 \quad 10^9\ \text{s}^{-1}$. Time taken for the collision = $d\ /\ v_{rms}$ = 4 × 10^{-13} s. Time taken between successive collisions = $l\ /\ v_{rms} = 2 \times 10^{-10}$ s. Thus the time taken between successive collisions is 500 times the time taken for a collision. Thus a molecule in a gas moves essentially free for most of the time.

13.11 Nearly 24 cm of mercury flows out, and the remaining 52 cm of mercury thread plus the 48 cm of air above it remain in equilibrium with the outside atmospheric pressure (We assume there is no change in temperature throughout).

13.12 Oxygen

13.14 Carbon[1.29 Å]; Gold [1.59 Å]; Liquid Nitrogen [1.77 Å]; Lithium [1.73 Å]; Liquid fluorine[1.88 Å]

Chapter 14

14.1 (b), (c)

14.2 (b) and (c): SHM; (a) and (d) represent periodic but not SHM [A polyatomic molecule has a number of natural frequencies; so in general, its vibration is a superposition of SHM's of a number of different frequencies. This superposition is periodic but not SHM].

14.3 (b) and (d) are periodic, each with a period of 2 s; (a) and (c) are not periodic. [Note in (c), repetition of merely one position is not enough for motion to be periodic; the entire motion during one period must be repeated successively].

14.4 (a) Simple harmonic, $T = (2\pi/\omega)$; (b) periodic, $T = (2\pi/\omega)$ but not simple harmonic; (c) simple harmonic, $T = (\pi/\omega)$; (d) periodic, $T = (2\pi/\omega)$ but not simple harmonic; (e) non-periodic; (f) non-periodic (physically not acceptable as the function $\to\infty$ as $t\to\infty$.

14.5 (a) 0, +, + ; (b) 0, –, – ; (c) –, 0,0 ; (d) –, –, – ; (e) +, +, + ; (f) –, –, –.

14.6 (c) represents a simple harmonic motion.

14.7 A = $\sqrt{2}$ cm, $\phi = 7\pi/4$; B = $\sqrt{2}$ cm, $a = \pi/4$.

14.8 219 N

14.9 Frequency 3.2 s^{-1}; maximum acceleration of the mass 8.0 m s^{-2}; maximum speed of the mass 0.4 m s^{-1}.

14.10 (a) $x = 2 \sin 20t$
(b) $x = 2 \cos 20t$
(c) $x = -2 \cos 20t$

where x is in cm. These functions differ neither in amplitude nor frequency. They differ in initial phase.

14.11 (a) $x = -3 \sin \pi t$ where x is in cm.

(b) $x = -2 \cos \frac{\pi}{2} t$ where x is in cm.

14.13 (a) F/k for both (a) and (b).

(b) $T = 2\pi$ for (a) and $2\pi\sqrt{\frac{m}{2k}}$ for (b)

14.14 100 m/min

14.15 8.4 s

14.16 (a) For a simple pendulum, k itself is proportional to m, so m cancels out.

(b) $\sin\theta < \theta$; if the restoring force, $mg \sin\theta$ is replaced by $mg\theta$, this amounts to effective reduction in g for large angles and hence an increase in time period T over that given by the formula $T = 2\pi\sqrt{\frac{l}{g}}$ where one assumes $\sin\theta = \theta$.

$\sqrt{\frac{m}{k}}$

(c) Yes, the motion in the wristwatch depends on spring action and has nothing to do with acceleration due to gravity.

(d) Gravity disappears for a man under free fall, so frequency is zero.

14.17 $T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + v^4/R^2}}}$. Hint: Effective acceleration due to gravity will get reduced due to radial acceleration v^2/R acting in the horizontal plane.

14.18 In equilibrium, weight of the cork equals the up thrust. When the cork is depressed by an amount x, the net upward force is $Ax\rho_l g$. Thus the force constant $k = A\rho_l g$.

Using $m = Ah\rho$, and $T = 2\pi\sqrt{\frac{m}{k}}$ one gets the given expression.

14.19 When both the ends are open to the atmosphere, and the difference in levels of the liquid in the two arms is h, the net force on the liquid column is $Ah\rho g$ where A is the area of cross-section of the tube and ρ is the density of the liquid. Since restoring force is proportional to h, motion is simple harmonic.

14.20 $T = 2\pi\sqrt{\frac{Vm}{Ba^2}}$ where B is the bulk modulus of air. For isothermal changes $B = P$.

14.21 (a) 5×10^4 N m^{-1}; (b) 1344.6 kg s^{-1}

14.22 Hint: Average K.E. = $\frac{1}{T}\int_0^T \frac{1}{2}mv^2 dt$; Average P.E.= $\frac{1}{T}\int_0^T \frac{1}{2}kx^2 dt$

14.23 Hint: The time period of a torsional pendulum is given by $T = 2\pi\sqrt{\frac{I}{\alpha}}$, where I is the moment of inertia about the axis of rotation. In our case $I = \frac{1}{2}MR^2$, where M is the mass of the disk and R its radius. Substituting the given values, $\alpha = 2.0$ N m rad^{-1}.

14.24 (a) $-5\pi^2$ m s^{-2} ; 0; (b) $-3\pi^2$ m s^{-2}; 0.4π m s^{-1}; (c) 0 ; $0.5\ \pi$ m s^{-1}

14.25 $\sqrt{\left(x_0^2 + \frac{v_0^2}{\omega^2}\right)}$

Chapter 15

15.1 0.5 s

15.2 8.7 s

15.3 2.06×10^4 N

15.4 Assume ideal gas law: $P = \frac{\rho RT}{M}$, where ρ is the density, M is the molecular mass, and T is the temperature of the gas. This gives $v = \sqrt{\frac{\gamma RT}{M}}$. This shows that v is:

(a) Independent of pressure.

(b) Increases as $\sqrt{T}$.

(c) The molecular mass of water (18) is less than that of N_2 (28) and O_2 (32).

Therefore as humidity increases, the effective molecular mass of air decreases and hence v increases.

15.5 The converse is not true. An obvious requirement for an acceptable function for a travelling wave is that it should be finite everywhere and at all times. Only function (c) satisfies this condition, the remaining functions cannot possibly represent a travelling wave.

15.6 (a) 3.4×10^{-4} m (b) 1.49×10^{-3} m

15.7 4.1×10^{-4} m

15.8 (a) A travelling wave. It travels from right to left with a speed of 20 ms^{-1}.

(b) 3.0 cm, 5.7 Hz

(c) $\pi/4$

(d) 3.5 m

15.9 All the graphs are sinusoidal. They have same amplitude and frequency, but different initial phases.

15.10 (a) $6.4\ \pi$ rad

(b) $0.8\ \pi$ rad

(c) π rad

(d) $(\pi/2)$ rad

15.11 (a) Stationary wave

(b) l = 3 m, n = 60 Hz, and υ = 180 m s^{-1} for each wave

(c) 648 N

15.12 (a) All the points except the nodes on the string have the same frequency and phase, but not the same amplitude.

(b) 0.042 m

15.13 (a) Stationary wave.

(b) Unacceptable function for any wave.

(c) Travelling harmonic wave.

(d) Superposition of two stationary waves.

15.14 (a) 79 m s^{-1}

(b) 248 N

15.15 347 m s^{-1}

Hint : $\upsilon_n = \dfrac{(2n-1)v}{4l}$; n = 1,2,3,….for a pipe with one end closed

15.16 5.06 km s^{-1}

15.17 First harmonic (fundamental); No.

15.18 318 Hz

15.20 (i) (a) 412 Hz, (b) 389 Hz, (ii) 340 m s^{-1} in each case.

15.21 400 Hz, 0.875 m, 350 m s^{-1}. No, because in this case, with respect to the medium, both the observer and the source are in motion.

15.22 (a) 1.666 cm, 87.75 cm s^{-1}; No, the velocity of wave propagation is –24 m s^{-1}

(b) All points at distances of $n\lambda$ ($n = \pm1, \pm2, \pm3, \ldots$) where λ = 12.6 m from the point x = 1 cm.

15.23 (a) The pulse does not have a definite wavelength or frequency, but has a definite speed of propagation (in a non-dispersive medium).

(b) No

15.24 $y = 0.05 \sin(\omega t - kx)$; here $\omega = 1.61 \times 10^3$ s^{-1}, $k = 4.84$ m^{-1}; x and y are in m.

15.25 45.9 kHz

15.26 1920 km

15.27 42.47 kHz

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GENERAL BOOKS

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